LIPSCHITZ MINORANTS OF BROWNIAN MOTION AND LÉVY PROCESSES

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ABSTRACT. For $\alpha > 0$, the α-Lipschitz minorant of a function $f : \mathbb{R} \to \mathbb{R}$ is the greatest function $m : \mathbb{R} \to \mathbb{R}$ such that $m \le f$ and $|m(s) - m(t)| \le \alpha|s - t|$ for all $s, t \in \mathbb{R}$, should such a function exist. If $X = (X_t)_{t\in\mathbb{R}}$ is a real-valued Lévy process that is not pure linear drift with slope $\pm\alpha$, then the sample paths of X have an α-Lipschitz minorant almost surely if and only if $|\mathbb{E}[X_1]| < \alpha$. Denoting the minorant by M, we investigate properties of the random closed set $\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$, which, since it is regenerative and stationary, has the distribution of the closed range of some subordinator "made stationary" in a suitable sense. We give conditions for the contact set $\mathcal{Z}$ to be countable or to have zero Lebesgue measure, and we obtain formulas that characterize the Lévy measure of the associated subordinator. We study the limit of $\mathcal{Z}$ as $\alpha \to \infty$ and find for the so-called abrupt Lévy processes introduced by Vigon that this limit is the set of local infima of X. When X is a Brownian motion with drift β such that $|\beta| < \alpha$, we calculate explicitly the densities of various random variables related to the minorant.

1. INTRODUCTION

A function $g : \mathbb{R} \to \mathbb{R}$ is α-Lipschitz for some $\alpha > 0$ if $|g(s) - g(t)| \le \alpha|s - t|$ for all $s, t \in \mathbb{R}$. If Γ is a set of α-Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\sup\{g(t_0) : g \in \Gamma\} < \infty$ for some $t_0 \in \mathbb{R}$, then the function $g^* : \mathbb{R} \to \mathbb{R}$ defined by $g^*(t) = \sup\{g(t) : g \in \Gamma\}$, $t \in \mathbb{R}$, is α-Lipschitz. Also, if $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function, then the set of α-Lipschitz functions dominated by f is non-empty if and only if f is bounded below on compact intervals and satisfies $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$ and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$. Therefore, in this case there is a unique greatest α-Lipschitz function dominated by f, and we call this function the *α-Lipschitz minorant* of f.

Denoting the α-Lipschitz minorant of f by m, an explicit formula for m is

$$\begin{aligned} m(t) &= \sup\{h \in \mathbb{R} : h - \alpha|t - s| \le f(s) \text{ for all } s \in \mathbb{R}\} \\ &= \inf\{f(s) + \alpha|t - s| : s \in \mathbb{R}\}. \end{aligned} \tag{1.1}$$

For the sake of completeness, we present a proof of these equalities in Lemma 8.1. The first equality says that for each $t \in \mathbb{R}$ we construct $m(t)$ by considering the set of "tent" functions $s \mapsto h - \alpha|t - s|$ that have a peak of height h at the position t and are dominated by f, and then taking the supremum of those peak heights – see Figure 2. The second equality is simply a rephrasing of the first.

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The property that the pointwise supremum of a suitable family of α-Lipschitz functions is also α-Lipschitz is reminiscent of the fact that the pointwise supremum of a suitable family of convex functions is also convex, and so the notion of the α-Lipschitz minorant of a function is analogous to that of the convex minorant. Indeed, there is a well-developed theory of abstract or generalized convexity that subsumes both of these concepts and is used widely in nonlinear optimization, particularly in the theory of optimal mass transport – see [Bal77, Die88, EN74, Lev03], Section 3.3 of [RR98] and Chapter 5 of [Vil09].

Furthermore, the second expression in (1.1) can be thought of as producing a function analogous to the smoothing of the function f by an integral kernel (that is, a function of the form $t \mapsto \int_{\mathbb{R}} K(|t-s|)\, ds$ for some suitable kernel $K : \mathbb{R} \to \mathbb{R}$) where one has taken the "min-plus" or "tropical" point of view and replaced the algebraic operations of $+$ and $\times$ by, respectively, $\wedge$ and $+$, so that integrals are replaced by infima. Note that if f is a continuous function that possesses an α_0-Lipschitz minorant for some α_0 (and hence an α-Lipschitz minorant for all $\alpha \geq \alpha_0$), then the α-Lipschitz minorants converge pointwise monotonically up to f as $\alpha \to +\infty$. Standard methods in optimization theory involve approximating a general function by a Lipschitz function and then determining approximate optima of the original function by finding optima of its Lipschitz approximant [HJL92a, HJL92b, HT93, NO05].

We investigate here the stochastic process $(M_t)_{t\in\mathbb{R}}$ obtained by taking the α-Lipschitz minorant of the sample path of a real-valued Lévy process $X = (X_t)_{t\in\mathbb{R}}$ for which the α-Lipschitz minorant almost surely exists, a condition that turns out to be equivalent to $|\mathbb{E}[X_1]| < \alpha$ when $X_0 = 0$ (excluding the trivial case where $X_t = \pm\alpha t$ for $t \in \mathbb{R}$) – see Proposition 2.1. See Figure 1 for an example of the minorant of a Brownian sample path. Our original motivation for this undertaking was the abovementioned analogy between α-Lipschitz minorants and convex minorants and the rich (and growing) literature on convex minorants of Brownian motion and Lévy processes in general [Gro83, Pit83, Bas84, Çin92, Ber00, CD01, Sui01, APRUB11, PU11].

In particular, we study properties of the *contact set* $\mathcal{Z} := \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$. This random set is clearly stationary and, as we show in Theorem 2.6, it is also regenerative. Consequently, its distribution is that of the closed range of a subordinator "made stationary" in a suitable manner. For a broad class of Lévy processes we are able to identify the associated subordinator in the sense that we can determined its Laplace exponent – see Theorem 3.8.

We show in Theorem 3.1 that if the paths of the Lévy process have either unbounded variation or bounded variation with drift d satisfying $|d| > \alpha$, then the associated subordinator has zero drift, and hence the random set $\mathcal{Z}$ has zero Lebesgue measure almost surely. Conversely, if the paths of the Lévy process have bounded variation and drift d satisfying $|d| < \alpha$, then the associated subordinator has positive drift, and hence the random set $\mathcal{Z}$ has infinite Lebesgue measure almost surely. In Proposition 3.7 we give conditions under which the Lévy measure of the subordinator associated to the set $\mathcal{Z}$ has finite total mass, which implies that $\mathcal{Z}$ is a discrete set in the case where it has zero Lebesgue measure.

If for the moment we write $\mathcal{Z}_\alpha$ instead of $\mathcal{Z}$ to stress the dependence on α, then it is clear that $\mathcal{Z}_{\alpha'} \subseteq \mathcal{Z}_{\alpha''}$ for $\alpha' \leq \alpha''$. We find in Theorem 4.1 that if the Lévy process is *abrupt*, that is, its paths have unbounded variation and "sharp" local

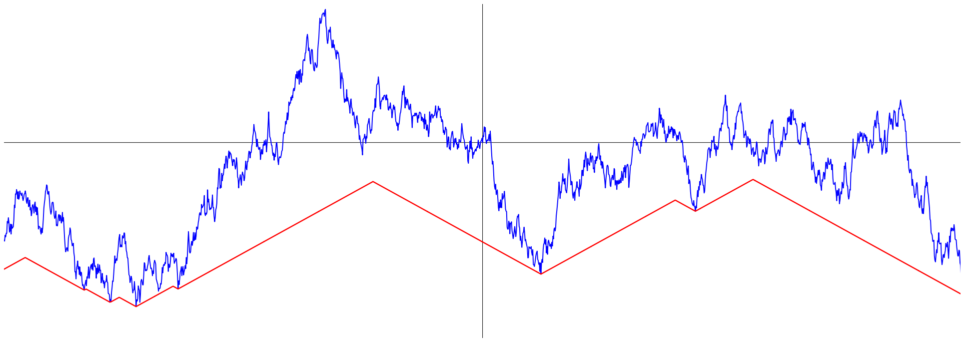

FIGURE 1. A typical Brownian motion sample path and its associated Lipschitz minorant.

extrema in a suitable sense (see Definition 3.3 for a precise definition), then the set $\bigcup_\alpha \mathcal{Z}_\alpha$ is almost surely the set of local infima of the Lévy process.

Lastly, when the Lévy process is a Brownian motion with drift, we can compute explicitly the distributions of a number of functionals of the α-Lipschitz minorant process. In order to describe these results, we first note that it follows from Lemma 8.3 below that the graph of the α-Lipschitz minorant M over one of the connected components of the complement of $\mathcal{Z}$ is almost surely a "sawtooth" that consists of a line of slope $+\alpha$ followed by a line of slope $-\alpha$. Set $G := \sup\{t < 0 : t \in \mathcal{Z}\}$, $D := \inf\{t > 0 : t \in \mathcal{Z}\}$, and put $K := D - G$. Let T be the unique $t \in [G, D]$ such that $M(t) = \max\{M(s) : s \in [G, D]\}$. That is, T is place where the peak of the sawtooth occurs. Further, let $H := X_T - M_T$ be the distance between the Brownian path and the α-Lipschitz minorant at the time where the peak occurs.

The following theorem summarizes a series of results that we establish in Section 7.

Theorem 1.1. *Suppose that X is a Brownian motion with drift β, where $|\beta| < \alpha$. Then, the following hold.*

(a) *The Lévy measure Λ of the subordinator associated to the contact set $\mathcal{Z}$ has finite mass and is characterized up to a scalar multiple by*

$$\frac{\int_{\mathbb{R}_+} 1 - e^{-\theta x}\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} = \frac{4(\alpha^2-\beta^2)\theta}{\left(\sqrt{2\theta+(\alpha-\beta)^2}+\alpha-\beta\right)\left(\sqrt{2\theta+(\alpha+\beta)^2}+\alpha+\beta\right)}$$

(b) *When $\beta = 0$ the measure Λ is absolutely continuous with respect to Lebesgue measure with*

$$\frac{\Lambda(dx)}{\Lambda(\mathbb{R}_+)} = \frac{2\alpha}{\sqrt{2\pi}}\left[x^{-\frac{1}{2}}e^{-\frac{\alpha^2 x}{2}} - 2\alpha^2\Phi(-\alpha x^{\frac{1}{2}})\right]\,dx,$$

where Φ is the standard normal cumulative distribution function (that is, $\Phi(z) := \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt$).

(c) *The distribution of T is characterized by*

$$\mathbb{E}[e^{-\theta T}] = 8\alpha(\alpha^2-\beta^2)\frac{1}{\theta}\left(\frac{1}{\sqrt{(\alpha+\beta)^2-2\theta}+3\alpha-\beta} - \frac{1}{\sqrt{(\alpha-\beta)^2+2\theta}+3\alpha+\beta}\right)$$

for $-\frac{(\alpha-\beta)^2}{2} \le \theta \le \frac{(\alpha+\beta)^2}{2}$. *Also,*

$$\mathbb{P}\{T > 0\} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right).$$

(d) *The random variable H has a* $\mathrm{Gamma}(2, 4\alpha)$ *distribution; that is, the distribution of H is absolutely continuous with respect to Lebesgue measure with density $h \mapsto (4\alpha)^2 h e^{-4\alpha h}$, $h \ge 0$.*

The rest of this article is organized as follows. In Section 2 we provide precise definitions and give some preliminary results relating to the nature of the contact set. In Section 3 we describe the subordinator associated with the contact set, and in Section 4 we describe the limit of the contact set as $\alpha \to \infty$. In order to prove Theorem 3.8 we need some preliminary results relating to the future infimum of a Lévy process, which we give in Section 5, and then we prove Theorem 3.8 in Section 6. In Section 7 we cover the special case when X is a two sided Brownian motion with drift in detail. Finally, in Section 8 we give some basic facts about the α-Lipschitz minorant of a function that are helpful throughout the paper.

2. Definitions and Preliminary Results

2.1. **Basic definitions.** Let $X = (X_t)_{t\in\mathbb{R}}$ be a real-valued Lévy process. That is, X has *càdlàg* sample paths, $X_0 = 0$, and $X_t - X_s$ is independent of $\{X_r : r \le s\}$ with the same distribution as X_{t-s} for $s, t \in \mathbb{R}$ with $s < t$.

The Lévy-Khintchine formula says that the characteristic function of X_t is given by $\mathbb{E}[e^{i\theta X_t}] = e^{-t\Psi(\theta)}$ for $\theta \in \mathbb{R}$, where

$$\Psi(\theta) = -ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{\{|x|<1\}})\,\Pi(dx)$$

with $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and Π a σ-finite measure concentrated on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2)\,\Pi(dx) < \infty$. We call σ^2 the *infinitesimal variance* of the Brownian component of X and Π the *Lévy measure* of X.

The sample paths of X have bounded variation if and only if $\sigma = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|)\,\Pi(dx) < \infty$. In this case Ψ can be rewritten as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}} (1 - e^{i\theta x})\,\Pi(dx).$$

We call $d \in \mathbb{R}$ the drift coefficient. For full details of these definitions see [Ber96].

2.2. **Existence of a minorant.**

Proposition 2.1. *Let X be a Lévy process. The α-Lipschitz minorant of X exists almost surely if and only if either $\sigma = 0$, $\Pi = 0$ and $|d| = \alpha$ (equivalently, $X_t = \alpha t$ for all $t \in \mathbb{R}$ or $X_t = -\alpha t$ for all $t \in \mathbb{R}$), or $\mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$.*

Proof. As we remarked in the Introduction, the α-Lipschitz minorant of a function $f : \mathbb{R} \to \mathbb{R}$ exists if and only if f is bounded below on compact intervals and satisfies $\liminf_{t\to-\infty} f(t) - \alpha t > -\infty$ and $\liminf_{t\to+\infty} f(t) + \alpha t > -\infty$.

Since the sample paths of a Lévy process are almost surely bounded on compact intervals, we need necessary and sufficient conditions for $\liminf_{t\to-\infty} X_t - \alpha t > -\infty$ and $\liminf_{t\to+\infty} X_t + \alpha t > -\infty$ to hold almost surely. This is equivalent to requiring that

$$\limsup_{t\to+\infty} X_t - \alpha t < +\infty \quad \text{a.s.} \qquad \text{and} \qquad \liminf_{t\to+\infty} X_t + \alpha t > -\infty \quad \text{a.s.}. \tag{2.1}$$

It is obvious that that the two conditions in (2.1) hold if $\sigma = 0$, $\Pi = 0$ and $|d| = \alpha$. It is clear from the strong law of large numbers that they also hold if $\mathbb{E}[|X_1|] < \infty$ and $|\mathbb{E}[X_1]| < \alpha$.

Consider the converse. Writing $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$ for $x \in \mathbb{R}$, the strong law of large numbers precludes any case where either $\mathbb{E}[X_1^+] = +\infty$ and $\mathbb{E}[X_1^-] < +\infty$ or $\mathbb{E}[X_1^+] < +\infty$ and $\mathbb{E}[X_1^-] = +\infty$. A result of Erickson [Don07, Chapter 4, Theorem 15] rules out the possibility $\mathbb{E}[X_1^+] = \mathbb{E}[X_1^-] = +\infty$, and so $\mathbb{E}[|X_1|] < \infty$. It now follows from the strong law of large numbers that $\lim_{t\to\infty} X_t/t = \mathbb{E}[X_1]$ and so $|\mathbb{E}[X_1]| \leq \alpha$. Suppose that X_t is non-degenerate for $t \neq 0$ (that is, that $\sigma \neq 0$ or $\Pi \neq 0$). Then, $\limsup_{t\to\infty} X_t - \mathbb{E}[X_1]t = +\infty$ a.s. and $\liminf_{t\to\infty} X_t - \mathbb{E}[X_1]t = -\infty$ a.s. (see, for example, [Kal02, Corollary 9.14]), and so $|\mathbb{E}[X_1]| < \alpha$ in this case. $\square$

Hypothesis 2.2. From now on we assume, unless we note otherwise, that the Lévy process $X = (X_t)_{t\in\mathbb{R}}$ has the properties:

- $X_0 = 0$;
- X_t is non-degenerate for $t \neq 0$;
- $\mathbb{E}[|X_1|] < \infty$;
- $|\mathbb{E}[X_1]| < \alpha$.

Notation 2.3. As in the Introduction, let $M = (M_t)_{t\in\mathbb{R}}$ be the α-Lipschitz minorant of X. Put $\mathcal{Z} = \{t \in \mathbb{R} : M_t = X_t \wedge X_{t-}\}$.

2.3. **The contact set is regenerative.** It follows fairly directly from our standing assumptions Hypothesis 2.2 that the random set $\mathcal{Z}$ is almost surely unbounded above and below. (Alternatively, it follows even more easily from Hypothesis 2.2 that $\mathcal{Z}$ is non-empty almost surely. We show below that $\mathcal{Z}$ is stationary, and any non-empty stationary random set is necessarily almost surely unbounded above and below.)

We now show that the contact set $\mathcal{Z}$ is stationary and that it is also regenerative in the sense of Fitzsimmons and Taksar [FT88]. For simplicity, we specialize the definition in [FT88] somewhat as follows by only considering random sets defined on probability spaces (rather than general σ-finite measure spaces).

Let Ω^0 denote the class of closed subsets of $\mathbb{R}$. For $t \in \mathbb{R}$ and $\omega^0 \in \Omega^0$, define

$$d_t(\omega^0) := \inf\{s > t : s \in \omega^0\}, \quad r_t(\omega^0) := d_t(\omega^0) - t,$$

and

$$\tau_t(\omega^0) = \mathbf{cl}\{s - t : s \in \omega^0 \cap (t, \infty)\} = \mathbf{cl}\left((\omega^0 - t) \cap (0, \infty)\right).$$

Here $\mathbf{cl}$ denotes closure and we adopt the convention $\inf \emptyset = +\infty$. Set $\mathcal{G}^0 = \sigma\{r_s : s \in \mathbb{R}\}$ and $\mathcal{G}^0_t = \sigma\{r_s : s \leq t\}$. Clearly $(d_t)_{t\in\mathbb{R}}$ is an increasing càdlàg process adapted to the filtration $(\mathcal{G}^0_t)_{t\in\mathbb{R}}$, and $d_t \geq t$ for all $t \in \mathbb{R}$.

A *random set* is a measurable mapping S from a measurable space $(\Omega, \mathcal{F})$ into $(\Omega^0, \mathcal{G}^0)$.

Definition 2.4. A probability measure $\mathbb{Q}$ on $(\Omega^0, \mathcal{G}^0)$ is regenerative with regeneration law $\mathbb{Q}^0$ if

(i) $\mathbb{Q}\{d_t = +\infty\} = 0$, for all $t \in \mathbb{R}$;
(ii) for all $t \in \mathbb{R}$ and for all $\mathcal{G}^0$-measurable nonnegative functions F,

$$\mathbb{Q}\left[F(\tau_{d_t}) \,|\, \mathcal{G}^0_{t+}\right] = \mathbb{Q}^0[F], \tag{2.2}$$

where we write $\mathbb{Q}[\cdot]$ and $\mathbb{Q}^0[\cdot]$ for expectations with respect to $\mathbb{Q}$ and $\mathbb{Q}^0$. A random set S defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a regenerative set if the push-forward of $\mathbb{P}$ by the map S (that is, the distribution of S) is a regenerative probability measure.

Remark 2.5. Suppose that the probability measure $\mathbb{Q}$ on $(\Omega^0, \mathcal{G}^0)$ is stationary; that is, if S^0 is the identity map on Ω^0, then the random set S^0 on $(\Omega^0, \mathcal{G}^0, \mathbb{Q})$ has the same distribution as $u + S^0$ for any $u \in \mathbb{R}$ or, equivalently, that the process $(r_t)_{t\in\mathbb{R}}$ has the same distribution as $(r_{t-u})_{t\in\mathbb{R}}$ for any $u \in \mathbb{R}$. Then, in order to check conditions (i) and (ii) of Definition 2.4 it suffices to check them for the case $t = 0$.

The probability measure $\mathbb{Q}^0$ is itself regenerative. It assigns all of its mass to the collection of closed subsets of $\mathbb{R}_+$. As remarked in [FT88], it is well known that any regenerative probability measure with this property arises as the distribution of a random set of the form $\mathbf{cl}\{Y_t : Y_t > Y_0,\, t \geq 0\}$, where $(Y_t)_{t\geq 0}$ is a subordinator (that is, a non-decreasing, real-valued Lévy process) with $Y_0 = 0$ – see [Mai71, Mai83]. Note that $\mathbf{cl}\{Y_t : Y_t > Y_0,\, t \geq 0\}$ has the same distribution as $\mathbf{cl}\{Y_{ct} : Y_{ct} > Y_{c0},\, t \geq 0\}$, and the distribution of the subordinator associated with a regeneration law can at most be determined up to linear time change (equivalently, the corresponding drift and Lévy measure can at most be determined up to a common constant multiple). It turns out that the distribution of the subordinator is unique except for this ambiguity – again see [Mai71, Mai83].

We refer the reader to [FT88] for a description of the sense in which a stationary regenerative probability measure $\mathbb{Q}$ with regeneration law $\mathbb{Q}^0$ can be regarded as $\mathbb{Q}^0$ "made stationary".

Theorem 2.6. *The random (closed) set $\mathcal{Z}$ is stationary and regenerative.*

Proof. We first show that $\mathcal{Z}$ is stationary. Note for $u \in \mathbb{R}$ that $u + \mathcal{Z} = \{t \in \mathbb{R} : X_{(t-u)} \wedge X_{(t-u)-} = M_{(t-u)}\}$. Define $(\check{X}_t)_{t\in\mathbb{R}}$ by $\check{X}_t = X_{t-u} - X_{(-u)}$ for $t \in \mathbb{R}$ and let $\check{M}$ be the α-Lipschitz minorant of $\check{X}$. Note that $\check{M}_t = M_{t-u} - X_{(-u)}$ for $t \in \mathbb{R}$. Therefore, $u + \mathcal{Z} = \{t \in \mathbb{R} : \check{X}_t \vee \check{X}_{t-} = \check{M}_t\}$ and hence $u + \mathcal{Z}$ has the same distribution as $\mathcal{Z}$ because $\check{X}$ has the same distribution as X.

We now show that $\mathcal{Z}$ is regenerative. For $t \in \mathbb{R}$ set

$$D_t := \inf\{s > t : X_s \wedge X_{s-} = M_s\} = d_t \circ \mathcal{Z},$$
$$R_t := D_t - t,$$
$$S_t := \inf\left\{s > t : X_s \wedge X_{s-} - \alpha(s-t) \leq \inf\{X_u - \alpha(u-t) : u \leq t\}\right\},$$
$$\check{S}_t := \inf\left\{s > t : X_{s-} - \alpha(s-t) \leq \inf\{X_u - \alpha(u-t) : u \leq t\}\right\},$$

and $\mathcal{F}_t := \bigcap_{s>t} \sigma\{X_u : u \leq s\}$. Note that S_t is a stopping time and $\check{S}_t$ is a predictable stopping time for the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. It follows from the quasi-left-continuity of X that $X_{\check{S}_t-} = X_{\check{S}_t}$ almost surely, and hence $X_{S_t} \leq X_{S_t-}$ almost surely. Lemma 8.4 then gives that

$$D_t = \inf\left\{s \geq S_t : X_t \wedge X_{t-} + \alpha(s - S_t) = \inf\{X_u + \alpha(u - S_t) : u \geq S_t\}\right\}$$

almost surely.

We have already remarked that $\mathcal{Z}$ is almost surely unbounded above and below, and hence condition (i) of Definition 2.4 holds. By Remark 2.5, in order to check condition (ii) of Definition 2.4, it suffices to consider the case $t = 0$.

For notational simplicity, set $S := S_0$ and $D := D_0$ – see Figure 3 for two illustrations of the construction of S and D from a sample path. For a random time U, let $\mathcal{F}_U$ be the σ-field generated by random variables of the form ξ_U, where $(\xi_t)_{t\in\mathbb{R}}$ is some optional process for the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ (cf. Millar [Mil77b, Mil78]). It follows from Corollary 8.2 (where we are thinking intuitively of removing the process to the right of D rather than to the right of zero) that $\bigcap_{\epsilon>0}\sigma\{R_s : s \le \epsilon\} \subseteq \mathcal{F}_D$.

Put

$$\tilde{X} = (\tilde{X}_s)_{s\ge 0} := ((X_{S+s} - X_S) + \alpha s)_{s\ge 0}\,.$$

By the strong Markov property at the stopping time S and the spatial homogeneity of X, the process $\tilde{X}$ is independent of $\mathcal{F}_S$ with the same distribution as the Lévy process $(X_t + \alpha t)_{t\ge 0}$. Suppose for the Lévy process $(X_t + \alpha t)_{t\ge 0}$ that zero is regular for the interval $(0,\infty)$. A result of Millar [Mil77a, Proposition 2.4] implies that almost surely there is a unique time $\tilde{T}$ such that $\tilde{X}_{\tilde{T}} = \inf\{\tilde{X}_s : s \ge 0\}$ and that if $\bar{T}$ is such that $\tilde{X}_{\bar{T}-} = \inf\{\tilde{X}_s : s \ge 0\}$, then $\bar{T} = \tilde{T}$. Thus, $\tilde{T} = \sup\{t \ge 0 : \tilde{X}_t \wedge \tilde{X}_{t-} = \inf\{\tilde{X}_s : s \ge 0\}\}$ and $D = S + \tilde{T}$. Combining this observation with the main result of Millar [Mil78] (see Remark 2.7 below) and the fact that $\tilde{X}_{\tilde{T}} = \inf\{\tilde{X}_s : s \ge 0\}$ gives that $(\tilde{X}_{\tilde{T}+t})_{t\ge 0}$ is conditionally independent of $\mathcal{F}_D$ given $\tilde{X}_{\tilde{T}}$. Thus, again by the spatial homogeneity of $\tilde{X}$, $(\tilde{X}_{\tilde{T}+t} - \tilde{X}_{\tilde{T}})_{t\ge 0}$ is independent of $\mathcal{F}_D$. This establishes condition (ii) of Definition 2.4 for $t = 0$.

If zero is not regular for the interval $(0,\infty)$ for the Lévy process $(X_t+\alpha t)_{t\ge 0}$, then zero is necessarily regular for the interval $(0,\infty)$ for the Lévy process $(X_{-t-}+\alpha t)_{t\ge 0}$ because this latter process the same distribution as $(-(X_t + \alpha t) + 2\alpha t)_{t\ge 0}$. The argument above then establishes that the random set $-\mathcal{Z}$ is regenerative. It follows from [FT88, Theorem 4.1] that $\mathcal{Z}$ is regenerative with the same distribution as $-\mathcal{Z}$. □

Remark 2.7. A key ingredient in the proof of Theorem 2.6 was the result of Millar from [Mil78] which says that, under suitable conditions, the future evolution of a càdlàg strong Markov process after the time it attains its global minimum is conditionally independent of the past up to that time given the value of the process and its left limit at that time. That result follows in turn from results in [GS74] on last exit decompositions or results in [PS72] on analogues of the strong Markov property at general coterminal times. We did not apply Millar's result directly; rather, we considered a random time $D = D_0$ that was the last time after a stopping time that a strong Markov process attained its infimum over times greater than the stopping time and combined Millar's result with the strong Markov property at the stopping time. An alternative route would have been to observe that the random time D is a *randomized coterminal time* in the sense of [Mil77b] for a suitable strong Markov process.

3. Identification of the associated subordinator

Let $Y = (Y_t)_{t\ge 0}$ be "the" subordinator associated with the regenerative set $\mathcal{Z}$. Write δ and Λ for the drift coefficient and Lévy measure of Y. Recall that these quantities are unique up to a common scalar multiple. The closed range of Y either has zero Lebesgue measure almost surely or infinite Lebesgue measure almost surely according to whether δ is zero or positive [Don07, Chapter 2, Theorem 3]. Consequently, the same dichotomy holds for the contact set $\mathcal{Z}$, and the following result gives necessary and sufficient conditions for each alternative.

Theorem 3.1. *If $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and $|d| = \alpha$, then the Lebesgue measure of $\mathcal{Z}$ is almost surely infinite. If X is not of this form, then the Lebesgue measure of $\mathcal{Z}$ is almost surely zero if and only if zero is regular for the interval $(-\infty, 0]$ for at least one of the Lévy processes $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$.*

Proof. Suppose first that $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$ and $|d| = \alpha$. In this case, the paths of X are piecewise linear with slope d. Our standing assumption $|\mathbb{E}[X_1]| < \alpha$ and the strong law of large numbers give $\lim_{t\to-\infty} X_t/t = \lim_{t\to+\infty} X_t/t = \mathbb{E}[X_1]$. It is now clear that $\mathcal{Z}$ has positive Lebesgue measure with positive probability and hence infinite Lebesgue measure almost surely.

Suppose now that X is not of this special form. It suffices by Fubini's theorem and the stationarity of $\mathcal{Z}$ to show that $\mathbb{P}\{0 \in \mathcal{Z}\} > 0$ if and only if zero is not regular for $(-\infty, 0]$ for both of the Lévy processes $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$.

Set $I^- := \inf\{X_t - \alpha t : t \leq 0\}$ and $I^+ := \inf\{X_t + \alpha t : t \geq 0\}$. Recall from (1.1) that $M_0 = I^- \wedge I^+$. Therefore,

$$\begin{aligned}\mathbb{P}\{0 \in \mathcal{Z}\} &= \mathbb{P}\{I^- \wedge I^+ = X_0 \wedge X_{0-} = 0\}\\ &= \mathbb{P}\{I^- = I^+ = 0\}\\ &= \mathbb{P}\{I^- = 0\}\mathbb{P}\{I^+ = 0\},\end{aligned}$$

and so $\mathbb{P}\{0 \in \mathcal{Z}\} > 0$ if and only if $\mathbb{P}\{I^- = 0\} > 0$ and $\mathbb{P}\{I^+ = 0\} > 0$.

Note that I^- has the same distribution as $\inf\{-X_t + \alpha t : t \geq 0\}$. From the formulas of Pecherskii and Rogozin [PR69] (or [Ber96, Theorem VI.5]),

$$\mathbb{E}[e^{\theta I^-}] = \exp\left(\int_0^\infty \int_{(-\infty,0]} (e^{\theta x} - 1)t^{-1}\mathbb{P}\{-X_t + \alpha t \in dx\}\, dt\right) \tag{3.1}$$

and

$$\mathbb{E}[e^{\theta I^+}] = \exp\left(\int_0^\infty \int_{(-\infty,0]} (e^{\theta x} - 1)t^{-1}\mathbb{P}\{X_t + \alpha t \in dx\}\, dt\right). \tag{3.2}$$

Taking the limit as $\theta \to \infty$ and applying monotone convergence in (3.1) and in (3.2) gives

$$\mathbb{P}\{I^- = 0\} = \exp\left(-\int_0^\infty t^{-1}\mathbb{P}\{-X_t + \alpha t < 0\}\, dt\right) \tag{3.3}$$

and

$$\mathbb{P}\{I^+ = 0\} = \exp\left(-\int_0^\infty t^{-1}\mathbb{P}\{X_t + \alpha t < 0\}\, dt\right). \tag{3.4}$$

Since we are assuming that it is not the case that $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$ and $|d| = \alpha$, we have $\mathbb{P}\{X_t + \alpha t = 0\} = \mathbb{P}\{-X_t + \alpha t = 0\} = 0$ for all $t > 0$. Moreover, by our standing assumption $|\mathbb{E}[X_1]| < \alpha$ it certainly follows that both $X_t + \alpha t$ and $-X_t + \alpha t$ drift to $+\infty$. Hence, by a result of Rogozin [Rog68] (or [Ber96, Theorem VI.12])

$$\int_1^\infty t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt < \infty \quad \text{and} \quad \int_1^\infty t^{-1}\mathbb{P}\{-X_t + \alpha t \leq 0\}\, dt < \infty. \tag{3.5}$$

The result now follows from Rogozin's regularity criterion [Rog68] (or [Ber96, Proposition VI.11]) which states that zero is not regular for the interval $(-\infty, 0]$

for both $(-X_t + \alpha t)_{t\geq 0}$ and $(X_t + \alpha t)_{t\geq 0}$ if and only if

$$\int_0^1 t^{-1}\mathbb{P}\{-X_t + \alpha t \leq 0\}\, dt < \infty \quad \text{and} \quad \int_0^1 t^{-1}\mathbb{P}\{X_t + \alpha t \leq 0\}\, dt < \infty. \tag{3.6}$$

□

Remark 3.2. (i) Note that zero is regular for the interval $(-\infty, 0]$ for both $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$ when X has unbounded variation, since then $\liminf_{t\to 0} t^{-1}X_t = -\infty$ [Rog68].

(ii) If X has bounded variation and drift coefficient d, then $\lim_{t\downarrow 0} t^{-1}(X_t+\alpha t) = d+\alpha$ and $\lim_{t\downarrow 0} t^{-1}(-X_t+\alpha t) = -d+\alpha$ [Sht65]. Thus, if $|d| < \alpha$, then zero is regular for $(-\infty, 0]$ for neither $(X_t + \alpha t)_{t\geq 0}$ or $(-X_t + \alpha t)_{t\geq 0}$, whereas if $|d| > \alpha$, then zero is regular for $(-\infty, 0]$ for exactly one of those two processes.

(iii) If X has bounded variation and $|d| = \alpha$, then an integral condition due to Bertoin involving the Lévy measure Π determines whether zero is regular for the interval $(-\infty, 0]$ for whichever of the processes $(X_t + \alpha t)_{t\geq 0}$ or $(-X_t + \alpha t)_{t\geq 0}$ has zero drift coefficient [Ber97].

Recall the notation $G = \sup\{t < 0 : t \in \mathcal{Z}\}$, $D = \inf\{t > 0 : t \in \mathcal{Z}\}$ and $K = D - G$ (note that $D = d_0 \circ \mathcal{Z}$). If the Lebesgue measure of $\mathcal{Z}$ is almost surely zero (equivalently when $\delta = 0$ [Don07, Chapter 2, Theorem 3]), then $0 \notin \mathcal{Z}$ and $G < 0 < D$, and the distribution of K is obtained by size-biasing the Lévy measure Λ; that is,

$$\mathbb{P}\{K \in dx\} = \frac{x\,\Lambda(dx)}{\int_{\mathbb{R}_+} y\,\Lambda(dy)}. \tag{3.7}$$

If the Lebesgue measure of $\mathcal{Z}$ is positive almost surely, then $\mathbb{P}\{K = 0\} > 0$ and we see by multiplying together (3.3) and (3.4) that

$$\mathbb{P}\{K = 0\} = \exp\left(-\int_0^\infty t^{-1}\left(\mathbb{P}\{X_t + \alpha t < 0\} + \mathbb{P}\{-X_t + \alpha t < 0\}\right) dt\right). \tag{3.8}$$

In this latter case, the conditional distribution of K given $K > 0$ is the size-biasing of Λ.

Theorem 3.1 and Remark 3.2 provide information about situations where the Lebesgue measure of the contact set $\mathcal{Z}$ is zero almost surely. It is of interest in such cases to determine whether the set $\mathcal{Z}$ is actually discrete almost surely or, equivalently, whether $\delta = 0$ and $\Lambda(\mathbb{R}_+) < \infty$. In order to state a result in this direction, we need to recall the definition of the so-called *abrupt* Lévy processes introduced by Vigon [Vig03].

We first write

$$\mathcal{M} := \bigcup_{\epsilon > 0}\{t \in \mathbb{R} : X_t \wedge X_{t-} = \inf\{X_s : s \in (t-\epsilon, t+\epsilon)\}\} \tag{3.9}$$

for the set of local infima of the path of X. As noted in [Vig03], if the paths of X have unbounded variation, then almost surely $X_{t-} = X_t$ for all $t \in \mathcal{M}$.

Definition 3.3. A Lévy process X is *abrupt* if its paths have unbounded variation and almost surely for all $t \in \mathcal{M}$

$$\limsup_{\varepsilon\uparrow 0} \frac{X_{t+\varepsilon} - X_{t-}}{\varepsilon} = -\infty \quad \text{and} \quad \liminf_{\varepsilon\downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = +\infty.$$

Remark 3.4. An equivalent definition may be made in terms of local maxima [Vig03, Remark 1.2]: a Lévy process X with unbounded variation is abrupt if almost surely for any t that is the time of a local maximum,

$$\liminf_{\varepsilon \uparrow 0} \frac{X_{t+\varepsilon} - X_{t-}}{\varepsilon} = +\infty \quad \text{and} \quad \limsup_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = -\infty.$$

Remark 3.5. A Lévy process X of unbounded variation is abrupt if and only if

$$\int_0^1 t^{-1} \mathbb{P}\{X_t \in [at, bt]\}\, dt < \infty, \quad \forall a < b, \tag{3.10}$$

(see [Vig03, Theorem 1.3]). Examples of abrupt Lévy processes include stable processes with stability parameter in the interval $(1, 2]$, processes with non-zero Brownian component, and any processes that creep upwards or downwards. An example of an unbounded variation process that is not abrupt is the Cauchy process.

Remark 3.6. The analytic condition given in Remark 3.5 (3.10) for a Lévy process X to be abrupt has an interpretation in terms of the smoothness of the the convex minorant of X over a finite interval. The results of Pitman and Uribe Bravo [PU11] imply that the number of segments of the convex minorant of X over a finite interval with slope between a and b is finite for all $a < b$ if and only if (3.10) holds.

Proposition 3.7. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 2.2. If X is either abrupt or has bounded variation with drift coefficient d satisfying $|d| > \alpha$, then $\Lambda(\mathbb{R}_+) < \infty$.*

Proof. Suppose first that X is abrupt. Then, every point of $\mathcal{Z}$ must be a local infimum of X by Theorem 4.1 below, and thus $\mathcal{Z}$ is countable. However, if $\Lambda(\mathbb{R}_+) = \infty$, then the closed ranged of any subordinator with Lévy measure Λ is a perfect set, and hence uncountable.

Suppose now that X has bounded variation with drift coefficient d satisfying $d > \alpha$. It suffices by the regenerativity property to show that $\inf\{t > D : t \in \mathcal{Z}\} > 0$ a.s..

Since $d > \alpha$, zero is not regular for $(0, \infty)$ for the Lévy process $(X_t + \alpha t)_{t \geq 0}$. A result of Millar states that any Lévy process for which zero is not regular for $(-\infty, 0)$ must jump into its global minimum – see [Mil77a, Theorem 3.1] and the remarks after that result. Thus, D is a jump times of X.

For $\delta > 0$, let $0 < J_1^\delta < J_2^\delta < \ldots$ be the successive nonnegative times at which X has jumps of size greater than δ in absolute value. The strong Markov property applied at the stopping time J_i^δ and Rogozin's celebrated result on the behavior of bounded variation processes at small times [Rog68] gives that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{J_i^\delta + \varepsilon} - X_{J_i^\delta}) = d$$

Hence, at any random time V such that $X_V \neq X_{V-}$ almost surely we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(X_{V+\varepsilon} - X_V) = d > \alpha.$$

Since D is a jump time of X, a.s. there exists $\varepsilon > 0$ such that $X_t > X_D + \alpha(t - D)$ for all $t \in (0, \varepsilon]$. Then, since $X_D = \inf_{t \geq D}\{X_t + \alpha t\}$ and since global minima of Lévy processes that are not compound Poisson processes with zero drift are unique [Ber96, Proposition VI.4], a.s. there exists a δ such that $(X_t + \alpha(t - D)) - \delta > 0$ for all $t > D + \varepsilon$. Since this implies that $\inf\{t > D : t \in \mathcal{Z}\} \geq \delta/2\alpha$, we can conclude that $\inf\{t > D : t \in \mathcal{Z}\} > 0$ a.s..

A time reversal argument then completes the proof for the case $d < -\alpha$. $\square$

In Section 6 we prove the following result, which characterizes Λ when X has paths of unbounded variation and satisfies certain extra conditions. In Corollary 5.4 we show that these condition hold when X has non-zero Brownian component.

Theorem 3.8. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 2.2. Suppose further that X_t has absolutely continuous distribution for all $t \neq 0$, and that the densities of the random variables $\inf_{t\geq 0}\{X_t + \alpha t\}$ and $\inf_{t\geq 0}\{X_{-t} + \alpha t\}$ are square integrable. Then, $\delta = 0$, $\Lambda(\mathbb{R}_+) < \infty$, and Λ is characterized by*

$$\begin{aligned}
&\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} \\
&\quad = 4\pi\alpha \int_{-\infty}^{\infty}\Bigg\{\exp\bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\big(e^{izX_t - iz\alpha t}-1\big)\,\mathbf{1}\{X_t \geq +\alpha t\} \\
&\qquad\qquad + \big(e^{izX_t + iz\alpha t}-1\big)\,\mathbf{1}\{X_t \leq -\alpha t\}\Big]\,dt\bigg) \\
&\qquad - \exp\bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\big(e^{-\theta t + izX_t - iz\alpha t}-1\big)\,\mathbf{1}\{X_t \geq +\alpha t\} \\
&\qquad\qquad + \big(e^{-\theta t + izX_t + iz\alpha t}-1\big)\,\mathbf{1}\{X_t \leq -\alpha t\}\Big]\,dt\bigg)\Bigg\}\,dz
\end{aligned}$$

for $\theta \geq 0$.

Note that the existence of the densities of the infima in the hypotheses of Theorem 3.8 comes from the assumption that X_t has absolutely continuous distribution for all $t \neq 0$ – see Lemma 5.1.

When the conditions of Theorem 3.8 are not satisfied, we are able to give the a characterization of Λ as a limit of integrals in the following way. Let $X^\varepsilon = X + \varepsilon B$, with B a (two-sided) standard Brownian motion independent of X, and let Λ^ε be the Lévy measure of the subordinator associated with the contact set for X^ε. Then

$$\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} = \lim_{\varepsilon\downarrow 0}\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda^\varepsilon(dx)}{\int_{\mathbb{R}_+} x\,\Lambda^\varepsilon(dx)}.$$

See Lemma 6.4 in Section 6 for details of this limit and a proof of the above equality.

4. The limit of the contact set for increasing slopes

We now investigate how $\mathcal{Z}$ changes as α increases. For the sake of clarity, let X be a fixed Lévy process with $X_0 = 0$ and $\mathbb{E}[|X_1|] < \infty$. Write $M^{(\alpha)} = (M_t^{(\alpha)})_{t\in\mathbb{R}}$ for the α-Lipschitz minorant of X for $\alpha > |\mathbb{E}[X_1]|$, and put $\mathcal{Z}_\alpha := \{t \in \mathbb{R} : X_t \wedge X_{t-} = M_t^{(\alpha)}\}$. For $|\mathbb{E}[X_1]| < \alpha' \leq \alpha''$, we have $M_t^{(\alpha')} \leq M_t^{(\alpha'')} \leq X_t$ for all $t \in \mathbb{R}$ (because any α'-Lipschitz function is also α''-Lipschitz), and so $\mathcal{Z}'_\alpha \subseteq \mathcal{Z}_{\alpha''}$.

If X has paths of bounded variation and drift coefficient d, then $|d| < \alpha$ for all α large enough. Since $\lim_{t\downarrow 0} t^{-1}X_t = -\lim_{t\downarrow 0} t^{-1}X_{-t} = d$, the law of large numbers implies that

$$\lim_{\alpha\to\infty} \mathbb{P}\{0 \in \mathcal{Z}_\alpha\} = \lim_{\alpha\to\infty} \mathbb{P}\{\inf_{t\geq 0}(X_t + \alpha t) = \inf_{t\leq 0}(X_t - \alpha t) = 0\} = 1,$$

and thus the set $\bigcup_{\alpha>|\mathbb{E}[X_1]|} Z_\alpha$ has full Lebesgue measure.

We now consider the case where X has paths of unbounded variation. Recall from (3.9) that $\mathcal{M}$ is the set of local infima of the path of X.

Theorem 4.1. *Let X be a Lévy process with $X_0 = 0$ and $|\mathbb{E}[X_1]| < \infty$. Then $\bigcup_{\alpha>|\mathbb{E}[X_1]|} \mathcal{Z}_\alpha \supseteq \mathcal{M}$. Furthermore, if X is abrupt, then $\bigcup_{\alpha>|\mathbb{E}[X_1]|} \mathcal{Z}_\alpha = \mathcal{M}$.*

Proof. Suppose that $t \in \mathcal{M}$ so that there exists $\epsilon > 0$ such that $\inf\{X_s : t - \epsilon < s < t + \epsilon\} = X_t = X_{t-}$. Fix any $\beta > |\mathbb{E}[X_1]|$. Then, by the strong law of large numbers, $\inf\{X_s + \beta s : s \geq 0\} > -\infty$ and $\inf\{X_s - \beta s : s \leq 0\} > -\infty$. It is clear that if $\alpha \in \mathbb{R}$ is such that

$$\alpha > -\frac{\inf\{X_s + \beta s : s \geq 0\} \vee \inf\{X_s - \beta s : s \leq 0\},}{\epsilon}$$

then $X_t = X_{t-} = M_t^{(\alpha)}$ and $t \in \mathcal{Z}_\alpha$. Hence $\bigcup_{\alpha>|\mathbb{E}[X_1]|} \mathcal{Z}_\alpha \supseteq \mathcal{M}$.

Now suppose that X is abrupt, and let $t \in \mathcal{Z}_\alpha$ for some $\alpha > |\mathbb{E}[X_1]|$. Then, one of the following three possibilities must occur:

(a) $X_t > X_{t-}$ and $\limsup_{\varepsilon\uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) \leq \alpha$;
(b) $X_{t-} > X_t$ and $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) \geq -\alpha$;
(c) $X_{t-} = X_t$ and $\limsup_{\varepsilon\uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) \leq \alpha$, $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) \geq -\alpha$.

We discount options (a) and (b) by assuming that t is a jump time of X and then showing that the lim inf or lim sup part of the statements cannot occur. Our argument borrows heavily from the proof of Property 2 in Proposition 1 of [PU11], which itself is based on the proof of Proposition 2.4 of [Mil77a], but is more detailed.

Arguing as in the proof of Proposition 3.7, for $\delta > 0$, let $0 < J_1^\delta < J_2^\delta < \ldots$ be the successive nonnegative times at which X has jumps of size greater than δ in absolute value. The strong Markov property applied at the stopping time J_i^δ and Rogozin's result on the behavior of unbounded variation processes at small times [Rog68] gives that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{J_i^\delta+\varepsilon} - X_{J_i^\delta}) = -\infty \quad \text{and} \quad \limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{J_i^\delta+\varepsilon} - X_{J_i^\delta}) = +\infty.$$

Hence, at any random time V such that $X_V \neq X_{V-}$ almost surely we have

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{V+\varepsilon} - X_V) = -\infty,$$

and, by a time reversal,

$$\limsup_{\varepsilon\uparrow 0} \varepsilon^{-1}(X_{V+\varepsilon} - X_{V-}) = +\infty.$$

Thus, neither of the possibilities (a) or (b) hold, and so (c) must hold. It then follows from Theorem 4.2 below that X must have a local minimum or maximum at t. However, X cannot have a local maximum at t by Remark 3.4, and so X must have a local minimum at t. □

The key to proving Theorem 4.1 in the abrupt case was the following theorem that describes the local behavior of an abrupt Lévy process at arbitrary times. This result is an immediate corollary of [Vig03, Theorem 2.6] once we use the fact that almost surely the paths of a Lévy processes cannot have both points of increase and points of decrease [Fou98].

Theorem 4.2. *Let X be an abrupt Lévy process. Then, almost surely for all t one of the following possibilities must hold:*

(i) $\limsup_{\varepsilon\uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) = +\infty$ *and* $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) = -\infty$*;*
(ii) $\limsup_{\varepsilon\uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) < +\infty$ *and* $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) = -\infty$*;*
(iii) $\limsup_{\varepsilon\uparrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_{t-}) = +\infty$ *and* $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(X_{t+\varepsilon} - X_t) > -\infty$*;*
(iv) *X has a local minimum or maximum at t.*

5. Future infimum of a Lévy process

For future use, we collect together in this section some preliminary results concerning the distribution of the infimum of a Lévy process $(Z_t)_{t\geq 0}$ and the time at which the infimum is attained.

Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process such that $Z_0 = 0$. Set $\underline{Z}_t := \inf\{Z_s : 0 \leq s \leq t\}$, $t \geq 0$. If Z is not a compound Poisson process (that is, either Z has a non-zero Brownian component or the Lévy measure of Z has infinite total mass or the Lévy measure has finite total mass but there is a non-zero drift coefficient), then

$$\mathbb{P}\{\exists 0 \leq s < t < u : \underline{Z}_s = \underline{Z}_t = Z_t \wedge Z_{t-} = \underline{Z}_u\} = 0 \tag{5.1}$$

– see, for example, [Ber96, Proposition VI.4]. Hence, almost surely for each $t \geq 0$ there is a unique time U_t such that $Z_{U_t} \wedge Z_{U_t -} = \underline{Z}_t$. If, in addition, $\lim_{t\to\infty} Z_t = +\infty$, then almost surely there is a unique time U_∞ such that $Z_{U_\infty} \wedge Z_{U_\infty -} = \underline{Z}_\infty := \inf\{Z_s : s \geq 0\}$.

Lemma 5.1. *Let Z be a Lévy process such that $Z_0 = 0$, Z_t has an absolutely continuous distribution for each $t > 0$, and $\lim_{t\to\infty} Z_t = +\infty$. Then, the distribution of $(U_\infty, \underline{Z}_\infty)$ restricted to $(0,\infty) \times (-\infty, 0]$ is absolutely continuous with respect to Lebesgue measure. Moreover, $\mathbb{P}\{(U_\infty, \underline{Z}_\infty) = (0,0)\} > 0$ if and only if zero is not regular for $(-\infty, 0)$.*

Proof. Because the random variable Z_t has an absolutely continuous distribution for each $t > 0$, it follows from [PU11, Theorem 2] that for all $t > 0$ the restriction of the distribution of the random vector $(U_t, \underline{Z}_t)$ is absolutely continuous with respect to Lebesgue measure on the set $(0,t] \times (-\infty, 0]$. Observe that

$$\mathbb{P}\left\{\exists s : (U_t, \underline{Z}_t) = (U_\infty, \underline{Z}_\infty)\ \forall t \geq s\right\} = 1.$$

Thus, if $A \subseteq (0,\infty) \times (-\infty, 0]$ is Borel with zero Lebesgue measure, then

$$\mathbb{P}\left\{(U_\infty, \underline{Z}_\infty) \in A\right\} = \lim_{t\to\infty} \mathbb{P}\left\{(U_t, \underline{Z}_t) \in A\right\} = 0.$$

The proof the claim concerning the atom at $(0,0)$ follows from the above formula, the fact that $\mathbb{P}\{(U_t, \underline{Z}_t) = (0,0)\} > 0$ if and only if zero is not regular for the interval $(-\infty, 0)$ [PU11, Theorem 2], and the hypothesis that $\lim_{t\to\infty} Z_t = +\infty$. □

Remark 5.2. Note that if the process Z has a non-zero Brownian component, then the random variable Z_t has an absolutely continuous distribution for all $t > 0$. Moreover, in this case zero is regular for the interval $(-\infty, 0)$

Let $\tau = (\tau_t)_{t\geq 0}$ be the local time at zero for the process $Z - \underline{Z}$. Write τ^{-1} for the inverse local time process. Set $\underline{H}_t := \underline{Z}_{\tau^{-1}(t)}$ The process $\underline{H} := (\underline{H}_t)_{t\geq 0}$ is the *descending ladder height process* for Z. If $\lim_{t\to\infty} Z_t = +\infty$, then $\hat{\underline{H}} := -\underline{H}$ is a subordinator killed at an independent exponential time (see, for example, [Ber96, Lemma VI.2]).

For the sake of completeness, we include the following observation that combines well-known results and probably already exists in the literature – it can be easily concluded from Theorem 19 and the remarks at the top of page 172 of [Ber96].

Lemma 5.3. *Let Z be a Lévy process such that $Z_0 = 0$ and $\lim_{t\to\infty} Z_t = +\infty$. Then, the distribution of random variable $\underline{Z}_\infty$ is absolutely continuous with a bounded density if and only if the (killed) subordinator $\underline{\hat{H}}$ has a positive drift coefficient.*

Proof. Let $S = (S_t)_{t\geq 0}$ be an (unkilled) subordinator with the same drift coefficient and Lévy measure as $\underline{\hat{H}}$, so that $-\underline{Z}_\infty$ has the same distribution as S_ζ, where ζ is an independent, exponentially distributed random time. Therefore, for some $q > 0$,

$$\mathbb{P}\{-\underline{Z}_\infty \in A\} = \int_0^\infty q e^{-qt}\mathbb{P}\{S_t \in A\}\, dt$$

for any Borel set $A \subseteq \mathbb{R}$. By a result of Kesten for general Lévy processes (see, for example, [Ber96, Theorem II.16]) the q-resolvent measure $\int_0^\infty e^{-qt}\mathbb{P}\{S_t \in \cdot\}\, dt$ of S is absolutely continuous with a bounded density for all $q > 0$ (equivalently, for some $q > 0$) if and only if points are not essentially polar for S. Moreover, points are not essentially polar for a bounded variation Lévy process (and, in particular, for a subordinator) if and only if the process has a non-zero drift coefficient [Ber96, Corollary II.20]. □

Corollary 5.4. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 2.2 and which has paths of unbounded variation almost surely. Then, the random variables $\inf\{X_t + \alpha t : t \geq 0\}$ and $\inf\{X_t - \alpha t : t \leq 0\}$ both have absolutely continuous distributions with bounded densities if and only if X has a non-zero Brownian component.*

Proof. By Lemma 5.3, the distributions in question are absolutely continuous with bounded densities if and only if the drift coefficients of the descending ladder processes for the two Lévy processes $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$ are non-zero. By the results of [Mil73] (see also [Ber96, Theorem VI.19]), this occurs if and only if both $(X_t + \alpha t)_{t\geq 0}$ and $(-X_t + \alpha t)_{t\geq 0}$ have positive probability of creeping down across x for some (equivalently, all) $x < 0$, where we recall that a Lévy process creeps down across $x < 0$ if the first passage time in $(-\infty, x)$ is not a jump time for the path of the process. Equivalently, both densities exist and are bounded if and only if the Lévy process $(X_t + \alpha t)_{t\geq 0}$ creeps downwards and the Lévy process $(X_t - \alpha t)_{t\geq 0}$ creeps upwards, where the latter notion is defined in the obvious way.

A result of Vigon [Vig02] (see also [Don07, Chapter 6, Corollary 9]) states that when X has unbounded variation, $(X_t + \alpha t)_{t\geq 0}$ creeps downward if and only if X creeps downward, and hence, in turn, if and only if $(X_t - \alpha t)_{t\geq 0}$ creeps downwards. A similar result applies to creeping upwards.

Thus, both densities exist and are bounded if and only if X creeps downwards and upwards. This occurs if and only if the ascending and descending ladder processes of X have positive drifts [Ber96, Theorem VI.19], which happens if and only if X has a non-zero Brownian component [Don07, Chapter 4, Corollary 4(i)] (or see the remark after the proof of [Ber96, Theorem VI.19]). □

6. The complementary interval straddling zero

6.1. **Distributions in the case of a non-zero Brownian component.** Suppose that $X = (X_t)_{t\in\mathbb{R}}$ is a Lévy process that satisfies our standing assumptions Hypothesis 2.2. Also, suppose until further notice that X has a non-zero Brownian component.

Recall that $M = (M_t)_{t\in\mathbb{R}}$ is the α-Lipschitz minorant of X and $\mathcal{Z}$ is the stationary regenerative set $\{t \in \mathbb{R} : X_t \wedge X_{t-} = M_t\}$. Recall also that $K = D - G$, where $G = \sup\{t < 0 : X_t \wedge X_{t-} = M_t\} = \sup\{t < 0 : t \in \mathcal{Z}\}$ and $D = \inf\{t > 0 : X_t \wedge X_{t-} = M_t\} = \inf\{t > 0 : t \in \mathcal{Z}\}$. Lastly, recall that T is the unique $t \in [G, D]$ such that $M_t = \max\{M_s : s \in [G, D]\}$.

Let f^+ (respectively, f^-) be the joint density of the random variables we denoted by $(U_\infty, \underline{Z}_\infty)$ in Lemma 5.1 in the case where the Lévy process Z is $(X_t + \alpha t)_{t\geq 0}$ (respectively, $(-X_t + \alpha t)_{t\geq 0}$).

Proposition 6.1. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 2.2. Suppose, moreover, that X has a non-zero Brownian component. Set $L := T - G$ and $R := D - T$. Then, the random vector (T, L, R) has a distribution that is absolutely continuous with respect to Lebesgue measure with joint density*

$$(\tau, \lambda, \rho) \mapsto 2\alpha \int_{-\infty}^{0} f^-(\lambda, h) f^+(\rho, h)\, dh, \quad \lambda, \rho > 0 \text{ and } \tau - \lambda < 0 < \tau + \rho.$$

Therefore, (T, G, D) also has an absolutely continuous distribution with joint density

$$(\tau, \gamma, \delta) \mapsto 2\alpha \int_{-\infty}^{0} f^-(\tau - \gamma, h) f^+(\delta - \tau, h)\, dh, \quad \gamma < 0 < \delta \text{ and } \gamma < \tau < \delta,$$

and K has an absolutely continuous distribution with density

$$\kappa \mapsto 2\alpha\kappa \int_0^{\kappa} \int_{-\infty}^{0} f^-(\xi, h) f^+(\kappa - \xi, h)\, dh\, d\xi, \quad \kappa > 0.$$

Proof. Observe that X is abrupt and so, by Proposition 3.7, $\mathcal{Z}$ is a stationary discrete random set with intensity

$$\left(\frac{\int_{\mathbb{R}_+} x\Lambda(dx)}{\Lambda(\mathbb{R}_+)}\right)^{-1} = \frac{\Lambda(\mathbb{R}_+)}{\int_{\mathbb{R}_+} x\Lambda(dx)} < \infty.$$

Hence, the set of times of peaks of the α-Lipschitz minorant M is also a stationary discrete random set with the same finite intensity. The point process consisting of a single point at time T is included in the set of times of peaks of M, and so for A a Lebesgue measurable set with Lebesgue measure $\lambda(A)$ we have

$$\begin{aligned}
\mathbb{P}\{T \in A\} &\leq \mathbb{P}\{\text{at least one peak of } M \text{ at a time } t \in A\} \\
&\leq \mathbb{E}[\text{number of times of peaks in } A] \\
&= \frac{\Lambda(\mathbb{R}_+)}{\int_{\mathbb{R}_+} x\Lambda(dx)}\, \lambda(A).
\end{aligned}$$

Thus, the distribution of T is absolutely continuous with respect to Lebesgue measure with density bounded above by $\Lambda(\mathbb{R}_+)/\int_{\mathbb{R}_+} x\Lambda(dx)$.

It follows from the observations made in the proof of Theorem 2.6 about the nature of the global minimum of the process $\tilde{X}$ that under our hypotheses, almost surely $X_G = X_{G-} = M_T - \alpha|G-T| = M_T - \alpha L$, $X_D = X_{D-} = M_T - \alpha|D-T| = M_T - \alpha R$, and $X_t \wedge X_{t-} > M_T - \alpha|t-T|$ for $t \notin \{G, D\}$. Thus,

$$0 = \inf\{t \geq 0 : X_{T+t} - (M_T - \alpha t)\} = X_{T+R} - (M_T - \alpha R)$$

and

$$\begin{aligned} 0 &= \inf\{t \leq 0 : X_{T+t} - (M_T + \alpha t)\} \\ &= \inf\{t \geq 0 : X_{T-t} - (M_T - \alpha t)\} = X_{T-L} - (M_T - \alpha L). \end{aligned}$$

Consequently,

$$\begin{aligned} X_{T-L} - X_T + \alpha L &= \inf\{t \geq 0 : X_{T-t} - X_T + \alpha t\} \\ &= \inf\{t \geq 0 : X_{T+t} - X_T + \alpha t)\} = X_{T+R} - X_T + \alpha R. \end{aligned} \tag{6.1}$$

Conversely, (T, L, R) is the unique triple with $T - L < 0 < T + R$ such that (6.1) holds.

Fix $\tau \in \mathbb{R}$ and $\lambda, \rho \in \mathbb{R}_+$ such that $\tau - \lambda < 0 < \tau + \rho$. Set

$$\begin{aligned} Z_t^- &:= X_{\tau-t} - X_\tau + \alpha t, \quad t \geq 0, \\ \underline{Z}^- &:= \inf\{Z_t^- : t \geq 0\}, \\ U^- &:= \inf\{t \geq 0 : Z_t^- = \underline{Z}^-\}. \end{aligned}$$

For $0 < \Delta\tau < \rho$ set

$$\begin{aligned} Z_t^+ &:= X_{t+\tau+\Delta\tau} - X_{\tau+\Delta\tau} + \alpha t, \quad t \geq 0, \\ \underline{Z}^+ &:= \inf\{Z_t^+ : t \geq 0\}, \\ U^+ &:= \inf\{t \geq 0 : Z_t^+ = \underline{Z}^+\}. \end{aligned}$$

From (6.1) we have for fixed $\Delta\lambda > 0$ and $\Delta\rho > 0$ that

$$\begin{aligned} &\mathbb{P}\{T \in [\tau, \tau + \Delta\tau],\, L \in [\lambda, \lambda + \Delta\lambda],\, R \in [\rho, \rho + \Delta\rho]\} \\ &\quad \approx \mathbb{P}(\{U^- \in [\lambda, \lambda + \Delta\lambda],\, U^+ \in [\rho, \rho + \Delta\rho]\} \\ &\qquad \cap \{\exists 0 \leq s \leq \Delta\tau : X_\tau + \underline{Z}^- + \alpha s = X_{\tau+\Delta\tau} + \underline{Z}^+ + \alpha(\Delta\tau - s)\}) \\ &\quad = \mathbb{P}(\{U^- \in [\lambda, \lambda + \Delta\lambda],\, U^+ \in [\rho, \rho + \Delta\rho]\} \\ &\qquad \cap \{\exists 0 \leq s \leq \Delta\tau : (\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) = 2\alpha s - \alpha\Delta\tau\}) \\ &\quad = \mathbb{P}(\{U^- \in [\lambda, \lambda + \Delta\lambda],\, U^+ \in [\rho, \rho + \Delta\rho]\} \\ &\qquad\quad \cap \{(\underline{Z}^+ - \underline{Z}^-) + (X_{\tau+\Delta\tau} - X_\tau) \in [-\alpha\Delta\tau, +\alpha\Delta\tau]\}) \end{aligned}$$

in the sense that the ratio of the two sides converges to 1 as $\Delta\tau \downarrow 0$.

Note that the random vectors $(U^-, \underline{Z}^-)$ and $(U^+, \underline{Z}^+)$ are independent with respective densities f^- and f^+, and so the joint density of $(U^-, U^+, \underline{Z}^+ - \underline{Z}^-)$ is

$$(u, v, w) \mapsto \int_{-\infty}^{\infty} f^-(u, h - w) f^+(v, h)\, dh.$$

Thus, using the fact that the random variable $\underline{Z}^+ - \underline{Z}^-$ is independent of $X_{\tau+\Delta\tau} - X_\tau$ and the latter random variable has the same distribution as $X_{\Delta\tau}$,

$$\begin{aligned}&\mathbb{P}\{T \in [\tau, \tau+\Delta\tau],\, L \in [\lambda, \lambda+\Delta\lambda],\, R \in [\rho, \rho+\Delta\rho]\}\\ &\quad\approx \int_\lambda^{\lambda+\Delta\lambda} du \int_\rho^{\rho+\Delta\rho} dv \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dh\\ &\qquad\times \mathbb{P}\{-w-\alpha\Delta\tau < X_{\Delta\tau} < -w+\alpha\Delta\tau\}\, f^-(u, h-w) f^+(v,h),\end{aligned}$$

again in the sense that the ratio of the two sides converges to 1 as $\Delta\tau \downarrow 0$.

By Fubini's theorem,

$$\begin{aligned}&\int_{-\infty}^{\infty} dw\, \mathbb{P}\{-w-\alpha\Delta\tau < X_{\Delta\tau} < -w+\alpha\Delta\tau\}\\ &\quad= \mathbb{E}\left[\int_{-\infty}^{\infty} dw\, \mathbf{1}\{-X_{\Delta\tau}-\alpha\Delta\tau < w < -X_{\Delta\tau}+\alpha\Delta\tau\}\right]\\ &\quad= \mathbb{E}\left[2\alpha\Delta\tau\right] = 2\alpha\Delta\tau.\end{aligned}$$

Moreover, for any $\epsilon > \Delta\tau$,

$$\begin{aligned}&\int_{-\infty}^{\infty} dw\, \mathbb{P}\{-w-\alpha\Delta\tau < X_{\Delta\tau} < -w+\alpha\Delta\tau\}\mathbf{1}\{|w|>\epsilon\}\\ &\quad= \mathbb{E}\left[\int_{-\infty}^{\infty} dw\, \mathbf{1}\{-X_{\Delta\tau}-\alpha\Delta\tau < w < -X_{\Delta\tau}+\alpha\Delta\tau,\, |w|>\epsilon\}\right]\\ &\quad= \mathbb{E}\left[(|X_{\Delta\tau}| - (\epsilon-\Delta\tau))_+ \wedge (2\Delta\tau)\right].\end{aligned}$$

Note that $(\Delta\tau)^{-1}[(|X_{\Delta\tau}| - (\epsilon-\Delta\tau))_+ \wedge (2\Delta\tau)] \le 2$ and that the random variable on the left of this inequality converges to 0 almost surely as $\Delta\tau \downarrow 0$. Hence, by bounded convergence,

$$\lim_{\Delta\tau\downarrow 0} \int_{-\infty}^{\infty} dw\, (\Delta\tau)^{-1}\mathbb{P}\{-w-\alpha\Delta\tau < X_{\Delta\tau} < -w+\alpha\Delta\tau\}\mathbf{1}\{|w|>\epsilon\} = 0.$$

Furthermore, the independent random variables $\underline{Z}^-$ and $\underline{Z}^+$ both have bounded densities by Corollary 5.4; that is, the functions $h \mapsto \int_0^\infty du\, f^-(u,h)$ and $h \mapsto \int_0^\infty dv\, f^+(v,h)$ both belong to $L^1 \cap L^\infty$. Therefore, the functions $h \mapsto \int_\lambda^{\lambda+\Delta\lambda} du\, f^-(u,h)$ and $h \mapsto \int_\rho^{\rho+\Delta\rho} dv\, f^+(v,h)$ both certainly belong to $L^1 \cap L^\infty$.

It now follows from the Lebesgue differentiation theorem that

$$\begin{aligned}&\lim_{\Delta\tau\downarrow 0} (\Delta\tau)^{-1} \int_{-\infty}^{\infty} dw\, \mathbb{P}\{-w-\alpha\Delta\tau < X_{\Delta\tau} < -w+\alpha\Delta\tau\} \int_\lambda^{\lambda+\Delta\lambda} du\, f^-(u, h-w)\\ &\quad= 2\alpha \int_\lambda^{\lambda+\Delta\lambda} du\, f^-(u,h)\end{aligned}$$

for Lebesgue almost every $h \in \mathbb{R}$. Moreover, the quantity on the left is bounded by $\sup_{h\in\mathbb{R}} 2\alpha \int_\lambda^{\lambda+\Delta\lambda} du\, f^-(u,h) < \infty$. Therefore, by bounded convergence,

$$\begin{aligned}&\lim_{\Delta\tau\downarrow 0} (\Delta\tau)^{-1}\mathbb{P}\{T \in [\tau, \tau+\Delta\tau],\, L \in [\lambda, \lambda+\Delta\lambda],\, R \in [\rho, \rho+\Delta\rho]\}\\ &\quad= 2\alpha \int_\lambda^{\lambda+\Delta\lambda} du \int_\rho^{\rho+\Delta\rho} dv \int_{-\infty}^{\infty} dh\, f^-(u,h) f^+(v,h).\end{aligned} \tag{6.2}$$

As we observed above, the measure $\mathbb{P}\{T \in d\tau\}$ is absolutely continuous with density bounded above by $\Lambda(\mathbb{R}_+) < \infty$, and so the same is certainly true of the measure

$\mathbb{P}\{T \in d\tau,\, L \in [\lambda, \lambda + \Delta\lambda],\, R \in [\rho, \rho + \Delta\rho]\}$ for fixed $\lambda, \Delta\lambda, \rho, \Delta\rho$. Therefore, by (6.2) and the Lebesgue differentiation theorem,

$$\begin{aligned}
&\mathbb{P}\{T \in A,\, L \in [\lambda, \lambda + \Delta\lambda],\, R \in [\rho, \rho + \Delta\rho]\} \\
&\quad = 2\alpha \int_{-\infty}^{\infty} d\tau \int_{\lambda}^{\lambda+\Delta\lambda} du \int_{\rho}^{\rho+\Delta\rho} dv \int_{-\infty}^{\infty} dh\, f^-(u,h) f^+(v,h) \mathbf{1}\{\tau \in A\}
\end{aligned}$$

for any Borel set $A \subseteq (-\rho, \lambda)$, and this establishes that (T, L, R) has the claimed density.

The remaining two claims follow immediately. □

Corollary 6.2. *Under the assumptions of Proposition 6.1,*

$$\begin{aligned}
\mathbb{E}[e^{-\theta K}] = -4\pi\alpha \frac{d}{d\theta} \int_{-\infty}^{\infty} \Bigg(&\exp\left\{ \int_0^\infty dt \int_0^\infty [e^{-\theta t + izx} - 1] t^{-1} \mathbb{P}\{X_t - \alpha t \in dx\} \right\} \\
&\times \exp\left\{ \int_0^\infty dt \int_0^\infty [e^{-\theta t - izx} - 1] t^{-1} \mathbb{P}\{-X_t - \alpha t \in dx\} \right\} \Bigg) dz.
\end{aligned}$$

Proof. From Proposition 6.1,

$$\begin{aligned}
&\mathbb{E}[e^{-\theta K}] \\
&\quad = 2\alpha \int_{-\infty}^{0} \int_0^\infty \kappa \int_0^\kappa f^-(\kappa - \xi, h) f^+(\xi, h) e^{-\theta\kappa}\, d\xi\, d\kappa\, dh \\
&\quad = 2\alpha \int_{-\infty}^{0} \int_0^\infty \int_\xi^\infty \kappa f^-(\kappa - \xi, h) f^+(\xi, h) e^{-\theta\kappa}\, d\kappa\, d\xi\, dh \\
&\quad = 2\alpha \int_{-\infty}^{0} \Bigg(\int_0^\infty f^+(\xi, h) e^{-\theta\xi} \int_\xi^\infty (\kappa - \xi) f^-(\kappa - \xi, h) e^{-\theta(\kappa - \xi)}\, d\kappa\, d\xi \\
&\qquad + \int_0^\infty \xi f^+(h, \xi) e^{-\theta\xi} \int_\xi^\infty f^-(h, \kappa - \xi) e^{-\theta(\kappa-\xi)}\, d\kappa\, d\xi \Bigg) dh \qquad (6.3) \\
&\quad = 2\alpha \int_{-\infty}^{0} \Bigg(\int_0^\infty f^+(\xi, h) e^{-\theta\xi} \int_0^\infty \kappa f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa\, d\xi \\
&\qquad + \int_0^\infty \xi f^+(\xi, h) e^{-\theta\xi} \int_0^\infty f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa\, d\xi \Bigg) dh \\
&\quad = -2\alpha \frac{d}{d\theta} \Bigg(\int_{-\infty}^{0} \Big(\int_0^\infty f^+(\xi, h) e^{-\theta\xi}\, d\xi \Big) \Big(\int_0^\infty f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa \Big) dh \Bigg)
\end{aligned}$$

Viewing $\int_0^\infty f^+(\xi, h) e^{-\theta\xi}\, d\xi$ and $\int_0^\infty f^-(\kappa, h) e^{-\theta\kappa}\, d\kappa$ as functions of h that belong to $L^1 \cap L^\infty \subset L^2$, we can use Plancherel's Theorem and then the Pecherskii-Rogozin formulas [Don07, p. 28] again to get that $\mathbb{E}[e^{-\theta K}]$ is

$$\begin{aligned}
&-2\alpha\frac{d}{d\theta}2\pi\int_{-\infty}^{\infty}\Bigg(\int_0^\infty\int_0^\infty f^+(\xi,-h)e^{izh-\theta\xi}\,d\xi\,dh \\
&\qquad\times\overline{\int_0^\infty\int_0^\infty f^-(\kappa,-h)e^{izh-\theta\kappa}\,d\kappa\,dh}\,dz\Bigg) \\
&=-4\pi\alpha\frac{d}{d\theta}\int_{-\infty}^{\infty}\Bigg(\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t+izx}-1]t^{-1}\mathbb{P}\{X_t-\alpha t\in dx\}\right\} \\
&\qquad\times\overline{\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t+izx}-1]t^{-1}\mathbb{P}\{-X_t-\alpha t\in dx\}\right\}}\Bigg)\,dz \\
&=-4\pi\alpha\frac{d}{d\theta}\int_{-\infty}^{\infty}\Bigg(\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t+izx}-1]t^{-1}\mathbb{P}\{X_t-\alpha t\in dx\}\right\} \\
&\qquad\times\exp\left\{\int_0^\infty dt\int_0^\infty[e^{-\theta t-izx}-1]t^{-1}\mathbb{P}\{-X_t-\alpha t\in dx\}\right\}\Bigg)\,dz \\
&=-4\pi\alpha\frac{d}{d\theta}\int_{-\infty}^{\infty}\exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\left(e^{-\theta t+izX_t-iz\alpha t}-1\right)\mathbf{1}\{X_t\geq+\alpha t\} \\
&\qquad+\left(e^{-\theta t+izX_t+iz\alpha t}-1\right)\mathbf{1}\{X_t\leq-\alpha t\}\Big]\,dt\Bigg)\,dz.
\end{aligned}$$

□

6.2. **Extension to more general Lévy processes.** Corollary 6.2 establishes Theorem 3.8 when X has a non-zero Brownian component. The next few results allow us establish the latter result for the class of Lévy processes described in its statement.

Recall the definitions

$$\begin{aligned}
G &:= \sup\{t<0 : X_t\wedge X_{t-}=M_t\}=\sup\{t<0 : t\in\mathcal{Z}\}, \\
D &:= \inf\{t>0 : X_t\wedge X_{t-}=M_t\}=\inf\{t>0 : t\in\mathcal{Z}\}, \\
T &:= \arg\max\{M_t : G\leq t\leq D\}, \\
S &:= \inf\{t>0 : X_t\wedge X_{t-}-\alpha t\leq\inf\{X_s-\alpha s : s\leq 0\}\}.
\end{aligned}$$

As in the proof of Theorem 2.6, it follows from Lemma 8.4 that almost surely

$$D=\inf\{t\geq S : X_t\wedge X_{t-}+\alpha(t-S)=\inf\{X_u+\alpha(u-S) : u\geq S\}\}.$$

Proposition 6.3. *Suppose that X is Lévy process satisfying our standing assumptions Hypothesis 2.2. Then, $\mathbb{P}\{0\notin\mathcal{Z},\, S=0\}=0$. In addition,*

(a) *If X has unbounded variation, then $G<T<S<D$ a.s.*
(b) *If X has bounded variation and drift coefficient d satisfying $d<-\alpha$, then $G\leq T<S<D$ a.s., and if X has bounded variation and drift coefficient d satisfying $d>\alpha$, then $G<T<S\leq D$ a.s..*
(c) *If X has bounded variation and drift coefficient d satisfying $|d|<\alpha$, then almost surely either $0\in\mathcal{Z}$ and $G=T=S=D=0$, or $0\notin\mathcal{Z}$ and $G\leq T\leq S\leq D$. Furthermore, $T=S=D$ almost surely on the event $\{T=S\}$.*

Proof. Firstly, if $0 \notin \mathcal{Z}$, then $\inf\{X_u - \alpha u : u \leq 0\} < 0$, and thus $S > 0$ a.s. on the event $\{0 \notin \mathcal{Z}\}$.

(a) Suppose that X has unbounded variation. We have from Theorem 3.1 (see Remark 3.2 (i)) that $0 \notin \mathcal{Z}$ almost surely. Rogozin's result on the small time behavior of unbounded variation processes [Rog68] implies that at the stopping time S

$$-\liminf_{\varepsilon \geq 0} \varepsilon^{-1}(X_{S+\varepsilon} - X_S) = \limsup_{\varepsilon \geq 0} \varepsilon^{-1}(X_{S+\varepsilon} - X_S) = \infty,$$

and hence it is not possible for the α-Lipschitz minorant to meet the path of X at time S. Thus, $T < S < D$ almost surely by Corollary 8.5. By time reversal, $G < T$ almost surely.

(b) Suppose X has bounded variation and drift coefficient d satisfying $|d| > \alpha$, then we have from Theorem 3.1 (see Remark 3.2 (ii)) that $0 \notin \mathcal{Z}$ almost surely. Therefore, by Corollary 8.5, if $T = S$ then $T = S = D$.

Suppose that $d < -\alpha$. Rogozin's result on the small time behavior of bounded variation processes [Rog68] implies that almost surely

$$\lim_{\varepsilon \geq 0} \varepsilon^{-1}(X_{S+\varepsilon} - X_S) = d.$$

Thus, $S \notin \mathcal{Z}$ and, in particular, $S < D$, so that $T < S < D$ a.s.

On the other hand, if $d > \alpha$, then the Lévy process $(X_t - \alpha t)_{t \geq 0}$ has positive drift and so the associated descending ladder process has zero drift coefficient [Don07, p. 56]. In that case, for any $x < 0$ we have $X_V < x$ almost surely, where $V := \inf\{t \geq 0 : X_t - \alpha t \leq x\}$ [Ber96, Theorem III.4]. Therefore,

$$X_S - \alpha S < \inf\{X_u - \alpha u : u \leq 0\} \quad \text{a.s.}$$

If $T = S$, then $T = S = D$ by Corollary 8.5, and then

$$\begin{aligned} X_S &= X_D \wedge X_{D-} \\ &= X_G \wedge X_{G-} + \alpha(D - G) \\ &= X_G \wedge X_{G-} + \alpha(S - G), \end{aligned}$$

which results in the contradiction

$$X_G \wedge X_{G-} - \alpha G = X_S - \alpha S < \inf\{X_u - \alpha u : u \leq 0\}.$$

Thus, $T < S \leq D$ a.s.

The results for G now follow by a time reversal argument.

(c) Suppose X has bounded variation and drift coefficient d satisfying $|d| > \alpha$. We know from Theorem 3.1 and Remark 3.2 that the subordinator associated with $\mathcal{Z}$ has non-zero drift and so $\mathcal{Z}$ has positive Lebesgue measure almost surely. The subset of points of $\mathcal{Z}$ that are isolated on either the left or the right is countable and hence has zero Lebesgue measure. It follows from the stationarity of $\mathbb{Z}$ that $G = T = S = D = 0$ almost surely on the event $\{0 \in \mathcal{Z}\}$. The remaining statements can be read from Corollary 8.5. □

Lemma 6.4. *Let X be a Lévy process that satisfies our standing assumptions Hypothesis 2.2. Suppose, moreover, that if X has paths of bounded variation, then $|d| \neq \alpha$. For $\varepsilon > 0$ set $X^\varepsilon = X + \varepsilon B$, where B is a standard Brownian motion on $\mathbb{R}$, independent of X. Define G^ε, D^ε and $K^\varepsilon = D^\varepsilon - G^\varepsilon$ to be the analogues of G, D and K with X replaced by X^ε. Then, $(G^\varepsilon, D^\varepsilon)$ converges almost surely to (G, D) as $\varepsilon \downarrow 0$, and so K^ε converges almost surely to K as $\varepsilon \downarrow 0$.*

Proof. By symmetry, it suffices to show that D^ε converges almost surely to D as $\varepsilon \downarrow 0$. We first show the convergence on the event $\{S > 0\}$.

Let S^ε be the analogue of the stopping time S with X replaced by X^ε. As we observed in the proof of Theorem 2.6, $X_S - \alpha S = X_S \wedge X_{S-} - \alpha S \le \inf\{X_u - \alpha u : u \le 0\}$. If X has unbounded variation or bounded variation with drift satisfying $d < \alpha$, then, since S is a stopping time, $\liminf_{u \downarrow S}(X_u - X_S - \alpha(u-S))/(u-S) < 0$. If X has bounded variation with drift satisfying $d > \alpha$, then by the remarks at the top of page 56 of [Don07], the downwards ladder height process of the process $(X_u - \alpha u)_{u \ge 0}$ (resp. $(-X_u + \alpha u)_{u \ge 0}$) has zero drift (resp. non-zero drift). By Lemma 5.3, the distribution of $\inf\{X_u - \alpha u : u \le 0\}$ is absolutely continuous with a bounded density, and hence

$$\mathbb{P}\{X_S - \alpha S = \inf\{X_u - \alpha u : u \le 0\}\} = 0$$

by Fubini's theorem and the fact that the range of a subordinator with zero drift has zero Lebesgue measure almost surely.

For all three of these cases, given any $\delta > 0$ we can, with probability one, thus find a time $t \in (S, S+\delta)$ such that

$$X_t \wedge X_{t-} - \alpha t < \inf\{X_u - \alpha u : u \le 0\}.$$

By the strong law of large numbers for the Brownian motion B,

$$\lim_{\varepsilon \downarrow 0} \inf\{X_u^\varepsilon - \alpha u : u \le 0\} = \inf\{X_u - \alpha u : u \le 0\}.$$

Hence, $X_t^\varepsilon \wedge X_{t-}^\varepsilon - \alpha t \le \inf\{X_u^\varepsilon - \alpha u : u \le 0\}$ for ε sufficiently small, and so $S^\varepsilon \le S + \delta$ for such an ε. Therefore, $\limsup_{\varepsilon \downarrow 0} S^\varepsilon \le S$.

On the other hand, for any $\delta > 0$ we have

$$\inf\{X_t \wedge X_{t-} - \alpha t - \inf\{X_u - \alpha u : u \le 0\} : t \in [0, (S-\delta)_+]\} > 0.$$

Thus, $X_t^\varepsilon \wedge X_{t-}^\varepsilon - \alpha t > \inf\{X_u^\varepsilon - \alpha u : u \le 0\}$ for all $t \in [0, (S-\delta)_+]$ for ε sufficiently small, so that $S^\varepsilon \ge (S-\delta)_+$. Therefore, $\liminf_{\varepsilon \downarrow 0} S^\varepsilon \ge S$. Consequently, $\lim_{\varepsilon \downarrow 0} S^\varepsilon = S$.

Now, as a result of the uniqueness of the global minima of Lévy processes that are not compound Poisson processes with zero drift [Ber96, Proposition VI.4], and the law of large numbers applied to B, we have

$$\lim_{\varepsilon \downarrow 0} \arg \inf_{u \ge S^\varepsilon} \{X_u^\varepsilon + \alpha(u - S^\varepsilon)\} = \arg \inf_{u \ge S} \{X_u + \alpha(u - S)\}.$$

It follows readily that D^ε converges to D almost surely as $\epsilon \downarrow 0$ on the event $\{S > 0\}$.

Suppose now that we are on the event $\{S = 0\}$. Then, by Proposition 6.3, $0 \in \mathcal{Z}$ almost surely, and we may suppose that X satisfies the conditions of part (c) of that result, so that $G = T = S = D = 0$ almost surely. Then, by the strong law of large numbers for the Brownian motion B, almost surely

$$\lim_{\varepsilon \downarrow 0} \inf_{u \le 0} \{X_u^\varepsilon - \alpha u\} = \lim_{\varepsilon \downarrow 0} \inf_{u \ge 0} \{X_u^\varepsilon + \alpha u\} = 0.$$

Therefore, D^ε also converges to D almost surely as $\epsilon \downarrow 0$ on the event $\{S = 0\}$. □

We are finally in a position to give the proof of Theorem 3.8. Suppose for the moment that X has a non-zero Brownian component. It follows from Theorem 3.1

and Proposition 3.7 that $\delta = 0$ and $\Lambda(\mathbb{R}_+) < \infty$. From (3.7) we have that

$$
\begin{aligned}
\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} &= \frac{\int_{\mathbb{R}_+} \left(\int_0^\theta x e^{-\varphi x}\,d\varphi\right)\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} \\
&= \int_0^\theta \left(\frac{\int_{\mathbb{R}_+} x e^{-\varphi x}\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)}\right) d\varphi = \int_0^\theta \mathbb{E}[e^{-\varphi K}]\,d\varphi.
\end{aligned}
\tag{6.4}
$$

By Corollary 6.2, this last integral is

$$
\begin{aligned}
4\pi\alpha \int_{-\infty}^{\infty} \Bigg\{ &\exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\left(e^{izX_t - iz\alpha t} - 1\right)\mathbf{1}\{X_t \geq +\alpha t\} \\
&\quad + \left(e^{izX_t + iz\alpha t} - 1\right)\mathbf{1}\{X_t \leq -\alpha t\}\Big]\,dt\Bigg) \\
&- \exp\Bigg(\int_0^\infty t^{-1}\mathbb{E}\Big[\left(e^{-\theta t + izX_t - iz\alpha t} - 1\right)\mathbf{1}\{X_t \geq +\alpha t\} \\
&\quad + \left(e^{-\theta t + izX_t + iz\alpha t} - 1\right)\mathbf{1}\{X_t \leq -\alpha t\}\Big]\,dt\Bigg)\Bigg\}\,dz,
\end{aligned}
\tag{6.5}
$$

as claimed in the theorem.

Now suppose X has zero Brownian component, but paths of unbounded variation almost surely. Let $X^\varepsilon = X + \varepsilon B$ and K^ε be as in Lemma 6.4, and let Λ^ε be the Lévy measure of the subordinator associated with the set of points where X^ε meets its α-Lipschitz minorant. By Lemma 6.4 we know that $K^\varepsilon \to K$ almost surely, and so

$$
\begin{aligned}
\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} &= \int_0^\theta \mathbb{E}[e^{-\varphi K}]\,d\varphi \\
&= \lim_{\varepsilon \downarrow 0} \int_0^\theta \mathbb{E}[e^{-\varphi K^\varepsilon}]\,d\varphi = \lim_{\varepsilon \downarrow 0} \frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\,\Lambda^\varepsilon(dx)}{\int_{\mathbb{R}_+} x\,\Lambda^\varepsilon(dx)}.
\end{aligned}
$$

Now, in the notation of of the proof of Corollary 6.2 , it can be seen that the square integrability of the densities of $\inf_{t\geq 0}\{X_t + \alpha t\}$ and $\inf_{t \geq 0}\{-X_t + \alpha t\}$ implies that

$$
\int_{-\infty}^0 \Big(\int_0^\infty f^+(\xi, h) e^{-\theta \xi}\,d\xi\Big)\Big(\int_0^\infty f^-(\kappa, h) e^{-\theta\kappa}\,d\kappa\Big)\,dh < \infty
$$

for all $\theta \geq 0$. Thus, by the same methods used in the proof of Corollary 6.2 from the last line of (6.3) onwards, it follows that (6.5) is finite. Then, since for each fixed value of z the integrand in (6.5) is a product of characteristic functions of certain infima, and hence not equal to zero, we can apply Fubini's theorem to swap the order of the integrals within the exponentials (here we are using the absolute continuity of the distribution of X_t for all $t > 0$). We now have that the integrand for each fixed value of z with X_t replaced by X_t^ε converges to the inegrand with just X_t as $\varepsilon \to 0$. Then, by finiteness of (6.5), we have that (6.5) with X_t replaced by X_t^ε converges to (6.5). $\square$

7. Lipschitz Minorants of Brownian Motion

7.1. **Williams' path decomposition for Brownian motion with drift.** We recall for later use a path composition due to David Williams that describes the distribution of a Brownian motion with positive drift in terms of the segment of the path up to the time the process achieves its global minimum and the segment of the path after that time – see [RW87, p. 436] or, for a concise description, [BS02, Section IV.5].

For $\mu \in \mathbb{R}$, let $Z^{(\mu)} = (Z_t^{(\mu)})_{t \geq 0}$ be a Brownian motion with drift μ started at zero. Take $\beta > 0$ and let E be a random variable that is independent of $Z^{(-\beta)}$ and has an exponential distribution with mean $(2\beta)^{-1}$. Set

$$T_E := \inf\{t \geq 0 : Z^{(-\beta)} = -E\}.$$

Then, there is a diffusion $W = (W_t)_{t \geq 0}$ with the properties

(i) W is independent of $Z^{(-\beta)}$ and E;
(ii) $W_0 = 0$;
(iii) $W_t > 0$ for all $t > 0$ a.s.;

such that if we define a process $(\tilde{Z}_t)_{t \geq 0}$ by

$$\tilde{Z}_t := \begin{cases} Z_t^{(-\beta)}, & 0 \leq t < T_E, \\ Z_{T_E}^{(-\beta)} + W_{t-T_E}, & t \geq T_E, \end{cases} \tag{7.1}$$

then $\tilde{Z}$ has the same distribution as $Z^{(\beta)}$. Thus, in particular,

$$-\inf\{Z_t^{(\beta)} : t \geq 0\} \sim \mathrm{Exp}(2\beta) \tag{7.2}$$

and the unique time that $Z^{(\beta)}$ achieves its global minimum is distributed as T_E. Recall also that

$$\mathbb{E}[\inf\{t \geq 0 : Z_t^{(-\beta)} = h\}] = \frac{h}{\beta} \tag{7.3}$$

for $h \leq 0$ (see, for example, [BS02, page 295, equation 2.2.0.1]).

7.2. **Random variables related to the Brownian Lipschitz minorant.**

Proposition 7.1. *Let X be a Brownian motion with drift β, where $|\beta| < \alpha$. Then, the distribution of K is characterized by*

$$\mathbb{E}[e^{-\theta K}] = \frac{8\alpha(\alpha^2 - \beta^2)\left(\frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}} + \frac{1}{\sqrt{2\theta+(\alpha-\beta)^2}}\right)}{\left(\sqrt{2\theta + (\alpha+\beta)^2} + \sqrt{2\theta + (\alpha-\beta)^2} + 2\alpha\right)^2}$$

for $\theta \geq 0$, and hence Λ is characterized by

$$\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)} = \frac{4(\alpha^2 - \beta^2)\theta}{\left(\sqrt{2\theta + (\alpha-\beta)^2} + \alpha - \beta\right)\left(\sqrt{2\theta + (\alpha+\beta)^2} + \alpha + \beta\right)}$$

for $\theta \geq 0$.

Proof. We have from [BS02, page 269, equation 1.14.3(1)] that

$$\int_{-\infty}^{0} f^-(\xi, h) e^{-\theta\xi}\, d\xi = 2(\alpha - \beta) e^{h(\sqrt{2\theta+(\alpha-\beta)^2}+(\alpha-\beta))}$$

and

$$\int_0^\infty f^+(\xi,h)e^{-\theta\xi}\,d\xi = 2(\alpha+\beta)e^{h(\sqrt{2\theta+(\alpha+\beta)^2}+(\alpha+\beta))}.$$

Thus, from (6.3),

$$\begin{aligned}
\mathbb{E}[e^{-\theta K}] &= -2\alpha\frac{d}{d\theta}\left(\int_{-\infty}^0 4(\alpha^2-\beta^2)e^{h(\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha)}\,dh\right) \\
&= 8\alpha(\alpha^2-\beta^2)\frac{d}{d\theta}\left(\frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha}\right) \\
&= \frac{8\alpha(\alpha^2-\beta^2)\left(\frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}}+\frac{1}{\sqrt{2\theta+(\alpha-\beta)^2}}\right)}{\left(\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha\right)^2},
\end{aligned}$$

as required.

Now, by (6.4),

$$\begin{aligned}
\frac{\int_{\mathbb{R}_+}(1-e^{-\theta x})\,\Lambda(dx)}{\int_{\mathbb{R}_+} x\,\Lambda(dx)} &= \int_0^\theta \mathbb{E}[e^{-\varphi K}]\,d\varphi \\
&= 8\alpha(\alpha^2-\beta^2)\Bigg[\frac{1}{\sqrt{(\alpha+\beta)^2}+\sqrt{(\alpha-\beta)^2}+2\alpha} \\
&\qquad - \frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha}\Bigg] \\
&= 8\alpha(\alpha^2-\beta^2)\Bigg[\frac{1}{4\alpha} \\
&\qquad - \frac{1}{\sqrt{2\theta+(\alpha+\beta)^2}+\sqrt{2\theta+(\alpha-\beta)^2}+2\alpha}\Bigg] \\
&= \frac{4(\alpha^2-\beta^2)\theta}{\left(\sqrt{2\theta+(\alpha-\beta)^2}+\alpha-\beta\right)\left(\sqrt{2\theta+(\alpha+\beta)^2}+\alpha+\beta\right)}
\end{aligned}$$

after a little algebra. □

Remark 7.2. There is an alternative way to verify that the Laplace transform for K presented in Proposition 7.1 is correct. Recall from the proof of Theorem 2.6 that $D = S + \tilde{T}$, where the independent random variables S and $\tilde{T}$ are defined by

$$S = \inf\{s > 0 : X_s - \alpha s = \inf\{X_u - \alpha u : u \le 0\}\}$$

and

$$\tilde{T} = \sup\{t \ge 0 : \tilde{X}_t = \inf\{\tilde{X}_s : s \ge 0\}\}$$

with

$$(\tilde{X}_s)_{s\ge 0} := ((X_{S+s} - X_S) + \alpha s)_{s\ge 0}\,.$$

Set $I^- := \inf\{X_u - \alpha u : u \le 0\}$. Because $(X_{-t} + \alpha t)_{t\ge 0}$ is a Brownian motion with drift $\alpha - \beta$, we know from Subsection 7.1 that $-I^-$ has an exponential distribution with mean $(2(\alpha-\beta))^{-1}$. Now $(X_t - \alpha t)_{t\ge 0}$ is a Brownian motion with drift $\beta - \alpha$, and so, again from Subsection 7.1, S is distributed as the time until this process achieves its global minimum. It follows that

$$\mathbb{E}[e^{-\theta S}] = \frac{2(\alpha - \beta)}{\sqrt{2\theta + (\alpha - \beta)^2} + \alpha - \beta}$$

and

$$\mathbb{E}[e^{-\theta \tilde{T}}] = \frac{2(\alpha + \beta)}{\sqrt{2\theta + (\alpha + \beta)^2} + \alpha + \beta}$$

– see, for example, [BS02, page 266, equation 1.12.3(2)].

By stationarity, D has the same distribution as $U(D-G) = UK$, where U is an independent random variable that is uniformly distributed on $[0,1]$. Thus,

$$\mathbb{E}[e^{-\theta D}] = \int_0^1 \mathbb{E}[e^{-u\theta K}]\, du = \mathbb{E}\left[\frac{1}{\theta K}\left(1 - e^{-\theta K}\right)\right] = \frac{1}{\theta} \frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)},$$

and

$$\begin{aligned}
\frac{\int_{\mathbb{R}_+} (1 - e^{-\theta x})\, \Lambda(dx)}{\int_{\mathbb{R}_+} x\, \Lambda(dx)} &= \theta \mathbb{E}[e^{-\theta D}] \\
&= \theta \mathbb{E}[e^{-\theta S}] \mathbb{E}[e^{-\theta \tilde{T}}] \\
&= \frac{4(\alpha^2 - \beta^2)\theta}{\left(\sqrt{2\theta + (\alpha - \beta)^2} + \alpha - \beta\right)\left(\sqrt{2\theta + (\alpha + \beta)^2} + \alpha + \beta\right)}.
\end{aligned}$$

This equality agrees with the one found in Proposition 7.1. Differentiating the expression on the right with respect to θ and recalling the observation (6.4), we arrive at the the expression for the Laplace transform of K in Proposition 7.1.

Proposition 7.3. *Let X be a Brownian motion with zero drift. Then,*

$$\mathbb{P}\{K \in d\kappa\} = \left(\frac{4\alpha^3}{\sqrt{2\pi}} \kappa^{1/2} e^{-\alpha^2 \kappa/2} - 4\alpha^4 \kappa \Phi(-\alpha \kappa^{1/2})\right) d\kappa,$$

where Φ is the standard normal cumulative distribution function. Thus,

$$\frac{\Lambda(dx)}{\Lambda(\mathbb{R}_+)} = \frac{2\alpha}{\sqrt{2\pi}} x^{-1/2} e^{-\alpha^2 x/2} - 2\alpha^2 \Phi(-\alpha x^{1/2})$$

Proof. We have from [BS02, page 269, equation 1.14.4(1)] that

$$f^-(\xi, h) = f^+(\xi, h) = \frac{-2\alpha h}{\sqrt{2\pi}\xi^{3/2}} \exp\left\{-\frac{(\alpha\xi - h)^2}{2\xi}\right\}.$$

Thus, by Proposition 6.1,

$$\begin{aligned}
\frac{\mathbb{P}\{K \in d\kappa\}}{d\kappa} &= \frac{4\alpha^3 \kappa e^{-\alpha^2\kappa/2}}{\pi} \int_0^\kappa \int_{-\infty}^0 \frac{h^2}{\xi^{3/2}(\kappa - \xi)^{3/2}} \exp\left\{2\alpha h - \frac{\kappa h^2}{2\xi(\kappa - \xi)}\right\} dh\, d\xi \\
&= \frac{4\alpha^3 e^{-\alpha^2\kappa/2}}{\pi\kappa} \int_{-\infty}^0 h^2 e^{2\alpha h} \left(\int_0^1 \frac{1}{\xi^{3/2}(1-\xi)^{3/2}} \exp\left\{-\frac{h^2/2\kappa}{\xi(1-\xi)}\right\} d\xi\right) dh.
\end{aligned}$$

The change of variable $y = \frac{1}{\xi(1-\xi)} - 4$ gives that

$$\int_0^{1/2} \frac{1}{\xi^{3/2}(1-\xi)^{3/2}} \exp\left\{-\frac{c}{\xi(1-\xi)}\right\} d\xi = e^{-4c} \int_0^\infty z^{-1/2} e^{-cz} dz = \frac{e^{-4c}\sqrt{\pi}}{\sqrt{c}}$$

for any $c > 0$, and hence

$$\begin{aligned}\frac{\mathbb{P}\{K \in d\kappa\}}{d\kappa} &= \frac{4\alpha^3 e^{-\alpha^2\kappa/2}}{\pi\kappa} \int_{-\infty}^{0} h^2 e^{2\alpha h} \frac{2e^{-2h^2/\kappa}\sqrt{\pi}}{\sqrt{h^2/2\kappa}}\, dh \\ &= -\frac{8\sqrt{2}\alpha^3 e^{-\alpha^2\kappa/2}}{\sqrt{\pi\kappa}} \int_{-\infty}^{0} h e^{2\alpha h - 2h^2/\kappa}\, dh.\end{aligned}$$

The further change of variable $z = 2\kappa^{-1/2}h - \alpha\kappa^{1/2}$ leads to

$$\begin{aligned}\frac{\mathbb{P}\{K \in d\kappa\}}{d\kappa} &= -4\alpha^3 \int_{-\infty}^{-\alpha\kappa^{1/2}} (\kappa^{1/2}z + \alpha\kappa)\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz \\ &= -4\alpha^3 \left(-\frac{\kappa^{1/2}}{\sqrt{2\pi}} e^{-\alpha^2\kappa/2} + \alpha\kappa\Phi(-\alpha\kappa^{1/2})\right) \\ &= \frac{4\alpha^3}{\sqrt{2\pi}}\kappa^{1/2} e^{-\alpha^2\kappa/2} - 4\alpha^4\kappa\Phi(-\alpha\kappa^{1/2}).\end{aligned}$$

Because $\Lambda(dx)$ is proportional to $x^{-1}\mathbb{P}\{K \in dx\}$, we need only find $\int_{\mathbb{R}_+} x^{-1}\mathbb{P}\{K \in dx\}$ to establish the claim for Λ, and this can be done using methods of integration similar to those used in Remark 7.4 below to check that the density of K integrates to one. □

Remark 7.4. We can check directly that the density given for K integrates to one. For the first term, we use the substitution $\eta = \alpha^2\kappa/2$, and for the second we use the substitution $\eta = \alpha^2\kappa$ and then change the order of integration to get that the integral of the claimed density is

$$\begin{aligned}&\frac{4}{\Gamma(3/2)} \int_0^\infty \eta^{1/2} e^{-\eta}\, d\eta - 4\int_0^\infty \eta\Phi(-\eta^{1/2})\, d\eta \\ &= 4 - \frac{4}{\sqrt{2\pi}} \int_0^\infty \int_{\eta^{1/2}}^\infty \eta e^{-y^2/2}\, dy\, d\eta \\ &= 4 - \frac{4}{\sqrt{2\pi}} \int_0^\infty \left(\int_0^{y^2} \eta\, d\eta\right) e^{-y^2/2}\, dy \\ &= 4 - \frac{2}{\sqrt{2\pi}} \int_0^\infty y^4 e^{-y^2/2}\, dy \\ &= 4 - \frac{3}{\Gamma(5/2)} \int_0^\infty x^{3/2} e^{-x}\, dx \;=\; 1.\end{aligned}$$

Proposition 7.5. *Let X be a Brownian motion with drift β, where $|\beta| < \alpha$. Recall that $T := \arg\max\{M_t : G \le t \le D\}$ and $H := X_T - M_T$. Then, H has a* Gamma$(2, 4\alpha)$ *distribution; that is, the distribution of H is absolutely continuous with respect to Lebesgue measure with density $h \mapsto (4\alpha)^2 h e^{-4\alpha h}$, $h \ge 0$. Also,*

$$\mathbb{P}\{T > 0\} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right),$$

and the distribution of T is characterized by

$$\mathbb{E}\left[e^{-\theta T}\right] = 8\alpha(\alpha^2 - \beta^2)\frac{1}{\theta}\left(\frac{1}{\sqrt{(\alpha+\beta)^2 - 2\theta} + 3\alpha - \beta} - \frac{1}{\sqrt{(\alpha-\beta)^2 + 2\theta} + 3\alpha + \beta}\right)$$

for $-\frac{(\alpha-\beta)^2}{2} \le \theta \le \frac{(\alpha+\beta)^2}{2}$.

Proof. Consider the claim regarding the distribution of H. A slight elaboration of the proof of Proposition 6.1 shows, in the notation of that result, that the random vector $(T, L, R, -H)$ has a distribution that is absolutely continuous with respect to Lebesgue measure with joint density $(\tau, \lambda, \rho, \eta) \mapsto 2\alpha f^-(\lambda, \eta) f^+(\rho, \eta)$, $\lambda, \rho > 0$, $\tau - \lambda < 0 < \tau + \rho$, $\eta < 0$. Therefore,

$$\mathbb{P}\{H \in dh\} = 2\alpha \int_0^\infty \int_0^\infty (\lambda + \rho) f^-(\lambda, -h) f^+(\rho, -h)\, d\lambda d\rho d\eta.$$

By (7.2),

$$\int_0^\infty f^-(\lambda, -h)\, d\lambda = 2(\alpha - \beta) e^{-2(\alpha-\beta)h}. \tag{7.4}$$

Combining this with (7.3) gives

$$\int_{-\infty}^0 \lambda f^-(\lambda, -h)\, d\lambda = \frac{-\eta}{\alpha - \beta} \times 2(\alpha - \beta) e^{-2(\alpha-\beta)h} = 2h\eta e^{-2(\alpha-\beta)h}. \tag{7.5}$$

Similarly,

$$\int_0^\infty f^+(\rho, -h)\, d\rho = 2(\alpha + \beta) e^{-2(\alpha+\beta)h} \tag{7.6}$$

and

$$\int_{-\infty}^0 \rho f^+(\rho, \eta)\, d\rho = 2he^{-2(\alpha+\beta)h}. \tag{7.7}$$

Thus,

$$\begin{aligned}\mathbb{P}\{H \in dh\} &= 2\alpha \Big[2he^{-2(\alpha-\beta)h} \times 2(\alpha + \beta) e^{-2(\alpha+\beta)h} \\ &\qquad + 2(\alpha - \beta) e^{-2(\alpha-\beta)h} \times 2he^{-2(\alpha+\beta)h} \Big]\, dh \\ &= (4\alpha)^2 h e^{-4\alpha h}\, dh.\end{aligned}$$

Note that $T > 0$ if and only if $I^+ > I^-$, where

$$I^+ := \inf\{X_t + \alpha t : t \geq 0\}$$

and

$$I^- := \inf\{X_t - \alpha t : t \leq 0\}.$$

Recall from Subsection 7.1 that the independent random variables I^+ and I^- are exponentially distributed with respective means $(2(\alpha + \beta))^{-1}$ and $(2(\alpha - \beta))^{-1}$. It follows that

$$\mathbb{P}\{T > 0\} = \frac{2(\alpha + \beta)}{2(\alpha + \beta) + 2(\alpha - \beta)} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right).$$

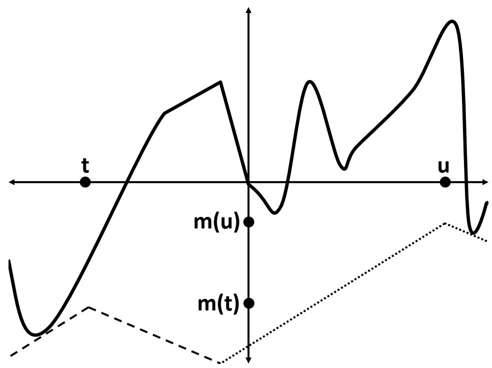

FIGURE 2. Lemma 8.1 shows that the height of the α-Lipschitz minorant of a function f at a fixed time t is given by $\sup\{h \in \mathbb{R} : h - \alpha|t-s| \le f(s) \text{ for all } s \in \mathbb{R}\}$.

We can also derive this last result from Proposition 6.1 as follows.

$$\begin{aligned}
\mathbb{P}\{T > 0\} &= 2\alpha \int_{-\infty}^{0} \int_{0}^{\infty} \int_{-\infty}^{0} \int_{\tau}^{\infty} f^-(\tau - \gamma, h) f^+(\delta - \tau, h)\, d\delta d\gamma d\tau dh \\
&= 2\alpha \int_{-\infty}^{0} \int_{0}^{\infty} \int_{-\infty}^{0} f^-(\tau - \gamma, h) \left(\int_{0}^{\infty} f^+(\eta, h)\, d\eta \right) dh d\tau d\gamma \\
&= 2\alpha \int_{-\infty}^{0} \left(\int_{0}^{\infty} \int_{-\infty}^{0} f^+(\tau - \gamma, h)\, d\gamma\, d\tau \right) \left(\int_{0}^{\infty} f^+(\eta, h)\, d\eta \right) dh \\
&= 2\alpha \int_{-\infty}^{0} \left(\int_{0}^{\infty} \eta f^-(\eta, h)\, d\eta \right) \left(\int_{0}^{\infty} f^+(\eta, h)\, d\eta \right) dh.
\end{aligned}$$

Substituting in (7.5) and (7.6), and then evaluating the resulting straightforward integral establishes the result.

The Laplace transform of T may be calculated using very similar methods. $\square$

8. Some facts about Lipschitz minorants

The following is a restatement of (1.1) accompanied by a proof.

Lemma 8.1. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Then,*

$$\begin{aligned}
m(t) &= \sup\{h \in \mathbb{R} : h - \alpha|t-s| \le f(s) \textit{ for all } s \in \mathbb{R}\} \\
&= \inf\{f(s) + \alpha|t-s| : s \in \mathbb{R}\}.
\end{aligned}$$

Proof. Consider the first equality. Fix $t \in \mathbb{R}$. Because m is α-Lipschitz, if $h \le m(t)$, then $h - \alpha|t-s| \le m(t) - \alpha|t-s| \le m(s) \le f(s)$ for all $s \in \mathbb{R}$. On the other hand, if $h > m(t)$, then $s \mapsto (h - \alpha|t-s|) \vee m(s)$ is an α-Lipschitz function that dominates m (strictly at t), and so $(h - \alpha|t-s|) \vee m(s) > f(s)$ for some $s \in \mathbb{R}$. This implies that $h - \alpha|t-s| > f(s)$, since $m(s) \le f(s)$. The second equality is simply a rephrasing of the first. $\square$

We leave the proof of the following straightforward consequence of Lemma 8.1 to the reader.

Corollary 8.2. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Define functions $f^{\leftarrow} : \mathbb{R} \to \mathbb{R}$ and $f^{\rightarrow} : \mathbb{R} \to \mathbb{R}$ by*

$$f^{\leftarrow}(t) := \begin{cases} f(t), & t < 0, \\ m(0) - \alpha t, & t \geq 0, \end{cases}$$

and

$$f^{\rightarrow}(t) := \begin{cases} m(0) + \alpha t, & t \leq 0, \\ f(t), & t > 0. \end{cases}$$

Denote the α-Lipschitz minorants of $f^{\leftarrow}$ and $f^{\rightarrow}$ by $m^{\leftarrow}$ and $m^{\rightarrow}$, respectively. Then, $m^{\leftarrow}(t) = m(t)$ for all $t \leq 0$ and $m^{\rightarrow}(t) = m(t)$ for all $t \geq 0$.

The next result says that if f is a càdlàg function with α-Lipschitz minorant m, then on an open interval in the complement of the closed set $\{t \in \mathbb{R} : m(t) = f(t) \wedge f(t-)\}$ the graph of the function m is either a straight line or a "sawtooth".

Lemma 8.3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. If $t' < t''$ are such that $f(t') \wedge f(t'-) = m(t')$, $f(t'') \wedge f(t''-) = m(t'')$, and $f(t) \wedge f(t-) > m(t)$ for $t' < t < t''$, then, setting $t^* = (f(t'') \wedge f(t''-) - f(t') \wedge f(t'-) + \alpha(t'' + t'))/(2\alpha)$,*

$$m(t) = \begin{cases} f(t') \wedge f(t'-) + \alpha(t - t'), & t' \leq t \leq t^*, \\ f(t'') \wedge f(t''-) + \alpha(t'' - t), & t^* \leq t \leq t''. \end{cases}$$

Proof. Define a function $\tilde{m} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{m}(t) := \begin{cases} f(t') \wedge f(t'-) + \alpha(t - t'), & t \leq t^*, \\ f(t'') \wedge f(t''-) + \alpha(t'' - t), & t^* \leq t. \end{cases}$$

That is, $\tilde{m}(t) = h^* - \alpha|t - t^*|$, where

$$h^* = (f(t'') \wedge f(t''-) + f(t') \wedge f(t'-) + \alpha(t'' - t'))/2.$$

Because $m(t') = \tilde{m}(t')$ $m(t'') = \tilde{m}(t'')$, and m is α-Lipschitz, we have $m(t) \leq \tilde{m}(t)$ for $t \in [t', t'']$ and $m(t) \geq \tilde{m}(t)$ for $t \notin [t', t'']$. Suppose for some $t_0 \in (t', t'')$ that $m(t_0) < \tilde{m}(t_0)$. We must have that $m(t_0) - \alpha|t' - t_0| \leq m(t') \leq f(t') \wedge f(t'-)$ and $m(t_0) - \alpha|t'' - t_0| \leq m(t'') \leq f(t'') \wedge f(t''-)$. Moreover, both of these inequalities must be strict, because otherwise we would conclude that $m(t_0) \geq \tilde{m}(t_0)$.

We can therefore choose $\epsilon > 0$ sufficiently small so that $m(t_0) + \epsilon - \alpha|t - t_0| < f(t) \wedge f(t-)$ for $t \in [t', t'']$. This implies that $m(t_0) + \epsilon - \alpha|t - t_0| < \tilde{m}(t) \leq m(t) \leq f(t) \wedge f(t-)$ for $t \notin [t', t'']$. Thus, $t \mapsto (m(t_0) + \epsilon - \alpha|t - t_0|) \vee m(t)$ is an α-Lipschitz function that is dominated everywhere by f and strictly dominates m at the point t_0, contradicting the definition of m. $\square$

We have a recipe for finding $\inf\{t > 0 : f(t) \wedge f(t-) = m(t)\}$ when f is a càdlàg function with α-Lipschitz minorant m. Figure 3 gives two examples of how the recipe applies to different paths (note that the value of α differs for the two examples).

Lemma 8.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Set*

$$\mathbf{d} := \inf\{t > 0 : f(t) \wedge f(t-) = m(t)\},$$
$$\mathbf{s} := \inf\left\{t > 0 : f(t) \wedge f(t-) - \alpha t \leq \inf\{f(u) - \alpha u : u \leq 0\}\right\},$$

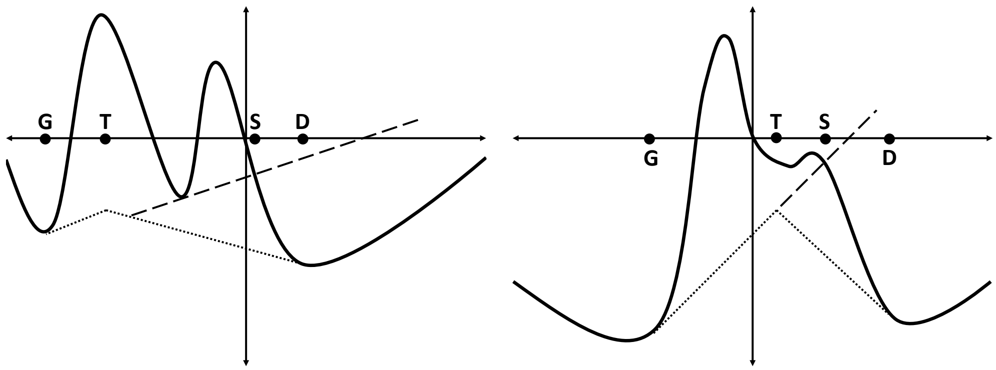

FIGURE 3. Two instances of the construction of Lemma 8.4.

and

$$\mathbf{e} := \inf\left\{t \geq \mathbf{s} : f(t) \wedge f(t-) + \alpha(t - \mathbf{s}) = \inf\{f(u) + \alpha(u - \mathbf{s}) : u \geq \mathbf{s}\}\right\}.$$

Suppose that $f(\mathbf{s}) \leq f(\mathbf{s}-)$. *Then,* $\mathbf{e} = \mathbf{d}$.

Proof. It suffices to show the following:

$$f(t) \wedge f(t-) > m(t) \text{ for } 0 < t < \mathbf{e}, \tag{8.1}$$

$$f(\mathbf{e}) \wedge f(\mathbf{e}-) \leq m(\mathbf{e}), \tag{8.2}$$

$$\mathbf{d} > 0 \Longrightarrow \mathbf{e} > 0. \tag{8.3}$$

For $0 < t < \mathbf{s}$, it follows from the definition of $\mathbf{s}$ that

$$\begin{aligned} f(t) \wedge f(t-) &> \inf\{f(u) - \alpha u : u \leq 0\} + \alpha t \\ &= \inf\{f(u) + \alpha(t - u) : u \leq 0\} \\ &\geq \inf\{f(u) + \alpha|t - u| : u \in \mathbb{R}\} = m(t). \end{aligned}$$

For $\mathbf{s} \leq t < \mathbf{e}$, it follows from the definition of $\mathbf{e}$ that

$$f(t) \wedge f(t-) + \alpha(t - \mathbf{s}) > \inf\{f(u) + \alpha(u - \mathbf{s}) : u \geq \mathbf{s}\},$$

and hence

$$\begin{aligned} f(t) \wedge f(t-) &> \inf\{f(u) + \alpha(u - \mathbf{s}) : u \geq \mathbf{s}\} - \alpha(t - \mathbf{s}) \\ &= \inf\{f(u) + \alpha(u - t) : u \geq \mathbf{s}\} \\ &\geq \inf\{f(u) + \alpha|t - u| : u \in \mathbb{R}\} = m(t). \end{aligned}$$

This completes the proof of (8.1)

Now $f(\mathbf{e}) \wedge f(\mathbf{e}-) + \alpha(\mathbf{e} - \mathbf{s}) = \inf\{f(u) + \alpha(u - \mathbf{s}) : u \geq \mathbf{s}\}$, and so $f(\mathbf{e}) \wedge f(\mathbf{e}-) = \inf\{f(u) + \alpha(u - \mathbf{e}) : u \geq \mathbf{s}\}$ This certainly gives

$$f(\mathbf{e}) \wedge f(\mathbf{e}-) \leq \inf\{f(u) + \alpha|\mathbf{e} - u| : u \geq \mathbf{s}\}. \tag{8.4}$$

Combined with the definition of $\mathbf{s}$, it also gives

$$\begin{aligned} f(\mathbf{e}) \wedge f(\mathbf{e}-) + \alpha(\mathbf{e} - \mathbf{s}) &\leq f(\mathbf{s}) + \alpha(\mathbf{s} - \mathbf{s}) \\ &\leq \inf\{f(s) - \alpha s : s \leq 0\} + \alpha \mathbf{s}. \end{aligned}$$

Thus, $f(\mathbf{e}) \wedge f(\mathbf{e}-) + 2\alpha(\mathbf{e}-\mathbf{s}) \le \inf\{f(s)+\alpha(\mathbf{e}-s) : s \le 0\}$ and hence, *a fortiori*,

$$f(\mathbf{e}) \wedge f(\mathbf{e}-) \le \inf\{f(s)+\alpha|\mathbf{e}-s| : s \le 0\}. \tag{8.5}$$

For $0 < s < \mathbf{s}$, $f(s) - s > \inf\{f(r) - \alpha r : r \le 0\}$, and so

$$\begin{aligned}
\inf\{f(s)+\alpha|\mathbf{e}-s| : 0 \le s < \mathbf{s}\} &= \inf\{f(s)+\alpha(\mathbf{e}-s) : 0 \le s < \mathbf{s}\} \\
&= \inf\{f(s)-\alpha s : 0 \le s < \mathbf{s}\} + \alpha\mathbf{e} \\
&\ge \inf\{f(r)-\alpha r : r \le 0\} + \alpha\mathbf{e} \\
&= \inf\{f(r)+\alpha(\mathbf{e}-r) : r \le 0\} \\
&= \inf\{f(r)+\alpha|\mathbf{e}-r| : r \le 0\}.
\end{aligned} \tag{8.6}$$

Combining (8.4), (8.5) and (8.6) gives (8.2).

The proof of (8.3) is a straightforward consequence of Lemma 8.3 and we leave it to the reader. □

Corollary 8.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with α-Lipschitz minorant $m : \mathbb{R} \to \mathbb{R}$. Define $\mathbf{d}$, $\mathbf{s}$, and $\mathbf{e}$ as in Lemma 8.4. Assume that $f(\mathbf{s}) \le f(\mathbf{s}-)$, so that $\mathbf{e} = \mathbf{d}$. Put $\mathbf{g} := \sup\{t < 0 : f(t) \wedge f(t-) = m(t)\}$ and assume that $f(0) \wedge f(0-) > m(0)$, so that $f(t) \wedge f(t-) > m(t)$ for $t \in (\mathbf{g}, \mathbf{d})$. Let $\mathbf{t} := (f(\mathbf{d}) \wedge f(\mathbf{d}-) - f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(\mathbf{d}+\mathbf{g}))/(2\alpha)$ be the point in $[\mathbf{g}, \mathbf{d}]$ at which the function m achieves its maximum. Then, $\mathbf{g} \le \mathbf{t} \le \mathbf{s} \le \mathbf{d}$. Moreover, if $\mathbf{t} = \mathbf{s}$, then $\mathbf{t} = \mathbf{s} = \mathbf{d}$.*

Proof. We first show that $\mathbf{g} \le \mathbf{t} \le \mathbf{s} \le \mathbf{d}$. We certainly have $\mathbf{g} \le \mathbf{s} \le \mathbf{d}$ and $\mathbf{g} \le \mathbf{t} \le \mathbf{d}$, so it suffices to prove that $\mathbf{t} \le \mathbf{s}$. Because $\mathbf{s} \ge 0$, this is clear when $\mathbf{t} < 0$, so it further suffices to consider the case where $\mathbf{t} \ge 0$. Suppose, then, that $\mathbf{g} \le 0 \le \mathbf{s} < \mathbf{t} \le \mathbf{d}$.

From Lemma 8.3 we have $m(u) = f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(u-\mathbf{g})$ for $\mathbf{g} \le u \le \mathbf{t}$ and $f(u) \wedge f(u-) \ge f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(u-\mathbf{g})$ for $u \le \mathbf{t}$. Therefore, $\inf\{f(u) \wedge f(u-) - \alpha u : u \le 0\} \ge f(\mathbf{g}) \wedge f(\mathbf{g}-) - \mathbf{g}$, and hence $\inf\{f(u) \wedge f(u-) - \alpha u : u \le 0\} = f(\mathbf{g}) \wedge f(\mathbf{g}-) - \alpha\mathbf{g}$. Now, by definition of $\mathbf{s}$, $f(\mathbf{s}) \wedge f(\mathbf{s}-) - \alpha\mathbf{s} \le \inf\{f(u) \wedge f(u-) - \alpha u : u \le 0\}$, and so

$$\begin{aligned}
f(\mathbf{s}) \wedge f(\mathbf{s}-) &\le f(\mathbf{g}) \wedge f(\mathbf{g}-) - \alpha\mathbf{g} + \alpha\mathbf{s} \\
&= f(\mathbf{g}) \wedge f(\mathbf{g}-) + \alpha(\mathbf{s}-\mathbf{g}) \\
&= m(\mathbf{s}),
\end{aligned}$$

which contradicts $\mathbf{d} = \inf\{u > 0 : f(u) \wedge f(u-) = m(u)\} = \inf\{u > 0 : f(u) \wedge f(u-) \le m(u)\}$ unless $\mathbf{s} = 0$ and $f(0) \wedge f(0-) = m(0)$, but we have assumed that this is not the case.

A similar argument shows that if $\mathbf{t} = \mathbf{s}$, then $\mathbf{t} = \mathbf{s} = \mathbf{d}$.

□

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