Confidence Intervals for Negative Binomial Random Variables of High Dispersion

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Abstract

This paper considers the problem of constructing confidence intervals for the mean of a Negative Binomial random variable based upon sampled data. When the sample size is large, it is a common practice to rely upon a Normal distribution approximation to construct these intervals. However, we demonstrate that the sample mean of highly dispersed Negative Binomials exhibits a slow convergence in distribution to the Normal as a function of the sample size. As a result, standard techniques (such as the Normal approximation and bootstrap) will construct confidence intervals for the mean that are typically too narrow and significantly undercover in the case of high dispersion. To address this problem, we propose methods based upon Bernstein's inequality along with the Gamma and Chi Square distributions as alternatives to the standard methods when the sample size is small and the dispersion is high. A confidence interval based upon Bernstein's inequality relies upon less stringent assumptions than those required of parametric models. Moreover, we prove a limit theorem demonstrating that the sample mean of Negative Binomials converges in distribution to a Gamma random variable under suitable hypotheses, and we use this observation to construct approximate confidence intervals. Furthermore, we investigate the applicability of the Chi Square distribution as a special case of the Gamma model. We then undertake a variety of simulation experiments to compare the proposed methods to standard techniques in terms of empirical coverage and provide concrete recommendations for the settings in which particular intervals are preferred. We also apply the proposed methods to examples arising in the serial analysis of gene expression and traffic flow in a communications network to illustrate both the strengths and weaknesses of these procedures along with those of standard techniques.

Keywords: Bernstein's Inequality, Chi Square Distribution, Confidence Intervals, Gamma Distribution, Negative Binomial Distribution, Serial Analysis of Gene Expression (SAGE).

1 Introduction

Given a sample of n independent, identically distributed (i.i.d.) random variables with finite variance, the Central Limit Theorem states that the distribution of the sample mean $\bar{X}$ is approximately Normal when the sample size n is large. As discussed in Rosenblum and van der Laan [2008], the Normal approximation and a bootstrap method are standard techniques used in the construction of confidence intervals for the mean μ in a variety of settings even for moderately small sample sizes (e.g. $n = 30$). However, it is important to note that such confidence intervals only perform well when the probability density function (PDF) of $\bar{X}$ is reasonably close to that of a Normal distribution. At moderate sample sizes, this cannot be assured in the case of random variables with highly skewed distributions [Wilcox, 2005], and the variability of estimates of the standard error of $\bar{X}$ provides an additional degree of uncertainty. In particular, we will demonstrate that a sample mean constructed from i.i.d. Negative Binomial random variables of high dispersion exhibits a PDF with an extremely heavy right tail. As a consequence of this skewness, standard techniques for constructing confidence intervals such as the Normal approximation and bootstrap may result in poor coverage and correspondingly poor inferences. In practice, research that relies on a relatively small number of independent samples (such as the investigation of [Lloyd-Smith et al., 2005] on the secondary transmission of infectious disease) should exercise caution to ensure that its conclusions do not rely upon biased estimates of variability. This paper investigates the performance of these methods through simulation studies and proposes a variety of improvements based upon Bernstein's Inequality, a Gamma model, and the Chi Square (χ^2) distribution for the construction of confidence intervals for the mean of Negative Binomial random variables of high dispersion.

For any *significance level* $\alpha \in (0, 1)$, standard techniques for constructing $1 - \alpha$ confidence intervals often rely upon inverting hypothesis testing procedures under specific parametric assumptions [Casella and Berger, 1990, Clopper and Pearson, 1934, Crow and Gardner, 1959, Sterne, 1954]. When these assumptions are satisfied, the resulting $1 - \alpha$ confidence intervals are *exact* in that the infimum coverage probability over all sample sizes is at least $1 - \alpha$ [Blyth and Still, 1963]. Rosenblum and van der Laan [2008] investigate the use of exact methods in constructing confidence intervals when the parametric assumptions underlying standard techniques are not valid. In particular, they rely upon tail probability bounds such as Bernstein's Inequality [Bernstein, 1934], Bennett's Inequality [Bennett, 1962], and methods based on the work of Hoeffding [1963] and Berry-Esseen [Berry, 1941, Esseen, 1942]. These bounds all rely upon much weaker hypotheses that do not involve distributional assumptions on the data. As a result, Rosenblum and van der Laan [2008] are able to construct confidence intervals for a wide variety of parameters based upon the corresponding estimators' influence curves. Such intervals will in general be more conservative than those based upon the Normal distribution but not necessarily exact due to the influence curve approximation.

However, determining confidence intervals for the mean of i.i.d. Negative Binomial random variables of high dispersion is not so straightforward, particularly in small sample sizes. Even the relatively weak assumptions underlying methods such as Bernstein's Inequality are not necessarily valid for Negative Binomials because the maximum deviation from the mean is not bounded. Since the assumptions required for the techniques described in Rosenblum and van der Laan [2008] do not hold in this setting, the resulting confidence intervals are not guaranteed to cover well. We will demonstrate in Section 4 that confidence intervals constructed from Bernstein's Inequality may exhibit poor coverage in the case of high dispersion. For this reason, we will also investigate the χ^2 and Gamma distributions as practical alternatives to standard techniques and refinements to Bernstein confidence intervals that can lead to improved coverage of $1 - \alpha$ confidence intervals. We will propose these new techniques and compare their performance to those of standard methods for constructing confidence intervals for the mean of i.i.d. Negative Binomial random variables when the dispersion is high and the sample size n is small.

Section 2 reviews the Negative Binomial distribution and demonstrates the slow convergence of the sample mean $\bar{X}$ to the Normal distribution as a function of the sample size n in the case of high dispersion. Section 3 describes Bernstein's Inequality's role in constructing $1 - \alpha$ confidence intervals for the mean, proves a limit theorem on the convergence of $\bar{X}$ to a Gamma distribution at large sample sizes and high dispersions, and also proposes the χ^2 distribution as an alternative approximation at moderate sample sizes. Section 4 summarizes a variety of simulation experiments that compare the coverage probabilities of the proposed methods of Section 3 to those of standard techniques such as the Wald (Normal approximation) and bootstrap [Efron and Tibshirani, 1994]. Section 4.4 will then investigate the extent to which Bernstein confidence intervals may be improved through a sensitivity analysis of the choice of the "upper bound" b. Section 5 considers the applicability of the proposed techniques in examples of serial analysis of gene expression (SAGE) and network traffic flow data. Finally, we will conclude the paper with a discussion in Section 6.

2 The Negative Binomial Distribution

A Negative Binomial random variable is conventionally used to compute the probability that a total of k failures will result before the rth success is observed when each trial is independent of all others and results in success with a fixed probability p. As described in Hilbe [2007], a Negative Binomial variable may instead be parameterized in terms of $\mu = r\left(\frac{1}{p} - 1\right)$ and $\theta = r$. (We will adopt this alternative parameterization for the remainder of this paper.) Then, for any $\mu \in \mathbb{R}^+$ and $\theta \in \mathbb{R}^+$, the resulting probability mass function for the Negative Binomial random variable $X \sim NB(\mu, \theta)$ is

$$P(X = k) = \frac{\mu^k}{k!} \frac{\Gamma(\theta + k)}{\Gamma(\theta)[\mu + \theta]^k} \frac{1}{\left(1 + \frac{\mu}{\theta}\right)^{\theta}}, k \in \mathbb{Z}^+. \tag{1}$$

Equation 1 can be shown to converge to the probability mass function of a Poisson random variable with mean parameter μ as $\theta \to \infty$ [Hilbe, 2007]. For this reason, the Negative Binomial may be considered an over-dispersed Poisson random variable with the dispersion controlled by the value of θ. Negative Binomial models are useful as robust alternatives to the Poisson that allow the variance parameter to exceed the mean. For instance, smaller values of θ result in a higher dispersion by adding more weight to the right tail of the probability mass function, which necessarily results in a higher variance. When the value of θ is very small, the Negative Binomial distribution exhibits a high degree of skewness. As a result of the extreme dispersion of the Negative Binomial from the Poisson in this case, the sample mean $\bar{X}$ of n i.i.d. $NB(\mu, \theta)$ observations may not be reasonably close to the Normal in distribution for small values of n. Wilcox [2005] warns that standard confidence intervals based upon a Normal approximation may result in poor coverage in scenarios such as this.

Figures 1 and 2 depict the estimated PDF of $\bar{X}$ for a variety of values of n in the case of i.i.d. copies of a $NB(\mu = 5, \theta = 0.1)$ random variable. Each of these densities was estimated from 10,000 independent simulations that computed $\bar{X}$ from n i.i.d. observations. These densities collectively demonstrate the heavy right tail of $\bar{X}$ at small sample sizes under high dispersion and the slow convergence of $\bar{X}$ in distribution to the Normal curve. Even at sample sizes of $n = 100$ and 150 in Figure 1, a disproportionately large percentage of values over 8 appear, and small remnants of this asymmetry persist even for sample sizes of 250 and 300. However, the density begins to look Normal at roughly $n = 75$, and subsequent increases in n largely serve to shrink the right tail. Figure 2, by contrast, investigates the PDF of $\bar{X}$ for small values of n in increments of 5 up to 40. At these sample sizes, the density plots portray stark asymmetries and extremely heavy right tails. Overall, the PDF of $\bar{X}$ does not bear much resemblance to the Normal distribution in these cases. Therefore, it is reasonable to believe that standard confidence intervals for μ such as the Wald (Normal approximation) method and bootstrap will result in poor coverage at small and moderate values of n. The next section will consider the use of confidence intervals based on Bernstein's Inequality, the Gamma approximation, and the χ^2 distribution as possible alternatives to these methods for small sample sizes and high dispersion.

3 Gamma, Chi Square, and Bernstein Confidence Intervals

3.1 The Gamma Model

We propose the Gamma distribution as an approximate PDF for the sample mean $\bar{X}$ of Negative Binomial random variables. The Gamma approximation may be established in a limit theorem based upon Laplace transforms. A Negative Binomial random variable X_i with parameters μ and θ and PMF (1) has a Laplace transform given by:

$$F_{X_i}(\lambda) \equiv E\left[\exp\left(-\lambda X_i\right)\right] = \left(1 + (1 - e^{-\lambda})\frac{\mu}{\theta}\right)^{-\theta}. \tag{2}$$

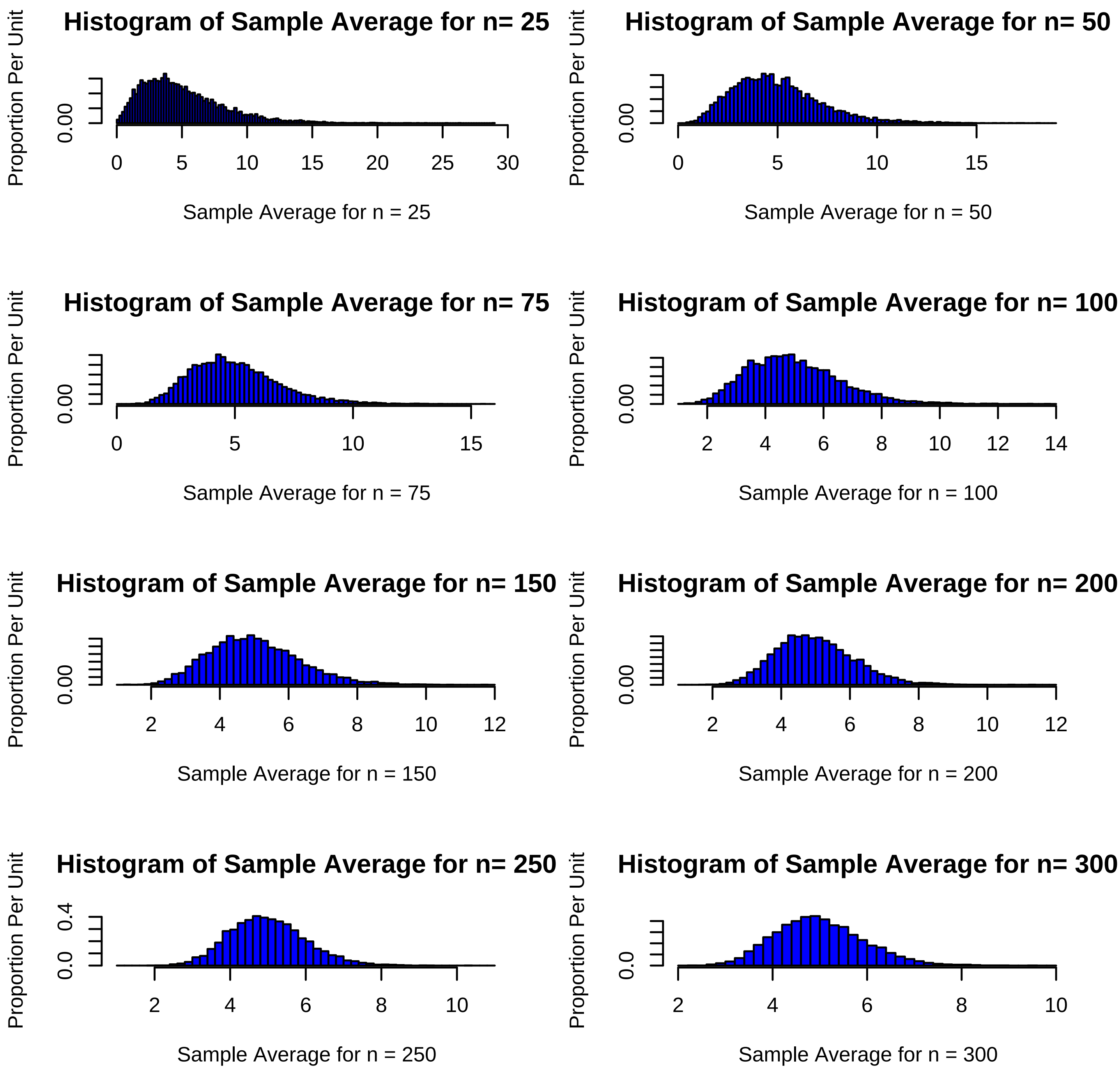

Figure 1: The probability density function (PDF) of the sample mean of n i.i.d. observations of a Negative Binomial random variable with parameters $\mu = 5$ and $\theta = 0.1$ at a number of sample sizes n. These PDFs are estimated above with histograms each produced from $10,000$ independent simulations. The figures collectively demonstrate the slow convergence of the sample mean's PDF to the Normal distribution when the dispersion is high (i.e. θ is small). Even at moderately large sample sizes ($100 \leq n \leq 200$), the sample mean exhibits a heavy right tail. As a result, confidence intervals for the mean μ based upon the Normal approximation are likely to result in poor coverage.

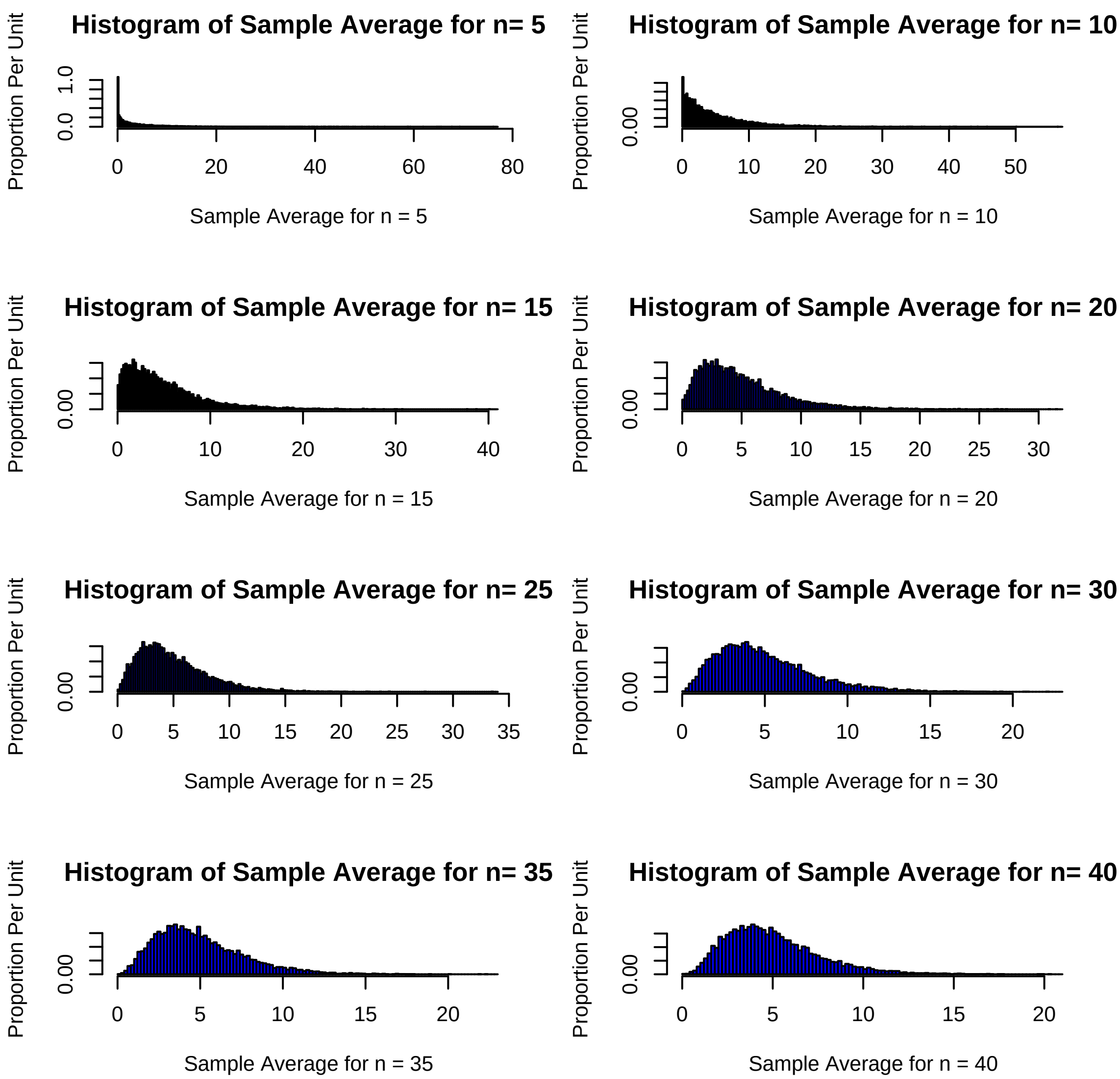

Figure 2: The probability density function (PDF) of the sample mean of n i.i.d. observations of a Negative Binomial random variable with parameters $\mu = 5$ and $\theta = 0.1$ at a number of sample sizes n. These PDFs are estimated above with histograms each produced from $10,000$ independent simulations.

Similarly, the sample mean $\bar{X}$ of n i.i.d. Negative Binomial random variables with parameters μ and θ has the Laplace transform

$$F_{\bar{X}}(\lambda) = \left(1 + (1 - e^{-\lambda/n})\frac{\mu}{\theta}\right)^{-\theta n}. \tag{3}$$

If θn converges to a positive constant γ as $n \to \infty$ and $\theta \to 0$, then the Laplace transform (3) converges to

$$F_{\bar{X}}(\lambda) = \left(1 + \frac{\lambda\mu}{\gamma}\right)^{-\gamma}. \tag{4}$$

Meanwhile, a Gamma random variable Y with a PDF given by

$$f_{a,b}(x) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)} \tag{5}$$

has a Laplace transform of

$$F_Y(\lambda) = \left(1 + \frac{\lambda}{b}\right)^{-a}. \tag{6}$$

The Laplace transforms (4) and (6) are identical when $a = \gamma$ and $b = \frac{\gamma}{\mu}$. Therefore, the sample mean of Negative Binomial random variables converges to a Gamma distribution with shape parameter a and rate parameter b as $n \to \infty$ and $\theta \to 0$.

Constructing a $1 - \alpha$ confidence interval using the Gamma distribution requires estimating the parameters $a = \gamma = \theta n$ and $b = \frac{\gamma}{\mu} = \frac{\theta n}{\mu}$ in terms of the estimates of μ and θ that are obtained from the data. Maximum likelihood estimates for these parameters may be obtained using numeric techniques, or the Method of Moments estimates $\hat{\mu} = \bar{X}$ and $\hat{\theta} = \bar{X}/((s^2/\bar{X}) - 1)$ may be employed. We will rely upon the latter choice as a default and provide a discussion of alternatives in Section 4.2. Once a and b are estimated, a $1 - \alpha$ confidence interval for μ is given by the $(\alpha/2)$nd and $(1 - \alpha/2)$th quantiles of the Gamma distribution.

3.2 The Chi Square Approximation

We also propose the Chi Square (χ^2) distribution as an approximate PDF for $\bar{X}$. Under the assumptions of Section 3.1, $\bar{X}$ approximately follows a Gamma distribution with parameters $a = \theta n$ and $b = \frac{\theta n}{\mu}$. The Chi Square family of distributions is a special case of the more general Gamma. The corresponding case occurs for the sample mean of Negative Binomial random variables when $\mu = 2n\theta$, which we will typically phrase in terms of the ratio quantity

$$\texttt{ratio} = \frac{\mu}{2n\theta}. \tag{7}$$

When ratio = 1, the Gamma parameters are $a = \theta n = \frac{\mu}{2}$ and $b = \frac{\theta n}{\mu} = \frac{1}{2}$, which collectively specify a χ^2 distribution with μ degrees of freedom.

Figure 1 depicts approximate densities for $\bar{X}$ in the case of n i.i.d. $NB(\mu = 5, \theta = 0.1)$ random variables at a variety of sample sizes. When $n = 25$, the ratio quantity is exactly 1, and the PDF of $\bar{X}$ closely follows the characteristic form of the χ^2 curve. However, these PDFs deviate from the Chi Square at both larger and smaller sample sizes. With this in mind, we emphasize that the Chi Square model's applicability should be carefully investigated before it is utilized in a particular context. In general, using a one–parameter Chi Square model to approximate a function of two–parameter Negative Binomial random variables lacks the two–parameter Gamma model's flexibility. This necessarily limits the use of Chi Square confidence intervals to situations in which the ratio quantity is reasonably close to 1. However, when the Chi Square model is reasonable, it allows for the construction of a confidence interval based only upon the estimator $\bar{X}$ of μ. Other techniques such as the Gamma and Normal approximation also require an estimate s^2 of σ^2 that is considerably more variable than $\bar{X}$ at small sample sizes. We will propose some guidelines in Section 4.3 for the use of the Chi Square approximation based upon a study of the ratio quantity (7)'s relationship to the method's coverage probability. When applicable, χ^2 confidence intervals for μ may be constructed by estimating the degrees of freedom with the sample mean $\bar{X}$ and then computing the $(\alpha/2)nd$ and $(1-\alpha/2)th$ percentiles of the corresponding χ^2 distribution.

The Chi Square distribution may also be used to construct confidence intervals for the success probability p under the traditional parametrization of the Negative Binomial distribution. On page 504, Casella and Berger [1990] demonstrate that the quantity $2pY$ converges in distribution to a χ^2 random variable with $2nr$ degrees of freedom, where Y is the sum of n i.i.d. Negative Binomial(r, p) random variables. When the number of successes r is known, a $1 - \alpha$ confidence interval for p may be written in terms of Y and the quantiles of the χ^2 distribution.

3.3 Bernstein's Inequality

Bernstein's Inequality [Bernstein, 1934] provides a tail probability bound on sums of bounded independent random variables. There are actually many variants of this inequality, and selecting an appropriate one requires an examination of the assumptions underlying a particular study. More classical versions of Bernstein's Inequality were derived for uniformly bounded random variables, but Negative Binomial random variables are not bounded above. We will first address this problem through a version of Bernstein's Inequality that does not rely upon an assumption of boundedness. We will then provide an alternative methodology based upon this assumption. Although it does not directly apply to Negative Binomial random variables, this Bounded Bernstein method may be appropriate when a Negative Binomial model is considered as an approximate distribution for bounded data. Moreover, this alternative method may also lead to improved confidence intervals.

We will begin by deriving a confidence limit using a variant of Bernstein's Inequality that does not require an assumption of boundedness. As part of Lemma 8 on pages 366 and 367, Birge and Massart [1998] show that a knowledge of a random variable's Laplace transform (which is also known as the Moment Generating Function) is sufficient to apply the classical version of Bernstein's Inequality given by Uspensky [1937]. Suppose that Z_i, $1 \leq i \leq n$, are i.i.d. negative binomial with parameters μ and θ, and let $Y_i = Z_i - \mu$. Then,

$$\log E\left[\exp(yY_i)\right] = -\theta \log\left(\frac{(1-e^y)\,\mu}{\theta} + 1\right) - y\mu. \tag{8}$$

From the classical version of Bernstein's Inequality [Uspensky, 1937],

$$P\left[\sum_{i=1}^{n} Y_i \geq n\epsilon\right] \leq \exp\left[\inf_{y\geq 0}\left(-ny\epsilon + \sum_{i=1}^{n}\log E[e^{yY_i}]\right)\right]. \tag{9}$$

Substituting (8) into (9), we obtain the following probability bounds:

$$P\left(\sum_{i=1}^{n} Y_i > n\epsilon\right) \leq \left(\frac{\theta+\mu}{n\epsilon+\theta+\mu}\right)^{-\theta}\left(\frac{(\theta+\mu)(n\epsilon+\mu)}{\mu(n\epsilon+\theta+\mu)}\right)^{-n\epsilon-\mu}, \tag{10}$$

and

$$P\left(\sum_{i=1}^{n} Y_i < -n\epsilon\right) \leq \left(\frac{\theta+\mu}{n\epsilon+\theta+\mu}\right)^{-\theta}\left(\frac{\mu(n\epsilon+\theta+\mu)}{(\theta+\mu)(n\epsilon+\mu)}\right)^{n\epsilon+\mu}. \tag{11}$$

Equation 11 comes from using the classical inequality with Y_i replaced by $-Y_i$. With $\bar{X}$ serving as the empirical mean of Z_i, $1 \leq i \leq n$, the symmetry of Equations 10 and 11 imply that we can construct a $1-\alpha$

confidence interval by setting the right side of either equation equal to $\alpha/2$. This amounts to constructing an interval of the form $\bar{X} \pm \epsilon$, where ϵ is the solution to the equation

$$\left(\frac{\theta + \mu}{n\epsilon + \theta + \mu}\right)^{-\theta} \left(\frac{(\theta + \mu)(n\epsilon + \mu)}{\mu(n\epsilon + \theta + \mu)}\right)^{-n\epsilon - \mu} = \alpha/2. \tag{12}$$

However, solving Equation 12 is not easily amenable to analytic methods. For the purposes of implementation, we instead rely upon a simple numeric root–finding procedure that selects the best among candidate values of ϵ at evenly spaced intervals over a range (e.g. searching in increments of 0.1 from 0 to 100) and then searches within a small neighborhood of this candidate for an improved solution. Although this procedure is not guaranteed to provide a good approximation of the true value of ϵ that solves Equation 12, in practice it often performs reasonably well without requiring significant computation. However, further investigation of the solution to Equation 12 may lead to improved performance of the confidence interval.

We will also construct a Bernstein confidence interval under an assumption of uniformly bounded data. For notational purposes, we will refer to the interval constructed for unbounded data as the "Unbounded Bernstein" method and that proposed for bounded data as the "Bounded Bernstein" procedure. As stated in van der Laan and Rubin [2005], suppose that $Z_1, \ldots, Z_n$ are independent random variables such that $Z_i \in [a, b] \in \mathbb{R}$ with probability one and $0 < \sum_{i=1}^{n} Var(Z_i)/n \leq \sigma^2$. Then, for all $\epsilon > 0$,

$$P\left(\frac{1}{n}\sum_{i=1}^{n}(Z_i - E[Z_i]) > \epsilon\right) \leq \exp\left[\frac{-1}{2}\left(\frac{n\epsilon^2}{\sigma^2 + \epsilon(b-a)/3}\right)\right], \tag{13}$$

which in turn implies that

$$P\left(\frac{1}{n}|\sum_{i=1}^{n}(Z_i - E[Z_i])| > \epsilon\right) \leq 2\exp\left[\frac{-1}{2}\left(\frac{n\epsilon^2}{\sigma^2 + \epsilon(b-a)/3}\right)\right]. \tag{14}$$

Here the value of ϵ can be considered the desired distance from the sample mean. We seek a $1-\alpha$ confidence interval (CI) of the form $\bar{X} \pm \epsilon$ so that the distance ϵ extends sufficiently far to ensure with a probability of at least $1 - \alpha$ that the experiment will result in a confidence interval containing the true mean μ. The appropriate value of ϵ may be selected by setting the right hand side of Equation (14) equal to α. Then, by applying the Quadratic Formula, the value of ϵ is given by

$$\epsilon = \frac{\frac{-2}{3}(b-a)\log(\alpha/2) \pm \sqrt{\frac{4}{9}(b-a)^2[\log(\alpha/2)]^2 - 8n\sigma^2\log(\alpha/2)}}{2n}. \tag{15}$$

We will select the value of ϵ that adds the square root in Equation (15). The appeal of using the Bounded Bernstein method in the construction of $1-\alpha$ confidence intervals is that it only requires three assumptions [Rosenblum and van der Laan, 2008]: (i) all observations are independent, (ii) the maximum deviation from the mean is bounded by a known constant, and (iii) the variance is bounded by a known constant. Standard techniques such as the Normal approximation depend upon stronger parametric assumptions (i.e. the sample mean is approximately Normal in distribution). Therefore, Bounded Bernstein confidence intervals may be applied more generally than those based upon the Normal approximation. Similarly, confidence intervals may also be constructed from other tail bounds such as Bennett's Inequality [Bennett, 1962, 1963]. Hoeffding's Inequality [Hoeffding, 1963] may in fact be applied under only assumptions (i) and (ii). The Berry-Esseen Inequality [Berry, 1941, Esseen, 1942, 1956, van Beek, 1972] also requires just the three above assumptions but only results in non-vacuous confidence intervals for $n \geq 1024$ [Rosenblum and van der Laan, 2008], which necessarily limits its application as an alternative to the Normal approximation.

The formulation for ϵ in Equation (15) depends upon the data's bounding range $[a, b]$ and the variance σ^2. In the case of i.i.d. observations of Negative Binomial random variables, the lower bound is $a = 0$

because these variables draw from a non-negative sample space. However, Negative Binomial variables are unbounded, which violates assumption (ii) underlying Bernstein's Inequality in the Bounded Bernstein method. Furthermore, in small sample sizes, the data-based unbiased estimate s^2 of the variance σ^2 exhibits a high degree of variability and therefore may greatly underestimate the value of σ^2. Without accurate upper bounds for b and σ^2, Bounded Bernstein $1-\alpha$ confidence intervals for μ are not necessarily exact. Rosenblum and van der Laan [2008] provide some practical recommendations to address these concerns by relying upon known information about b and σ^2 collected in previous studies. Other possibilities include selecting b via a heuristic such as the 99.99th percentile of the Negative Binomial distribution with μ and θ estimated from the data. The only strict requirement for the Bounded Bernstein method is that we select a value of b at least as large as the maximum observed value. By default, we will rely upon the following data–based heuristic:

$$b = \frac{n+1}{n} \max\left(X_1, \ldots, X_n\right). \tag{16}$$

This heuristic was selected to provide an estimated upper bound in terms of the data and the sample size n. This choice of b will be considered in the simulation studies of Section 4, and Section 4.4 will examine b's impact on the coverage probability of corresponding Bounded Bernstein $1-\alpha$ confidence intervals.

Although the Bounded Bernstein method is not appropriate for unbounded data, the Negative Binomial model is often nonetheless considered as an approximate PDF for bounded data. The case studies of Section 5 provide examples in the serial analysis of gene expression and an examination of traffic flow in a communications network in which the underlying data are bounded but are reasonably approximated by Negative Binomial models.

4 Simulation Studies: Comparing the Wald, Bootstrap, χ^2, Gamma, and Bernstein Confidence Intervals for μ

4.1 Coverage Probabilities of the Proposed Methods

We designed two simulation studies to compare the proposed methods for efficacy. The first simulation compared the Wald (Normal Approximation), bootstrap, χ^2, Gamma, and both the Bounded and Unbounded Bernstein methods of constructing $1-\alpha$ confidence intervals for μ. We selected the computational parameter sets $\mu = \{5, 10\}$, $\theta = \{0.1, 1, 10, 10000\}$, and $n = \{10, 20, \ldots, 100\}$, which are summarized in Table 1. Each combination of values for μ, θ, and n led to a unique and independent simulation experiment. We selected these values of θ to allow for both high dispersion (when θ is low) and low dispersion (when θ is high), and we considered both small and moderate values of n to determine cut-off points at which standard methods like the Wald and bootstrap would overtake the proposed methods in terms of coverage. Each experiment consisted of 10,000 independent trials, and on each trial we randomly generated n i.i.d. $NB(\mu, \theta)$ random variables in the **R** statistical programming language. With $\alpha = 0.05$, we then computed 95% confidence intervals for μ based upon the data collected in the trial. The Wald method constructed confidence intervals by adding and subtracting 1.96 estimated standard errors to the sample mean. Bootstrap confidence intervals were computed according to the Bias Corrected and Accelerated (BCA) method [Efron and Tibshirani, 1994] based upon $B = 10,000$ resamplings of $\bar{X}$ from the data collected in each trial. We estimated the coverage probability of each method at each choice of parameters by computing the empirical proportion of trials within the experiment that resulted in a confidence interval containing the true value of μ.

In general, it appears that the Wald and bootstrap confidence intervals perform well for large values of θ (i.e. $\theta \geq 1$) but significantly under-cover μ when $\theta = 0.1$. Just as the limit theorem of Section 3.1 suggests, the Gamma confidence intervals improve in coverage as n increases and θ decreases. By contrast, the Bounded Bernstein and χ^2 methods exhibit more accurate coverage for small values of θ. Meanwhile, the Unbounded Bernstein method exhibits erratic behavior and actually decreases in coverage as the sample size increases.

At large values of θ, this decrease in coverage also occurs for the Gamma method. We suspect that this behavior may be the result of poor performance in the Method of Moments estimator of θ. For the Unbounded Bernstein method, it is also possible that the root–finding procedure used to estimate the distance ϵ to extend the interval from $\bar{X}$ may also adversely affect coverage when the candidate solution does not solve Equation 12 reasonably well.

The performance of the methods in Simulation 1 may be compared by reviewing the results in Tables 6 through 29 of the Appendix. These tables contain estimated coverage probabilities along with average, median, and standard deviations of the lengths of the intervals produced by each method in the 10,000 trials of a given simulation experiment. For instance, when $\mu = 5$, $\theta = 0.1$, and $n = 30$, the Wald method produces a coverage of 0.7935 with an average length of 9.7, median value of 8.3, and a standard deviation of 6.2. By contrast, the Chi Square method results in a coverage of 0.9592 with average, median, and standard deviation length values of 11.5, 11.3, and 3.6, respectively. In this particular context, the longer Chi Square interval results in improved coverage over the Normal approximation. This should not be surprising because the Chi Square method's ratio quantity is $\mu/(2n\theta) = 5/(2 * 30 * 0.1) = 5/6$, which is reasonably close to 1.

In general, the Wald, Gamma, and bootstrap results were reasonably similar for most values of μ, θ, and n. Meanwhile, the Bounded Bernstein and χ^2 methods greatly improve upon the Wald and bootstrap in terms of coverage probability at small values of θ. In comparing any two coverage probabilities at a specific value of n, μ, and θ, the difference in proportions has a margin of error of no more than 1.39% for a two-sided test. This worst-case error margin is obtained under the extreme assumption that the true coverage of each method is actually 50%. If the coverage of each method is actually 95%, then this margin of error drops to 0.6%. Any observed difference that is larger than the margin of error may be considered significant at the 5% level.

We then undertook a second independent simulation experiment that considered a greater variety of θ values and sample sizes. As summarized in Table 1, we considered values of $\theta \in \{0.025, 0.05, 0.075, 0.1, 0.2, 0.3, 0.4, 0.5\}$ at sample sizes $n \in \{5, 10, \ldots, 100\}$ while maintaining the μ values and number of trials as in the first simulation. In this second simulation, the bootstrap method was not considered because of its heavy computational requirements and its similarity in coverage to the Wald method in the first simulation. Computing a single coverage probability with the bootstrap requires generating $100, 000, 000n$ random numbers, where n is the sample size of the experiment. All told, Simulations 1 and 2 required nearly a week of continuous computation, of which all but a few hours were spent on the bootstrap. Tables 30 through 49 contain estimated coverage probabilities along with average, median, and standard deviation values for the interval lengths of each method over all experiments.

The small values of θ in the second simulation study correspond to extremely high dispersion in the data, and in these cases the Wald method severely under-covers μ. When $\mu = 5$, $\theta = 0.025$, and $n = 50$, the Wald method's estimated coverage is 67.64%, and this coverage only improves to 78.02% when $n = 100$. By contrast, the Bounded Bernstein method covers at rates of 79.3% and 89.7% at sample sizes of 50 and 100, respectively. Although the Bounded Bernstein and χ^2 confidence intervals outperform the Wald and Gamma at small sample sizes under high dispersion, it is important to note that neither of these confidence intervals are exact, and they also severely under-cover at the smallest values of n and θ. In the case of the Bounded Bernstein confidence interval, it is possible that the method can be improved or perhaps even made exact by refining the estimates of the variance σ^2 and the upper bound b. We will explore a sensitivity analysis of the coverage across a broad spectrum of b values in Section 4.4.

Figures 3 and 4 provide concrete recommendations on which confidence interval to select across all values of μ, θ, and n considered in the two simulation experiments. These recommendations are based upon which method exhibited a coverage closest to 95% in the simulation experiments. It is important to note that these recommendations allow for under-coverage; in the case of $\mu = 5$, $\theta = 0.5$, and $n = 100$, the Wald Method covers with an estimated probability of 93.53% and the Gamma at 92.34% whereas the Bounded Bernstein and χ^2 methods cover at 99.39% and 100%, respectively. Also, it is important to remember that none of the proposed methods perform particularly well when both n and θ are extremely small.

One caveat to the simulation results presented here is the special case of a data set containing all zeros. Tables 50 through 53 show the percentage of all–zero data sets generated in the simulation experiments. When such data arose, we adopted the convention that all methods should produce a confidence interval containing only the point zero, and therefore the interval would not cover the mean in this circumstance. This only affected the results at small values of both n and θ; for instance, in the experiment with $\mu = 10$, $\theta = 0.025$, and $n = 5$, 51.44% of all data sets produced contained only zeros. However, most scientific studies would not produce a confidence interval for μ based upon a data set of all zeros. If one is willing to ignore these cases, then the coverage probabilities at these small values of n and θ may be updated by computing the proportion of intervals that covered μ among the non–zero data set. For instance, the Bounded Bernstein's total coverage was 33.66% when $\mu = 10, \theta = 0.025$, and $n = 5$, and so its coverage among the non–zero data sets is actually $3366/(10000 - 5144) = 69.316\%$.

Overall, it appears that the Bounded Bernstein and Chi Square methods considerably improve upon the Wald and bootstrap at small sample sizes and high dispersion while the Gamma model closely tracks the standard methods. However, it is important to remember the conditions under which each method is applicable. The Bounded Bernstein method assumes that the data are uniformly bounded, which may be appropriate for scenarios in which the Negative Binomial model is an approximation to the distribution of bounded data but not specifically for Negative Binomial models themselves. The Chi Square approximation only applies when the ratio quantity of Equation 7 is reasonably close to 1. Therefore, while these proposed techniques may lead to improved coverage, some investigation of the conditions of the study in question should be considered in selecting an appropriate method.

Sim.	μ	θ	n	**trials**
1	$\{5, 10\}$	$\{0.1, 1, 10, 10000\}$	$\{10, 20, \ldots, 100\}$	10000
2	$\{5, 10\}$	$\{0.025, 0.05, 0.075, 0.1, 0.2, \ldots, 0.5\}$	$\{5, 10, \ldots, 100\}$	10000

Table 1: Parameter values for μ, θ, n, and the number of trials in the two simulation experiments of Section 4. The first simulation compared the Unbounded and Bounded Bernstein, χ^2, Gamma, Wald, and bootstrap confidence interval methods at each combination of the first set of parameter values. The second simulation compared the Wald, Unbounded and Bounded Bernstein, Gamma, and χ^2 methods at each combination of the second set of parameter values.

4.2 The Accuracy of θ Estimates

Both the Gamma model and the Unbounded Bernstein method rely upon an estimate of θ to produce a $1 - \alpha$ confidence interval for μ. In addition to the Method of Moments estimator $\hat{\theta} = \bar{X}/((s^2/\bar{X}) - 1)$, a variety of iterative maximum likelihood estimation (MLE) procedures have been proposed [Clark and Perry, 1989, Piegorsch, 1990]. Ara and Ferreri [1997] provide conditions for the existence and uniqueness of the MLE. Meanwhile, Pieters et al. [1977] compares an MLE procedure to the Method of Moments at small sample sizes. Although the MLE estimator was preferred, implementations such as that in the **glm.nb** function of the **R** statistical programming language tend to produce computational errors that prevented its application in the simulation experiments of the previous section. MLE methods appear to break down when the estimate s^2 of σ^2 is less than or equal to the estimate $\bar{X}$ of μ. Similarly, the Method of Moments estimator results in a non–positive approximation of the strictly positive parameter θ in this situation. For the purposes of the simulations, we chose to avoid this issue by truncating all non–positive estimates of θ to the value of 0.001 before applying the confidence interval procedures.

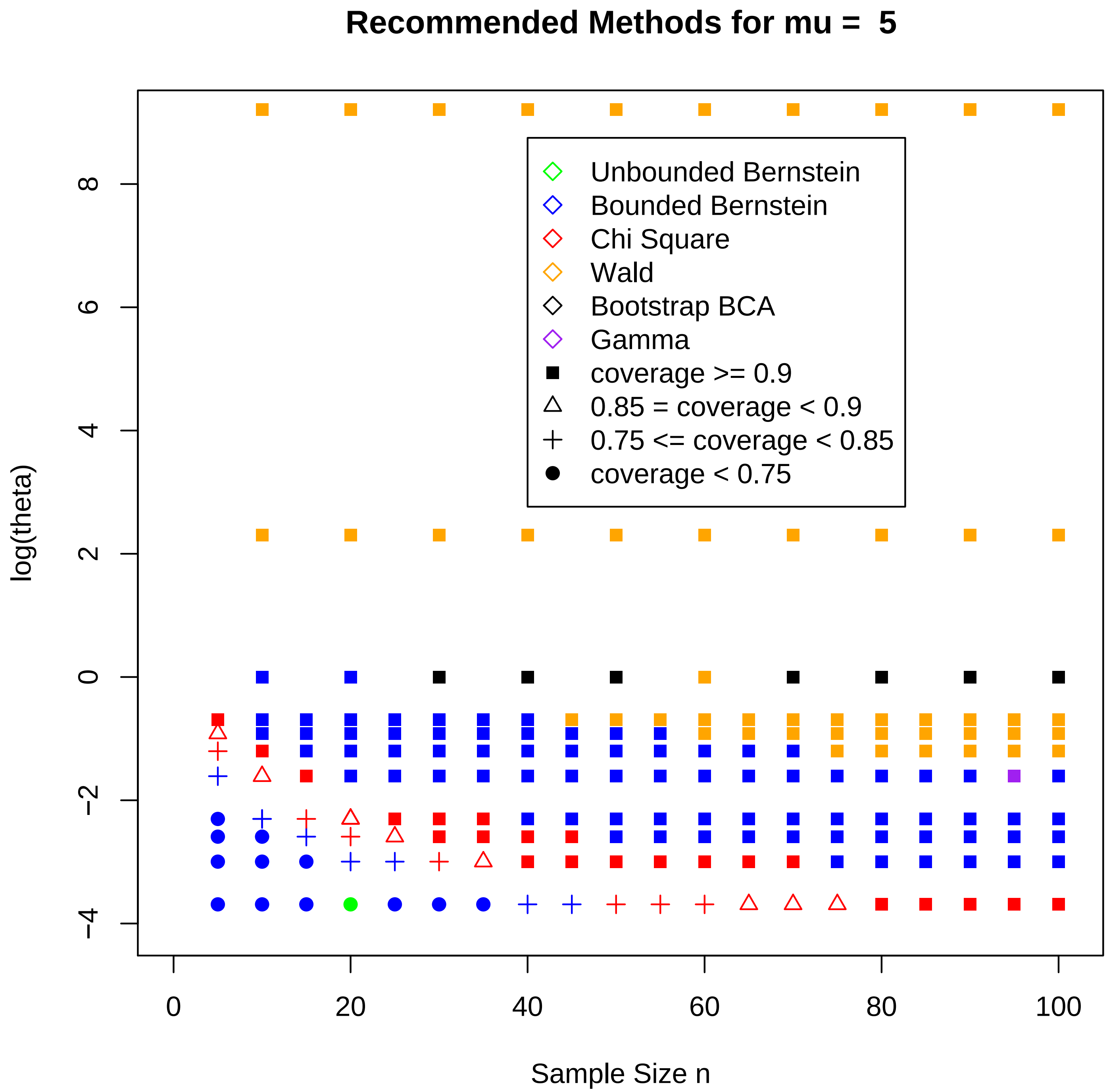

Figure 3: Recommended methods for constructing 95% confidence intervals for $\mu = 5$ as a function of the sample size n and θ (log scale). Each recommendation is presented in terms of a color signifying the preferred method and a symbol indicating this method's coverage quality. These recommendations are based upon the estimated coverage probabilities in the simulation study of Section 4. We selected among the candidate procedures based upon which method resulted in an estimated coverage closest to 95%.

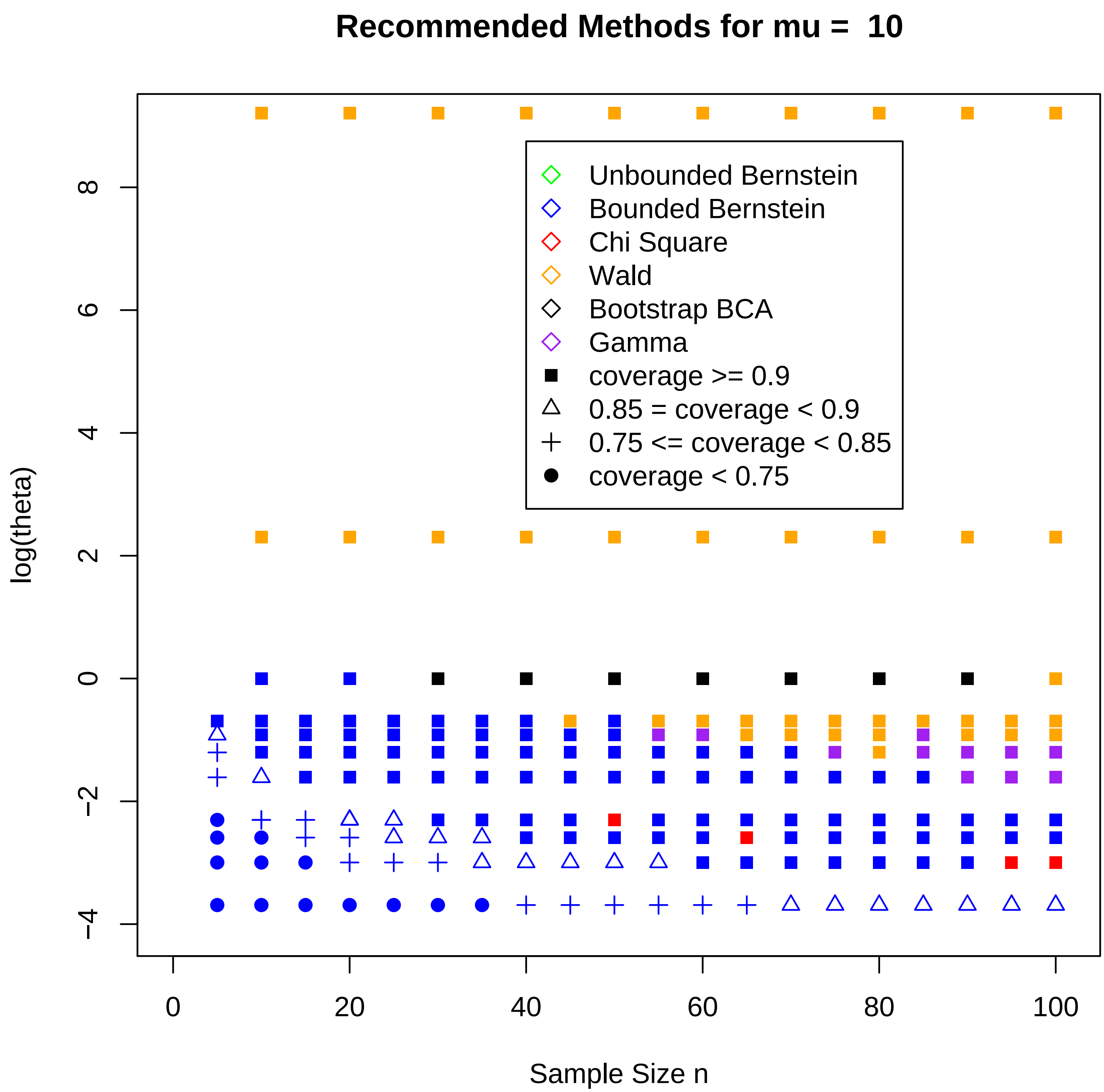

Figure 4: Recommended methods for constructing 95% confidence intervals for $\mu = 10$ as a function of the sample size n and θ (log scale). Each recommendation is presented in terms of a color signifying the preferred method and a symbol indicating this method's coverage quality. These recommendations are based upon the estimated coverage probabilities in the simulation study of Section 4. We selected among the candidate procedures based upon which method resulted in an estimated coverage closest to 95%.

Tables 54 through 65 provide summary information about the Method of Moments (non–truncated) estimates of θ over the range of experiments conducted in Simulations 1 and 2. For each combination of μ, θ, and n in the simulations, we provide the percentage of negative estimates along with the average, median, and standard deviation of the 10,000 values. The percentage of θ estimates below zero grows as a function of θ because it shrinks the data's standard deviation. For small values of θ, the median estimate is reasonably reliable even at small sample sizes, but the average and standard deviation can be greatly affected by extreme data sets. In general, it appears that a sample size of at least 30 or 40 is required to ensure that the Method of Moments estimator is not highly vulnerable to extreme data sets and may need to be as large as 85 to ensure reliability in the smallest values of θ studied in the simulations.

While MLE or other improved estimators of θ may lead to stronger performance of the Gamma and Unbounded Bernstein confidence intervals, it appears that their simulation results were not greatly impacted by the use of the Method of Moments estimator. At moderate sample sizes, the Method of Moments typically produced estimates that were reasonably close to the true value of θ, and its more erratic performance at smaller sample sizes corresponds to cases in which the MLE is also expected to have problems.

4.3 The Applicability of the χ^2 Method

Section 3.2 introduced the Chi Square approximation to the Gamma model of Section 3.1. This special case occurs when the ratio quantity of Equation 7 is equal to one. The principle advantage of the Chi Square method is that it allows for the construction of $1-\alpha$ confidence intervals without relying upon the extremely variable estimator s^2 of the variance σ^2, which is especially useful at small sample sizes. We are interested in determining how robust the Chi Square approximation is to deviations away from a ratio quantity of one. We conducted a third simulation to gain insight on the Chi Square's coverage at a variety of ratio quantities:

Each experiment consisted of selecting μ uniformly on $(1, 50)$, θ uniformly on $(0, 1)$, and the sample size n uniformly on the integers in $\{5, 6, \ldots, 150\}$. For each combination of n, μ, and θ, we randomly generated $10,000$ data sets of n i.i.d. $NB(\mu, \theta)$ random variables, applied the Chi Square method to each data set, and estimated the method's coverage probability by the empirical proportion of Chi Square 95% intervals that contained the selected value of μ. We conducted a total of $100,000$ such experiments to collect data at a wide range of ratio quantities.

Figure 5 displays boxplots of the ratio distribution for the simulation data partitioned into coverage groups. For magnification purposes, the plot restricts attention to the cases that resulted in a coverage of at least 50%. The 4% of the simulations not pictured generated anomalously large ratio quantities: approximately 1% of all simulations resulted in a ratio larger than 30, and the maximum observed value was $75,170$. Among the simulations with ratios less than 8, the correlation between the ratio quantity and the Chi Square method's coverage probability was -0.98, indicating a very strong negative association. Figure 5 suggests that ratio values less than 1 typically result in coverages greater than 95%. It also appears that the Chi Square method will cover at a rate of at least 90% when the ratio quantity is below 2.

With these observations in mind, it appears that the ratio quantity of Equation 7 can provide some insight into the applicability of the Chi Square method. Ratios less than 1 will typically result in a confidence interval that overcovers because it is too wide, whereas ratios greater than 1 indicate intervals that are too narrow and will undercover μ. Because of its strong relationship with coverage, the ratio quantity can be used as a guide in selecting among candidate confidence intervals even when the Chi Square procedure is not applicable. For instance, if the ratio quantity is 3, the Chi Square method might only produce an interval with a coverage of 75%, but the length of this interval can be used as a reference in the comparison of other candidate procedures so that a wider interval is ultimately selected. Furthermore, when the ratio quantity is less than 1, the Chi Square interval's length can be viewed as a maximum range so that any wider interval may be immediately excluded.

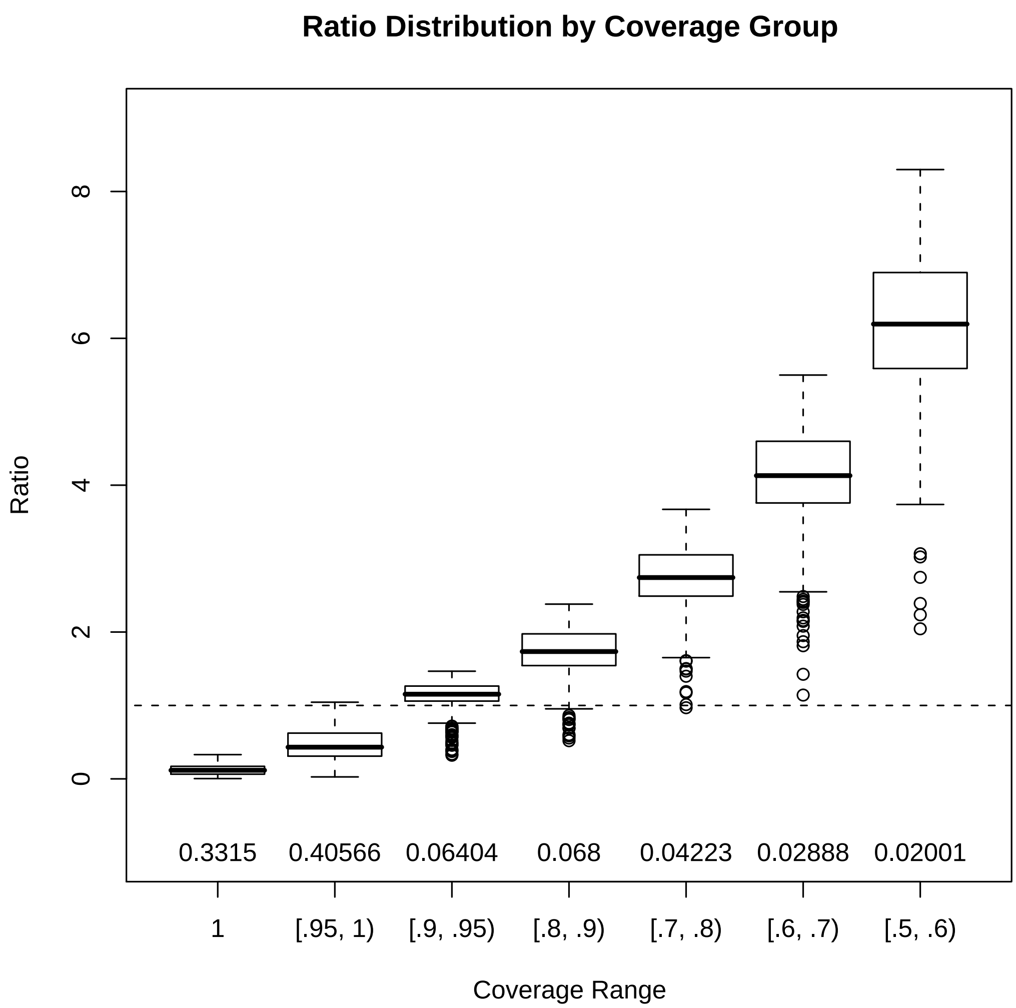

Figure 5: The distribution of ratio quantities (7) by coverage group for χ^2 95% confidence intervals. This box plot was constructed from data collected in the simulation of Section 4.3. The dashed lined represents a ratio quantity of 1, at which the Chi Square approximation to the PDF of $\bar{X}$ is exact. The proportion of simulations falling into each category are displayed below the box plots. The remaining 4% of the simulations that produced coverages under 50% are not displayed for magnification purposes.

4.4 The Upper Bound b in Bounded Bernstein Confidence Intervals

The simulation results of Section 4 show multiple examples in which the Bounded Bernstein confidence intervals as implemented result in estimated coverage probabilities well below 95%. These results were based upon the unbiased estimate s^2 of the variance σ^2 and an upper bound b given by Equation (16). However, because s^2 is highly variable at small sample sizes, it may result in values much smaller than σ^2. Likewise, it is unclear how to optimally select the value of b because the Negative Binomial random variables in question are unbounded and highly skewed. If we are primarily concerned with producing exact confidence intervals through the Bounded Bernstein method, there is no harm in greatly overestimating σ^2 and b. However, in practice we would prefer to construct intervals that are as narrow as possible while maintaining the minimum desired coverage. In this section we will examine the impact of selecting various choices of b in the context of the first simulation study of Section 4.

A fourth simulation, which was conducted independently of the two considered in Section 4 and that investigating the applicability of the Chi Square approximation in Section 4.3, consisted of repeating Simulation 1 for only the Bounded Bernstein method. In this case, a variety of b values were substituted in place of the heuristic of Equation (16) used previously. The values of b range from extremely small (such as the somewhat pathological case of $b = 0$) to extremely large (e.g. $b = 10^6$). The estimated coverage probability at each value of μ, θ, n, and b are presented in Tables 66 – 73.

In the case of low dispersion (e.g. $\theta = 10$ or $\theta = 10000$), the Bounded Bernstein method results in exact confidence intervals for all considered values of μ, n, and b. Likewise, when $\theta = 1$, the Bounded Bernstein method results in coverages over 95% except for the single case of $\mu = 5$, $n = 10$, and $b = 0$. These values of θ largely correspond to cases in which the Wald method performs reasonably well, so we are primarily concerned with higher dispersion, which we consider in the case of $\theta = 0.1$. Here the choice of b can greatly alter the coverage of the Bounded Bernstein confidence interval. This method resulted in an estimated coverage of just under 77% when $\mu = 5$, $\theta = 0.1$, and $n = 10$ for each of the two independent simulation experiments of Section 4. In this case, the heuristic of Equation (16) typically produced a value of about 10 for b. However, Table 66 shows how this coverage improves as a function of b. The choice of $b = 20$ instead of 10 would increase the estimated coverage of the Bounded Bernstein method from 77.7% to 97.95%. Because coverage grows monotonically as a function of b, similar improvements may be seen for all choices of μ and θ. Therefore, we recommend that the researcher fine-tune the Bounded Bernstein confidence interval in a situation-dependent manner through the selection of the upper bound b while relying upon the estimate s^2 of σ^2. Looking qualitatively at the results, this suggests that perhaps doubling the value of the heuristic of Equation (16) may produce the desired improvements when $n \leq 30$.

5 Data Analysis

The Negative Binomial model is particularly applicable as a generalization of the Poisson random variable that allows for the variance parameter to differ from the mean. In this section we will consider examples from the serial analysis of gene expression (SAGE) and network traffic flow data and explore the utility of the proposed methods as alternatives to the Wald and bootstrap confidence intervals. In doing so, we seek to better elucidate the strengths and weaknesses of the candidate procedures.

5.1 SAGE Data

A serial analysis of gene expression (SAGE) is used in molecular biology to estimate the relative abundance of messenger ribonucleic acid (mRNA) molecules based upon the frequency of corresponding 14 base pair *tag* sequences that are extracted from a cell [Velculescu et al., 1995]. Because the cost of sequencing can be prohibitive, the sample size is often limited to a small quantity. Robinson and Smyth [2008] propose a

Negative Binomial model for the tag counts of SAGE data and consider the problem of small sample estimation of the dispersion parameter. In this model, the tag counts are assumed to be independent Negative Binomial random variables with common dispersion for the purposes of estimation in spite of the possibility of related biological functions and expression co-regulation [Robinson and Smyth, 2008]. We consider this Negative Binomial model in the context of Sample GSM15034 of SAGE data stored at the National Center for Biotechnology Information website for the United States' National Institutes of Health:

(`http://www.ncbi.nlm.nih.gov/geo/query/acc.cgi?acc=GSM15034`).

The data, which are shown in Table 2, depict the $n = 20$ most frequent tags and their corresponding counts in a sample taken from the cells of *mus musculus*. The sample mean and standard deviation are $\bar{X} = 306.1$ and $s = 786.15$, respectively. We estimated the value of θ to be 0.6269 with a standard error of 0.1676. These estimates were obtained by applying an iterative maximum likelihood estimation (MLE) procedure within the **glm.nb** method of the **R** statistical programming language to the data. It should also be noted that MLE procedures typically underestimate variance parameters (and therefore the dispersion) [Robinson and Smyth, 2008], so θ may in fact be smaller than 0.6269.

However, even if the tag counts in the SAGE data may be assumed to be independent with a common dispersion, it is not at all clear that they are identically distributed. Robinson and Smyth [2008] consider a model in which each tag count has its own mean parameter, and so the data are only i.i.d. if each tag has the same mean. The χ^2, Gamma, Wald, and bootstrap confidence intervals may not necessarily be applicable when the data are not i.i.d., but both Bernstein methods only require independent data. Furthermore, the Bounded Bernstein method only applies if the data are uniformly bounded. Such an assumption seems reasonable in this context because the tag counts cannot exceed the length of the mRNA sequence. Therefore, the Negative Binomial model may be seen as merely an approximation to the true distribution of the tag counts, and the Bounded Bernstein method seems reasonable in this context. Based upon the underlying assumptions of each technique, Table 3 displays 95% confidence intervals for the mean μ tag count computed according to the Unbounded and Bounded Bernstein, χ^2, Wald, bootstrap, and Gamma methods based upon the SAGE tag count data of Table 2.

The 95% confidence intervals of Table 3 simultaneously illustrate many of the strengths and weaknesses of each method. Both the Bounded Bernstein and Wald intervals include a range of negative numbers as possible values for the mean μ, which is unreasonable for Negative Binomial random variables because they draw from a non-negative sample space. By contrast, the χ^2, Gamma, and bootstrap results are assured to be non-negative. One advantage of using the Gamma approximation over the Wald method is that it produces a confidence interval of similar width that is also guaranteed to be positive. The simulation results of Section 4 suggest that the Wald, Gamma, and bootstrap confidence intervals will under-cover μ at small values of n and θ, and so it is not surprising that the Bounded Bernstein interval is much wider. However, the χ^2 confidence interval is considerably more narrow than those of the Wald and bootstrap. This is not surprising because the ratio quantity (7) has a value of 12.21. Because this ratio is much larger than 1, we expect the χ^2 method to significantly undercover μ. The findings of Section 4.3 suggest that a ratio of 12.21 will result in a coverage well below 50%. Although the χ^2 method performs poorly in this circumstance, this interpretation of the ratio quantity provides a strong indication that the wider intervals produced by other methods are more reasonable for this context.

In selecting among the candidate confidence intervals of Table 3, we recommend choosing the Bounded Bernstein result for two reasons: first, the simulation studies suggest that this method improves upon the coverage of the Wald, Gamma, and bootstrap procedures at small values of n and θ. Second, the assumption of independent data is more reasonable than that of i.i.d. data, and only the Unbounded and Bounded Bernstein methods remain robust in this setting. Although its negative left endpoint limits the interpretation of the confidence interval, the Bounded Bernstein method at least suggests that a wider range of values for μ should be considered than those reported by the Wald, Gamma, and bootstrap results. In selecting among the remaining confidence intervals, the bootstrap and Gamma results are expected to exhibit similar coverage to that of the Wald while maintaining a positive left endpoint. By contrast, the χ^2 confidence interval does not appear to be a good fit for this combination of n, μ, and θ.

Tag	Count
TGAGCAAAGCCACC	3581
CATGTCGACCAGCC	657
TGAGCGCATGGGTC	428
CAGGCAGTGACAGC	170
CAGGTCGCGAAGGG	143
CTGGAGGACCCATG	138
CCACCAGGCAGCTC	122
CGAGCAAATGCCAG	116
CCATGCCAGGCAAT	98
CCCAGCCATCCCAT	78
CCAAAGGAGAGGGC	74
CACCTGGCGTCATG	74
TTAAACGGCGGCTG	66
TGGCCTGAAGAGCA	65
CTAACGGCCGAGAT	62
TGACCTTGCATGTA	54
CTACCGATGGCTGT	53
CAGGACACCACATC	50
CTGGGAGGTCAGGC	48
CTGCCCAATTTGCC	45

Table 2: SAGE Sample GSM15034 taken from *mus musculus* displaying the 20 most frequent tags and their corresponding counts in the SAGE sample.

5.2 Traffic Flow Data

We now consider an example arising from the analysis of traffic flow data in an internet communications network. Sanchez and He [2005] seek to estimate the mean packets per second (PPS) flowing through the network and propose a Negative Binomial model for the packet counts. The data, which are available at the Lawrence Berkeley National Laboratory's website (`http://ita.ee.lbl.gov/html/contrib/DEC-PKT.html`), consist of packet counts at each of $n = 102$ consecutive seconds. It is presumed that packets arrive according to a Poisson process with dispersion, so a Negative Binomial model is suitable for this analysis. This again raises the question of whether the data are actually bounded. Communications networks typically operate under a capacity constraint that suggests bounded data; therefore, it seems reasonable to assume that the Bounded Bernstein method is also appropriate in this scenario. The sample mean and standard deviation of the data are $\bar{X} = 310.31$ and $s = 94.54$, respectively. We again used the **glm.nb** method in **R** to estimate the dispersion as $\hat{\theta} = 10.59$ with a standard error of 1.52. Although this example has a similar sample mean to that of the SAGE data above, the values of n and θ are considerably larger in this case.

In the simulation studies of Section 4, the most similar case to the current example is that of $n = 100$ and $\theta = 10$. At larger values of n and θ, the simulation results generally suggest that the Wald and bootstrap methods perform well in terms of coverage, whereas the Bounded Bernstein and χ^2 techniques generally over–cover the mean μ. By contrast, the Gamma approximation and Unbounded Bernstein methods may be

	Lower Limit	Upper Limit	Other Quantities
Unbounded Bernstein	182.14	430.06	ϵ quality = 6.90e-07
Bounded Bernstein	-455.02	1067.22	–
Chi Square	259.53	356.46	$\hat{\mu}/(2n\hat{\theta}) = 12.21$
Gamma	160.82	497.35	–
Wald	-38.00	650.20	–
Bootstrap	111.95	1015.75	–

Table 3: 95% confidence intervals for the mean tag count based upon the SAGE data of Table 2. The χ^2, Wald, bootstrap, and Gamma confidence intervals were computed under the assumption of i.i.d. data, whereas the Unbounded and Bounded Bernstein intervals only assume that the data are independent. The estimated ratio quantity of 12.21 indicates that the Chi Square method is likely to significantly undercover μ. The ϵ quality metric shows that the selected value of ϵ in the Unbounded Bernstein method solves Equation 12 to within 6.90e-07.

impacted by the estimate of the dispersion θ. Table 4 displays 95% confidence intervals for the mean PPS computed according to the five methods. The Wald and bootstrap methods result in very similar confidence intervals. Here the Bounded Bernstein and χ^2 intervals are considerably wider than those of the Wald, Gamma, and bootstrap, which all offer similar results. The Unbounded Bernstein method actually produces the most narrow interval. While its selected value of ϵ appears to almost exactly solve Equation 12, the simulation results suggest that this Unbounded Bernstein method tends to undercover while the Wald and bootstrap results are reasonably accurate. With a ratio quantity is 0.14, the Chi Square method is expected to severely overcover because its approximation is not sufficiently close to the Gamma model.

The simulation results of Section 4 suggest that the Wald and bootstrap methods cover μ with a probability very close to the desired 95% for larger values of n and θ while the Bounded Bernstein and χ^2 methods tend to over-cover the mean. Because the Wald and bootstrap results are similar, we recommend selecting either as the preferred confidence interval in this setting. Therefore, it seems reasonable to believe that the mean traffic flow of the network is somewhere between approximately 292 and 329 packets per second. The question remains as to why the Gamma method also produced a similar interval but exhibited poor coverage in the simulations. It is possible that the Gamma approximation is sensitive to the estimated value of θ and performs poorly in some circumstances on account of this estimator's variability.

	Lower Limit	Upper Limit	Other Quantities
Unbounded Bernstein	306.97	313.66	ϵ quality = 2.29e-08
Bounded Bernstein	276.02	344.60	–
Chi Square	263.41	361.01	$\hat{\mu}/(2n\hat{\theta})$ 0.14
Gamma	292.08	329.09	–
Wald	291.97	328.66	–
Bootstrap	292.91	329.49	–

Table 4: 95% confidence intervals for the mean packets per second (PPS) flowing through a communications network. The data were collected from $n = 102$ seconds of traffic flow. The ϵ quality metric shows that the selected value of ϵ in the Unbounded Bernstein method solves Equation 12 to within 2.29e-08. Similarly, the estimated ratio value of 0.14 suggests that the Chi Square method is likely to significantly overcover the mean μ.

The two data analysis examples presented in this section are an important reminder that the question of which confidence interval to select should be addressed in the context of the problem at hand. When the sample size is large and the dispersion is small, standard techniques like the Wald and bootstrap perform well and lead to reasonable inferences. At small sample sizes and larger dispersions, the Bounded Bernstein confidence interval often results in improved coverage. The χ^2 method may also be considered as an alternative when the ratio quantity is reasonably close to 1. These proposed techniques generally result in wider confidence intervals than those of the standard methods, and these results serve as an important reminder that it is difficult to draw strong inferences from a limited amount of highly dispersed data.

6 Discussion

The simulation studies of Section 4 establish the Bounded Bernstein method as a clear improvement over the Wald and bootstrap in the construction of $1-\alpha$ confidence intervals for the mean μ of Negative Binomial random variables when n and θ are small. The Gamma Model typically results in similar results to those of standard techniques. While the Chi Square method can be extremely accurate, it ony applies as a special case of the Gamma when the ratio quantity is reasonably close to 1. The Unbounded Bernstein method, while a promising idea in theory, appears to exhibit problems due to its reliance on highly variable estimates of θ and numeric techniques to select the value of ϵ that solves Equation 12. However, in total, the recommendations presented in Figures 3 and 4 depict a wide range of values of n and θ for which the proposed techniques improve upon standard methods in terms of fiducial limits. The subsequent simulations of Section 4.4 in turn show the plausibility of further improvements in the Bounded Bernstein method through refining the estimate of the data's upper bound b. In light of these results, it is clear that researchers should consider the efficacy of employing the Wald and bootstrap methods to produce confidence intervals in the case of high dispersion and small or moderate sample sizes.

A number of other considerations apply in selecting among candidate procedures for constructing $1-\alpha$ confidence intervals. These concerns are summarized in Table 6. In terms of computational speed, the bootstrap method requires B bootstrap resamplings of the data and corresponding sample mean calculations plus a final sort of the results, which leads to a computational complexity of $O\left(Bn + B\log(B)\right)$. In practice, the value of B should be a reasonably large number such as 10,000, which renders the bootstrap method significantly more costly than the alternatives. However, in many cases a single bootstrap confidence interval may be computed in no more than a minute. Furthermore, compared to the time required to design and gather data in a scientific study, even a computation requiring several hours to compute a bootstrap confidence interval is reasonable. We are also concerned with the interpretability of the confidence intervals produced by each method. In the case of a small value of μ, the resulting Wald, Unbounded Bernstein, and Bounded Bernstein confidence intervals may result in a left endpoint that is less than zero; such a result is of course an implausible value of μ for non-negative data. By contrast, χ^2 and Gamma confidence intervals always result in positive left endpoints because these distributions have positive sample spaces. Similarly, the bootstrap produces its confidence interval from repeated computations of $\bar{X}$ based upon resampled data, so these intervals are also assured to be non-negative. In terms of applicability, the bootstrap and Unbounded and Bounded Bernstein methods only require very mild assumptions, although the Bounded Bernstein's supposition of uniformly bounded data is violated when the data truly follow a Negative Binomial distribution. By contrast, the Chi Square, Gamma, and Wald confidence intervals require specific parametric assumptions about the distribution of the sample mean $\bar{X}$, with the Chi Square approximation requiring the additional assumption that the ratio quantity of Equation 7 is equal to 1. Meanwhile, the Normal approximation of the Wald method is well justified by the Central Limit Theorem for large values of n and θ but results in severe under-coverage at smaller values. Likewise, the Gamma approximation is justified by the limit theorem of Section 3.1 at large values of n and small values of θ but appears to perform poorly at small sample sizes and lower dispersions. Finally, it should be emphasized that the χ^2, Wald, Gamma, and bootstrap methods assume i.i.d. data, whereas both Bernstein confidence intervals only require that the data be independent.

The simulation results clearly demonstrate that the Bounded Bernstein and χ^2 methods are useful alternatives to the Wald and bootstrap at small sample sizes under high dispersion. However, it is also important to consider whether these methods' respective coverage probabilities asymptotically converge to the desired fiduciary limit. The Wald method is well justified at large sample sizes by the Central Limit Theorem, and bootstrap confidence intervals can be shown to converge in coverage to $1 - \alpha$ as the sample size n and number of resamplings B grow large. This does not appear to be the case for the Bounded Bernstein and χ^2 methods, though. The simulation results suggest that these techniques will largely over-cover μ for large sample sizes. Although the coverage probability of each technique is the most informative measure of the method's reliability, these other aspects should be considered in selecting among candidate procedures for constructing $1 - \alpha$ confidence intervals.

The proposed Bounded Bernstein and χ^2 methods greatly enhance our ability to construct $1 - \alpha$ confidence intervals for μ in the case of moderate and small samples of highly dispersed Negative Binomial random variables. The Negative Binomial model is increasingly used in scientific applications such as genomic high throughput analysis. The resulting confidence intervals from the proposed methods will in practice be wider than those of the standard techniques, and these results will in turn allow us to draw improved inferences about the underlying scientific phenomena in question.

Future investigation in this area may explore a variety of questions raised by this study. The two Bernstein confidence intervals may be refined through improvements in probability tail bounds, improved procedures for calculating ϵ to solve Equation 12, and improved estimates of the upper limit b, variance σ^2, and the dispersion parameter θ, particularly in the case of small sample sizes. The limits of the χ^2 distribution's applicability as a probability model for $\bar{X}$ may be better substantiated through both analytical and empirical techniques. The length of Chi Square confidence intervals may also be reduced; for instance, Tate and Klett [1959] demonstrate a variety of approaches that reduce the length of a Chi Square interval for the variance of a Normal distribution over that obtained from a procedure allocating equal probability mass to each tail. A more thorough investigation of the Gamma model for $\bar{X}$ would provide greater insight into values of n and θ for which this approximation would be reasonable, the variability of estimating its parameters a and b, and the coverage quality of its resulting confidence intervals. Finally, the proposed techniques may be generalized to construct confidence intervals for other parameters of a sample of n i.i.d. Negative Binomial random variables using techniques based upon the data's empirical influence curve.

	Un. Bernstein	**B. Bernstein**	χ^2	**Wald**	**Boot**	**Gamma**
Computationally Fast	Yes	Yes	Yes	Yes	No	Yes
Positive CIs Assured	No	No	Yes	No	Yes	Yes
Parametric Assumptions	No	No	Yes	Yes	No	Yes
Useful at Small n and θ	No	Yes	Yes	No	No	No
Over-covers at Large n	No	Yes	Yes	No	No	No
Over-covers for High θ	No	Yes	Yes	No	No	No
Requires Independent Data	Yes	Yes	Yes	Yes	Yes	Yes
Requires i.i.d. Data	No	No	Yes	Yes	Yes	Yes

Table 5: A comparison of the Unbounded Bernstein, Bounded Bernstein, χ^2, Wald (Normal Approximation), bootstrap, and Gamma methods for computing $1 - \alpha$ confidence intervals of the mean μ based upon n i.i.d. observations of a Negative Binomial random variable in terms of a variety of concerns about the applicability, feasibility, and interpretability of these methods.

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7 Appendix: Simulation Tables

n/θ	0.1	1	10	10000
10	0.6544	0.8743	0.9088	0.9164
20	0.7460	0.9057	0.9307	0.9303
30	0.7935	0.9196	0.9374	0.9421
40	0.8199	0.9248	0.9426	0.9409
50	0.8401	0.9316	0.9434	0.9453
60	0.8524	0.9378	0.9437	0.9424
70	0.8616	0.9343	0.9453	0.9471
80	0.8742	0.9371	0.9475	0.9471
90	0.8715	0.9404	0.9497	0.9475
100	0.8871	0.9379	0.9453	0.9444

Table 6: Simulation 1 results for the Wald method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	13.9 (9.4) 14.3	6.3 (5.8) 2.6	3.3 (3.2) 0.9	2.7 (2.7) 0.7
20	11.2 (9) 8.7	4.6 (4.4) 1.4	2.4 (2.3) 0.4	1.9 (1.9) 0.3
30	9.7 (8.3) 6.2	3.8 (3.7) 0.9	1.9 (1.9) 0.3	1.6 (1.6) 0.2
40	8.7 (7.7) 4.9	3.3 (3.2) 0.7	1.7 (1.7) 0.2	1.4 (1.4) 0.2
50	7.9 (7.1) 4	3 (2.9) 0.6	1.5 (1.5) 0.2	1.2 (1.2) 0.1
60	7.3 (6.7) 3.4	2.7 (2.7) 0.5	1.4 (1.4) 0.1	1.1 (1.1) 0.1
70	6.9 (6.4) 3	2.5 (2.5) 0.4	1.3 (1.3) 0.1	1 (1) 0.1
80	6.5 (6.1) 2.7	2.4 (2.3) 0.4	1.2 (1.2) 0.1	1 (1) 0.1
90	6.1 (5.8) 2.3	2.2 (2.2) 0.3	1.1 (1.1) 0.1	0.9 (0.9) 0.1
100	5.9 (5.6) 2.1	2.1 (2.1) 0.3	1.1 (1.1) 0.1	0.9 (0.9) 0.1

Table 7: Simulation 1 length results for the Wald method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.7446	0.9995	1.0000	1.0000
20	0.8992	1.0000	1.0000	1.0000
30	0.9592	1.0000	1.0000	1.0000
40	0.9824	1.0000	1.0000	1.0000
50	0.9920	1.0000	1.0000	1.0000
60	0.9964	1.0000	1.0000	1.0000
70	0.9985	1.0000	1.0000	1.0000
80	0.9991	1.0000	1.0000	1.0000
90	0.9996	1.0000	1.0000	1.0000
100	0.9998	1.0000	1.0000	1.0000

Table 8: Simulation 1 results for the Chi-Square method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	10.8 (10.1) 5.8	11.8 (11.7) 2.2	12 (12) 1.1	12 (12) 0.9
20	11.2 (10.9) 4.4	11.9 (11.9) 1.6	12 (12) 0.8	12 (12) 0.6
30	11.5 (11.3) 3.6	11.9 (12) 1.3	12 (12) 0.6	12 (12) 0.5
40	11.6 (11.4) 3.2	11.9 (11.9) 1.1	12 (12) 0.6	12 (12) 0.5
50	11.7 (11.5) 2.8	12 (12) 1	12 (12) 0.5	12 (12) 0.4
60	11.7 (11.6) 2.6	12 (12) 0.9	12 (12) 0.4	12 (12) 0.4
70	11.8 (11.7) 2.4	12 (12) 0.8	12 (12) 0.4	12 (12) 0.3
80	11.8 (11.7) 2.3	12 (12) 0.8	12 (12) 0.4	12 (12) 0.3
90	11.8 (11.8) 2.1	12 (12) 0.7	12 (12) 0.4	12 (12) 0.3
100	11.8 (11.8) 2	12 (12) 0.7	12 (12) 0.3	12 (12) 0.3

Table 9: Simulation 1 length results for the Chi-Square method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.6732	0.8350	0.8073	0.8595
20	0.7648	0.8694	0.7500	0.8260
30	0.8141	0.8918	0.7279	0.7984
40	0.8387	0.8954	0.7205	0.7841
50	0.8574	0.9047	0.7131	0.7748
60	0.8667	0.9147	0.7123	0.7549
70	0.8762	0.9074	0.7072	0.7583
80	0.8848	0.9112	0.7158	0.7504
90	0.8837	0.9129	0.7151	0.7435
100	0.8979	0.9092	0.7146	0.7405

Table 10: Simulation 1 results for the Gamma method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	13.1 (8.8) 13.7	5.6 (5.1) 2.8	7.7 (2.7) 9.4	13.8 (19.3) 11.2
20	10.7 (8.6) 8.4	4.1 (3.9) 1.4	8.4 (1.5) 16.7	26.4 (41) 23.9
30	9.3 (8) 6	3.4 (3.3) 1	6.4 (1.2) 16	30.3 (49.2) 27.6
40	8.4 (7.4) 4.8	3 (2.9) 0.8	4.3 (1) 13.3	30.8 (50.9) 28.4
50	7.7 (6.9) 4	2.7 (2.6) 0.6	3 (0.9) 10.7	29.9 (50.6) 28.2
60	7.2 (6.6) 3.4	2.5 (2.4) 0.5	2 (0.8) 8	28.8 (49.5) 27.6
70	6.7 (6.2) 2.9	2.3 (2.3) 0.4	1.6 (0.7) 6.8	28.2 (48.9) 26.8
80	6.4 (6) 2.6	2.2 (2.1) 0.4	1.1 (0.7) 4.9	27.5 (47.7) 25.9
90	6 (5.7) 2.3	2 (2) 0.3	1.1 (0.6) 4.7	26.7 (46.5) 25.1
100	5.8 (5.4) 2.1	1.9 (1.9) 0.3	0.8 (0.6) 3.4	25.9 (45.4) 24.3

Table 11: Simulation 1 length results for the Gamma method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.7245	0.8126	0.8429	0.9149
20	0.7708	0.7176	0.6640	0.8248
30	0.7842	0.6642	0.5502	0.7749
40	0.7865	0.5903	0.4734	0.7415
50	0.7792	0.5548	0.4168	0.7179
60	0.7710	0.5208	0.3722	0.6894
70	0.7508	0.4794	0.3419	0.6838
80	0.7422	0.4680	0.3172	0.6740
90	0.7302	0.4348	0.3144	0.6673
100	0.7163	0.4198	0.2863	0.6576

Table 12: Simulation 1 results for the Unbounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	22.4 (14.2) 23.4	6.2 (4.9) 7.9	28 (2.4) 43.1	56.3 (100) 48.8
20	14.2 (10.5) 12.8	2.9 (2.7) 1.2	16.2 (1.1) 35.7	54.1 (100) 49.4
30	10.5 (8.3) 8.2	2 (1.9) 0.7	10.4 (0.7) 29.5	54.1 (100) 49.6
40	8.3 (6.8) 5.9	1.5 (1.4) 0.4	6.5 (0.5) 23.7	53.5 (100) 49.7
50	6.9 (5.8) 4.5	1.2 (1.2) 0.3	4.3 (0.4) 19.4	52.5 (100) 49.8
60	5.9 (5.1) 3.6	1 (1) 0.2	2.6 (0.3) 14.9	51.7 (100) 49.8
70	5.1 (4.5) 2.9	0.9 (0.8) 0.2	2 (0.3) 13	52.2 (100) 49.8
80	4.6 (4.1) 2.5	0.8 (0.7) 0.2	1.2 (0.2) 9.5	52.5 (100) 49.8
90	4.1 (3.7) 2	0.7 (0.7) 0.1	1.2 (0.2) 9.7	52.7 (100) 49.8
100	3.7 (3.3) 1.8	0.6 (0.6) 0.1	0.7 (0.2) 7.2	52.9 (100) 49.8

Table 13: Simulation 1 length results for the Unbounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.7742	0.9746	0.9983	0.9993
20	0.8640	0.9888	0.9982	0.9991
30	0.9091	0.9917	0.9990	0.9996
40	0.9312	0.9947	0.9986	0.9989
50	0.9459	0.9947	0.9991	0.9991
60	0.9523	0.9948	0.9991	0.9992
70	0.9599	0.9950	0.9981	0.9996
80	0.9637	0.9957	0.9987	0.9991
90	0.9664	0.9966	0.9987	0.9986
100	0.9713	0.9952	0.9988	0.9991

Table 14: Simulation 1 results for the Bounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	30.7 (20.4) 31.8	13.9 (12.9) 5.9	7.9 (7.7) 1.9	6.8 (6.7) 1.4
20	23.5 (18.8) 18.6	9.3 (8.8) 2.9	5 (4.9) 0.9	4.2 (4.2) 0.6
30	19.9 (16.8) 13.1	7.4 (7.1) 2	3.9 (3.8) 0.6	3.2 (3.2) 0.4
40	17.4 (15.3) 10.2	6.3 (6.1) 1.5	3.2 (3.2) 0.4	2.7 (2.7) 0.3
50	15.7 (14) 8.4	5.5 (5.4) 1.2	2.8 (2.8) 0.3	2.3 (2.3) 0.2
60	14.4 (13.1) 7.1	5 (4.9) 1	2.5 (2.5) 0.3	2.1 (2.1) 0.2
70	13.3 (12.1) 6.1	4.6 (4.5) 0.8	2.3 (2.3) 0.2	1.9 (1.9) 0.2
80	12.5 (11.6) 5.4	4.2 (4.2) 0.7	2.1 (2.1) 0.2	1.7 (1.7) 0.1
90	11.7 (10.9) 4.8	4 (3.9) 0.6	2 (2) 0.2	1.6 (1.6) 0.1
100	11.1 (10.4) 4.3	3.7 (3.7) 0.6	1.9 (1.9) 0.2	1.5 (1.5) 0.1

Table 15: Simulation 1 length results for the Bounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.6991	0.8641	0.8802	0.8821
20	0.8035	0.9062	0.9175	0.9125
30	0.8518	0.9233	0.9301	0.9318
40	0.8729	0.9283	0.9369	0.9320
50	0.8888	0.9342	0.9370	0.9368
60	0.8953	0.9372	0.9393	0.9355
70	0.9004	0.9364	0.9406	0.9416
80	0.9064	0.9398	0.9447	0.9423
90	0.9087	0.9416	0.9463	0.9429
100	0.9185	0.9390	0.9407	0.9387

Table 16: Simulation 1 results for the Bootstrap method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	15 (9.7) 16.2	6.3 (5.7) 2.9	3.2 (3.1) 0.9	2.6 (2.5) 0.6
20	13.3 (10.3) 10.9	4.7 (4.4) 1.5	2.3 (2.3) 0.4	1.9 (1.9) 0.3
30	11.5 (9.5) 8	3.9 (3.7) 1	1.9 (1.9) 0.3	1.6 (1.5) 0.2
40	10.1 (8.8) 6.2	3.4 (3.3) 0.8	1.7 (1.6) 0.2	1.4 (1.4) 0.2
50	9.2 (8) 5.1	3 (3) 0.6	1.5 (1.5) 0.2	1.2 (1.2) 0.1
60	8.4 (7.5) 4.3	2.8 (2.7) 0.5	1.4 (1.4) 0.1	1.1 (1.1) 0.1
70	7.7 (7) 3.7	2.6 (2.5) 0.4	1.3 (1.3) 0.1	1 (1) 0.1
80	7.3 (6.7) 3.3	2.4 (2.4) 0.4	1.2 (1.2) 0.1	1 (1) 0.1
90	6.8 (6.3) 2.9	2.3 (2.2) 0.3	1.1 (1.1) 0.1	0.9 (0.9) 0.1
100	6.5 (6) 2.6	2.1 (2.1) 0.3	1.1 (1.1) 0.1	0.9 (0.9) 0.1

Table 17: Simulation 1 length results for the Bootstrap method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.6541	0.8717	0.9103	0.9166
20	0.7501	0.9020	0.9353	0.9307
30	0.7927	0.9180	0.9376	0.9403
40	0.8287	0.9266	0.9444	0.9449
50	0.8385	0.9329	0.9406	0.9414
60	0.8580	0.9360	0.9414	0.9419
70	0.8618	0.9362	0.9438	0.9423
80	0.8750	0.9365	0.9455	0.9454
90	0.8775	0.9404	0.9448	0.9453
100	0.8883	0.9411	0.9445	0.9504

Table 18: Simulation 1 results for the Wald method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	27.7 (18.7) 28.6	12 (11.2) 5	5.3 (5.2) 1.4	3.8 (3.8) 0.9
20	22.1 (18) 17	8.8 (8.5) 2.6	3.9 (3.8) 0.7	2.7 (2.7) 0.5
30	19.1 (16.4) 12.2	7.3 (7.1) 1.8	3.2 (3.1) 0.5	2.2 (2.2) 0.3
40	17.5 (15.5) 9.7	6.4 (6.2) 1.4	2.8 (2.7) 0.4	1.9 (1.9) 0.2
50	15.7 (14.5) 7.8	5.7 (5.6) 1.1	2.5 (2.5) 0.3	1.7 (1.7) 0.2
60	14.7 (13.6) 6.7	5.2 (5.1) 0.9	2.3 (2.2) 0.2	1.6 (1.6) 0.2
70	13.7 (12.7) 5.8	4.8 (4.8) 0.8	2.1 (2.1) 0.2	1.5 (1.5) 0.1
80	12.9 (12.2) 5.2	4.5 (4.5) 0.7	2 (1.9) 0.2	1.4 (1.4) 0.1
90	12.2 (11.5) 4.7	4.3 (4.2) 0.6	1.8 (1.8) 0.2	1.3 (1.3) 0.1
100	11.7 (11.1) 4.3	4.1 (4) 0.6	1.7 (1.7) 0.1	1.2 (1.2) 0.1

Table 19: Simulation 1 length results for the Wald method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.5857	0.9920	1.0000	1.0000
20	0.7685	0.9996	1.0000	1.0000
30	0.8573	0.9999	1.0000	1.0000
40	0.9184	1.0000	1.0000	1.0000
50	0.9478	1.0000	1.0000	1.0000
60	0.9661	1.0000	1.0000	1.0000
70	0.9800	1.0000	1.0000	1.0000
80	0.9876	1.0000	1.0000	1.0000
90	0.9924	1.0000	1.0000	1.0000
100	0.9946	1.0000	1.0000	1.0000

Table 20: Simulation 1 results for the Chi-Square method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	15.4 (14.4) 8.2	17 (17) 3	17.2 (17.1) 1.3	17.2 (17.2) 0.9
20	16.1 (15.7) 6.1	17.1 (17.1) 2.1	17.2 (17.2) 0.9	17.2 (17.2) 0.6
30	16.4 (16.1) 5	17.1 (17.1) 1.7	17.2 (17.2) 0.7	17.2 (17.2) 0.5
40	16.8 (16.6) 4.4	17.2 (17.1) 1.5	17.2 (17.2) 0.6	17.2 (17.2) 0.4
50	16.8 (16.6) 4	17.2 (17.2) 1.3	17.2 (17.2) 0.6	17.2 (17.2) 0.4
60	16.9 (16.8) 3.6	17.2 (17.2) 1.2	17.2 (17.2) 0.5	17.2 (17.2) 0.4
70	16.9 (16.8) 3.3	17.2 (17.2) 1.1	17.2 (17.2) 0.5	17.2 (17.2) 0.3
80	17 (16.9) 3.1	17.2 (17.2) 1	17.2 (17.2) 0.4	17.2 (17.2) 0.3
90	17 (16.9) 3	17.2 (17.2) 1	17.2 (17.2) 0.4	17.2 (17.2) 0.3
100	17 (16.9) 2.8	17.2 (17.2) 0.9	17.2 (17.2) 0.4	17.2 (17.2) 0.3

Table 21: Simulation 1 length results for the Chi-Square method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.6767	0.8558	0.7974	0.8609
20	0.7767	0.8917	0.7976	0.8291
30	0.8180	0.9049	0.7866	0.8010
40	0.8508	0.9146	0.7952	0.7885
50	0.8584	0.9220	0.8097	0.7724
60	0.8777	0.9253	0.8045	0.7560
70	0.8777	0.9222	0.8128	0.7504
80	0.8887	0.9248	0.8154	0.7461
90	0.8886	0.9278	0.8180	0.7392
100	0.9004	0.9290	0.8204	0.7332

Table 22: Simulation 1 results for the Gamma method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	26.3 (17.7) 27.3	11.2 (10.4) 5	9.3 (4.1) 14.3	27.4 (41.5) 22.5
20	21.3 (17.3) 16.4	8.3 (8) 2.7	5.9 (2.7) 16.9	54.2 (87.7) 47.4
30	18.6 (15.9) 11.8	6.9 (6.7) 1.8	3.5 (2.2) 11.9	59.7 (101.7) 55.3
40	17 (15.2) 9.5	6 (5.9) 1.4	2.4 (1.9) 7.2	61.8 (105.9) 56.8
50	15.4 (14.1) 7.6	5.4 (5.3) 1.1	2 (1.7) 5.1	60.2 (105) 56.4
60	14.4 (13.3) 6.6	5 (4.9) 0.9	1.6 (1.6) 1.5	58.5 (103.4) 55.2
70	13.4 (12.5) 5.7	4.6 (4.5) 0.8	1.5 (1.5) 2.6	56.2 (100.3) 53.6
80	12.7 (12) 5.1	4.3 (4.2) 0.7	1.4 (1.4) 1.1	54.3 (97.4) 51.9
90	12 (11.4) 4.6	4.1 (4) 0.6	1.3 (1.3) 0.2	52.4 (94.8) 50.3
100	11.5 (10.9) 4.2	3.9 (3.8) 0.6	1.2 (1.2) 0.2	50.6 (92.1) 48.7

Table 23: Simulation 1 length results for the Gamma method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.7149	0.8091	0.7703	0.8965
20	0.7755	0.7306	0.5919	0.8150
30	0.7877	0.6538	0.4819	0.7594
40	0.7960	0.6011	0.4317	0.7332
50	0.7752	0.5546	0.3932	0.7095
60	0.7756	0.5242	0.3578	0.6814
70	0.7597	0.4953	0.3312	0.6677
80	0.7475	0.4586	0.3134	0.6596
90	0.7338	0.4461	0.2884	0.6476
100	0.7147	0.4158	0.2844	0.6391

Table 24: Simulation 1 results for the Unbounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	38.3 (28) 32.2	10.7 (9.5) 6	15.8 (3.3) 32.4	57.2 (100) 48.4
20	27.3 (20.9) 22.1	5.6 (5.2) 2.2	5.1 (1.6) 18.2	56.2 (100) 49.1
30	20.7 (16.4) 15.6	3.8 (3.6) 1.2	2.3 (1) 11	53.6 (100) 49.5
40	16.8 (13.9) 11.7	2.9 (2.7) 0.8	1.2 (0.8) 6.5	53.9 (100) 49.6
50	13.7 (11.8) 8.5	2.3 (2.2) 0.6	0.9 (0.6) 4.7	53 (100) 49.7
60	11.8 (10.2) 7	1.9 (1.9) 0.5	0.6 (0.5) 1.4	52.7 (100) 49.8
70	10.3 (9) 5.8	1.6 (1.6) 0.4	0.5 (0.5) 2.4	52.1 (100) 49.8
80	9.2 (8.1) 4.8	1.4 (1.4) 0.3	0.4 (0.4) 1	52 (100) 49.8
90	8.2 (7.3) 4.1	1.3 (1.3) 0.3	0.4 (0.4) 0	51.8 (100) 49.9
100	7.4 (6.7) 3.7	1.2 (1.1) 0.2	0.3 (0.3) 0	51.6 (100) 49.9

Table 25: Simulation 1 length results for the Unbounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.7748	0.9795	0.9986	1.0000
20	0.8683	0.9895	0.9991	0.9999
30	0.9112	0.9912	0.9991	0.9998
40	0.9347	0.9940	0.9993	0.9996
50	0.9430	0.9951	0.9992	0.9998
60	0.9573	0.9963	0.9990	0.9996
70	0.9586	0.9956	0.9993	0.9994
80	0.9654	0.9961	0.9988	0.9995
90	0.9678	0.9956	0.9994	0.9995
100	0.9726	0.9967	0.9989	0.9991

Table 26: Simulation 1 results for the Bounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	61 (41) 63.5	26.8 (24.9) 11.2	13.6 (13.3) 3	10.8 (10.7) 1.8
20	46.5 (37.4) 36.6	17.9 (17.2) 5.6	8.5 (8.3) 1.4	6.4 (6.4) 0.9
30	39.1 (33.1) 25.7	14.2 (13.7) 3.8	6.5 (6.4) 0.9	4.9 (4.8) 0.6
40	35.1 (30.9) 20.4	12.1 (11.7) 2.8	5.4 (5.4) 0.7	4 (4) 0.4
50	31.1 (28.3) 16.1	10.6 (10.4) 2.3	4.7 (4.7) 0.6	3.5 (3.5) 0.3
60	28.7 (26.2) 13.9	9.6 (9.4) 1.9	4.2 (4.2) 0.5	3.1 (3.1) 0.3
70	26.5 (24.4) 12.1	8.7 (8.5) 1.6	3.8 (3.8) 0.4	2.8 (2.8) 0.2
80	24.9 (23.1) 10.6	8.1 (7.9) 1.4	3.5 (3.5) 0.3	2.6 (2.6) 0.2
90	23.3 (21.7) 9.5	7.6 (7.5) 1.2	3.3 (3.3) 0.3	2.4 (2.4) 0.2
100	22.1 (20.6) 8.7	7.2 (7) 1.1	3.1 (3.1) 0.3	2.2 (2.2) 0.2

Table 27: Simulation 1 length results for the Bounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.7014	0.8703	0.8833	0.8855
20	0.8151	0.9102	0.9238	0.9165
30	0.8570	0.9205	0.9298	0.9310
40	0.8798	0.9290	0.9386	0.9366
50	0.8892	0.9368	0.9369	0.9343
60	0.9036	0.9404	0.9367	0.9376
70	0.9070	0.9382	0.9397	0.9383
80	0.9093	0.9390	0.9437	0.9406
90	0.9132	0.9417	0.9424	0.9420
100	0.9191	0.9391	0.9424	0.9474

Table 28: Simulation 1 results for the Bootstrap method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.1	1	10	10000
10	30.4 (19.8) 32.9	12.1 (11) 5.4	5.1 (5) 1.4	3.7 (3.6) 0.9
20	26.4 (20.9) 21.6	9 (8.6) 3	3.8 (3.8) 0.7	2.7 (2.6) 0.5
30	22.7 (18.9) 15.6	7.4 (7.2) 2	3.1 (3.1) 0.5	2.2 (2.2) 0.3
40	20.5 (17.7) 12.5	6.5 (6.3) 1.5	2.7 (2.7) 0.4	1.9 (1.9) 0.2
50	18.2 (16.3) 9.8	5.8 (5.7) 1.2	2.4 (2.4) 0.3	1.7 (1.7) 0.2
60	16.8 (15.2) 8.5	5.3 (5.2) 1	2.2 (2.2) 0.2	1.6 (1.6) 0.2
70	15.5 (14.1) 7.4	4.9 (4.8) 0.8	2.1 (2.1) 0.2	1.5 (1.5) 0.1
80	14.5 (13.4) 6.5	4.6 (4.5) 0.8	1.9 (1.9) 0.2	1.4 (1.4) 0.1
90	13.6 (12.6) 5.8	4.3 (4.3) 0.7	1.8 (1.8) 0.2	1.3 (1.3) 0.1
100	13 (12) 5.3	4.1 (4.1) 0.6	1.7 (1.7) 0.1	1.2 (1.2) 0.1

Table 29: Simulation 1 length results for the Bootstrap method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2781	0.3979	0.4783	0.5280	0.6408	0.7006	0.7288	0.7502
10	0.4061	0.5286	0.6027	0.6492	0.7426	0.7906	0.8128	0.8374
15	0.4733	0.5976	0.6635	0.7101	0.7915	0.8329	0.8444	0.8584
20	0.5356	0.6485	0.7071	0.7511	0.8214	0.8524	0.8644	0.8830
25	0.5692	0.6910	0.7389	0.7699	0.8427	0.8712	0.8735	0.8897
30	0.6061	0.7141	0.7659	0.7918	0.8437	0.8817	0.8911	0.8980
35	0.6234	0.7317	0.7773	0.8042	0.8628	0.8880	0.8973	0.9065
40	0.6486	0.7559	0.7981	0.8190	0.8717	0.8917	0.9055	0.9067
45	0.6643	0.7612	0.8061	0.8297	0.8809	0.8914	0.9066	0.9150
50	0.6764	0.7757	0.8171	0.8423	0.8885	0.8955	0.9110	0.9158
55	0.6995	0.7841	0.8244	0.8498	0.8907	0.9056	0.9100	0.9183
60	0.7128	0.7972	0.8316	0.8544	0.8890	0.9062	0.9140	0.9215
65	0.7138	0.8000	0.8428	0.8513	0.8973	0.9125	0.9251	0.9188
70	0.7377	0.8095	0.8413	0.8634	0.8933	0.9104	0.9224	0.9274
75	0.7430	0.8241	0.8397	0.8720	0.8957	0.9140	0.9236	0.9256
80	0.7523	0.8202	0.8520	0.8709	0.9088	0.9156	0.9217	0.9237
85	0.7525	0.8263	0.8606	0.8756	0.9054	0.9198	0.9282	0.9343
90	0.7604	0.8332	0.8615	0.8766	0.9074	0.9197	0.9297	0.9289
95	0.7757	0.8353	0.8628	0.8853	0.9118	0.9211	0.9239	0.9342
100	0.7802	0.8390	0.8687	0.8863	0.9082	0.9241	0.9288	0.9323

Table 30: Simulation 2 results for the Wald method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	35.8 (9.9) 64.5	25.9 (9.4) 42.6	20.7 (9.4) 29.2	18.6 (9.8) 24.2	14.1 (9.4) 14.5	12.3 (9.3) 10.9	11.2 (8.8) 8.9	10.4 (8.6) 7.5
10	24 (8.1) 40.4	17.3 (8.6) 23.2	15.1 (9.3) 17.5	14.2 (9.6) 14.9	11.2 (9) 8.8	9.7 (8.3) 6.3	8.8 (7.7) 5.2	8.1 (7.3) 4.3
15	19 (7.9) 28.2	15.2 (9.1) 17.9	13.2 (9.3) 13	12.2 (9.3) 10.6	9.7 (8.4) 6.2	8.4 (7.5) 4.5	7.5 (6.9) 3.6	6.9 (6.4) 3
20	17.1 (8.4) 23.2	13.7 (9.4) 14.1	12.2 (9.2) 10.6	11.2 (9.2) 8.5	8.7 (7.8) 4.9	7.4 (6.8) 3.5	6.7 (6.2) 2.8	6.1 (5.7) 2.3
25	15.6 (8.3) 19.7	12.9 (9.4) 12	11.4 (9.2) 9	10.4 (8.7) 7.2	8 (7.2) 4	6.8 (6.3) 2.9	6 (5.6) 2.3	5.5 (5.2) 1.9
30	14.8 (8.9) 17.3	12.2 (9.3) 10.4	10.7 (8.8) 7.8	9.7 (8.4) 6.1	7.3 (6.8) 3.4	6.2 (5.8) 2.4	5.6 (5.3) 1.9	5.1 (4.9) 1.6
35	14.1 (9.1) 15.4	11.7 (9.2) 9.6	10.1 (8.6) 6.8	9.1 (8) 5.4	6.9 (6.4) 3	5.9 (5.6) 2.1	5.1 (4.9) 1.7	4.7 (4.5) 1.4
40	13.7 (9.2) 14.1	11.2 (9.2) 8.4	9.7 (8.4) 6.1	8.6 (7.6) 4.8	6.5 (6.1) 2.7	5.5 (5.3) 1.9	4.9 (4.6) 1.5	4.4 (4.3) 1.3
45	13.3 (9.1) 13.2	10.6 (8.8) 7.7	9.2 (8.1) 5.4	8.3 (7.4) 4.5	6.2 (5.9) 2.4	5.2 (5) 1.7	4.6 (4.4) 1.3	4.2 (4.1) 1.1
50	12.8 (9.3) 12.1	10.3 (8.7) 7.1	8.9 (7.9) 5.1	7.9 (7.2) 3.9	5.9 (5.6) 2.2	5 (4.8) 1.6	4.4 (4.2) 1.2	4 (3.9) 1
55	12.5 (9.3) 11.3	9.9 (8.5) 6.5	8.6 (7.7) 4.7	7.6 (7) 3.6	5.7 (5.4) 2	4.8 (4.6) 1.4	4.2 (4.1) 1.1	3.8 (3.7) 0.9
60	12.2 (9.3) 10.7	9.6 (8.3) 5.9	8.3 (7.5) 4.3	7.4 (6.8) 3.4	5.5 (5.2) 1.8	4.6 (4.4) 1.3	4 (3.9) 1	3.7 (3.6) 0.8
65	11.7 (9) 10	9.2 (8.1) 5.6	8 (7.3) 4	7.1 (6.6) 3.2	5.3 (5.1) 1.7	4.4 (4.3) 1.2	3.9 (3.8) 1	3.5 (3.4) 0.8
70	11.7 (9.2) 9.4	9.1 (8) 5.3	7.7 (7.1) 3.7	6.9 (6.4) 3	5.1 (4.9) 1.6	4.2 (4.1) 1.1	3.7 (3.6) 0.9	3.4 (3.3) 0.7
75	11.4 (9.2) 8.9	8.9 (7.9) 5	7.4 (6.8) 3.6	6.7 (6.3) 2.8	4.9 (4.8) 1.5	4.1 (4) 1.1	3.6 (3.5) 0.8	3.3 (3.2) 0.7
80	11.2 (9.1) 8.5	8.7 (7.7) 4.8	7.3 (6.7) 3.4	6.5 (6.1) 2.6	4.8 (4.6) 1.4	4 (3.9) 1	3.5 (3.4) 0.8	3.2 (3.1) 0.6
85	10.9 (9) 8.1	8.4 (7.5) 4.5	7.2 (6.6) 3.2	6.3 (5.9) 2.5	4.7 (4.5) 1.4	3.9 (3.8) 0.9	3.4 (3.4) 0.7	3.1 (3) 0.6
90	10.6 (8.8) 7.6	8.2 (7.3) 4.3	7 (6.5) 3.1	6.1 (5.8) 2.3	4.5 (4.4) 1.3	3.8 (3.7) 0.9	3.3 (3.3) 0.7	3 (3) 0.6
95	10.5 (8.9) 7.3	8 (7.3) 4	6.8 (6.4) 2.9	6 (5.7) 2.2	4.4 (4.3) 1.2	3.7 (3.6) 0.9	3.2 (3.2) 0.7	2.9 (2.9) 0.5
100	10.3 (8.7) 7.1	7.9 (7.2) 4	6.6 (6.2) 2.7	5.9 (5.6) 2.2	4.3 (4.2) 1.2	3.6 (3.5) 0.8	3.1 (3.1) 0.6	2.9 (2.8) 0.5

Table 31: Simulation 2 length results for the Wald method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2319	0.3698	0.4829	0.5528	0.7411	0.8444	0.8887	0.9273
10	0.3717	0.5543	0.6610	0.7387	0.8992	0.9544	0.9782	0.9898
15	0.4738	0.6638	0.7728	0.8413	0.9559	0.9835	0.9952	0.9978
20	0.5485	0.7467	0.8437	0.8953	0.9820	0.9950	0.9986	0.9994
25	0.6111	0.8056	0.8886	0.9318	0.9902	0.9988	0.9992	0.9998
30	0.6756	0.8480	0.9218	0.9588	0.9966	0.9995	0.9998	1.0000
35	0.7044	0.8772	0.9423	0.9729	0.9979	0.9999	1.0000	1.0000
40	0.7385	0.9045	0.9576	0.9819	0.9990	0.9998	1.0000	1.0000
45	0.7645	0.9240	0.9698	0.9861	0.9997	0.9999	1.0000	1.0000
50	0.7970	0.9400	0.9766	0.9917	0.9998	1.0000	1.0000	1.0000
55	0.8261	0.9481	0.9838	0.9934	0.9998	1.0000	1.0000	1.0000
60	0.8444	0.9625	0.9884	0.9965	1.0000	1.0000	1.0000	1.0000
65	0.8565	0.9681	0.9923	0.9979	1.0000	1.0000	1.0000	1.0000
70	0.8783	0.9744	0.9946	0.9986	1.0000	1.0000	1.0000	1.0000
75	0.8910	0.9797	0.9965	0.9985	1.0000	1.0000	1.0000	1.0000
80	0.9006	0.9830	0.9969	0.9994	1.0000	1.0000	1.0000	1.0000
85	0.9111	0.9865	0.9973	0.9997	1.0000	1.0000	1.0000	1.0000
90	0.9229	0.9876	0.9976	0.9995	1.0000	1.0000	1.0000	1.0000
95	0.9290	0.9911	0.9985	0.9999	1.0000	1.0000	1.0000	1.0000
100	0.9414	0.9919	0.9992	1.0000	1.0000	1.0000	1.0000	1.0000

Table 32: Simulation 2 results for the Chi-Square method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	12.6 (8.5) 11.5	11.6 (8.8) 9.5	11.1 (9.1) 8	11 (9.5) 7.3	10.8 (10.1) 5.8	11.1 (10.7) 5	11.2 (10.9) 4.5	11.2 (10.9) 4.1
10	10.9 (8.1) 9.3	10.3 (8.8) 7.3	10.5 (9.6) 6.4	10.8 (10.1) 5.9	11.2 (10.9) 4.4	11.4 (11.2) 3.7	11.6 (11.3) 3.3	11.7 (11.6) 2.9
15	10.3 (8.2) 8.1	10.5 (9.5) 6.5	10.7 (10) 5.5	11 (10.7) 5	11.5 (11.3) 3.7	11.6 (11.6) 3	11.7 (11.6) 2.7	11.8 (11.7) 2.4
20	10.2 (8.6) 7.4	10.6 (9.9) 5.8	11 (10.5) 4.9	11.2 (11) 4.4	11.6 (11.5) 3.2	11.7 (11.7) 2.6	11.8 (11.7) 2.3	11.9 (11.8) 2.1
25	10.2 (8.8) 6.9	10.9 (10.4) 5.3	11.2 (10.9) 4.5	11.4 (11.1) 4	11.7 (11.6) 2.9	11.7 (11.7) 2.3	11.8 (11.7) 2.1	11.8 (11.8) 1.9
30	10.3 (9.2) 6.4	11 (10.6) 4.9	11.3 (11) 4.1	11.5 (11.3) 3.6	11.7 (11.6) 2.6	11.8 (11.7) 2.1	11.9 (11.8) 1.9	11.9 (11.9) 1.7
35	10.4 (9.6) 6.1	11.2 (10.8) 4.7	11.4 (11.2) 3.8	11.6 (11.4) 3.4	11.8 (11.7) 2.4	11.9 (11.8) 2	11.9 (11.8) 1.7	11.9 (11.9) 1.6
40	10.6 (9.9) 5.8	11.3 (11) 4.3	11.5 (11.4) 3.6	11.6 (11.4) 3.1	11.8 (11.7) 2.3	11.9 (11.8) 1.9	11.9 (11.8) 1.6	11.9 (11.9) 1.5
45	10.7 (10) 5.7	11.3 (11) 4.1	11.5 (11.4) 3.4	11.6 (11.5) 3	11.8 (11.8) 2.1	11.9 (11.9) 1.8	11.9 (11.9) 1.6	11.9 (11.9) 1.4
50	10.8 (10.3) 5.4	11.4 (11.1) 3.9	11.6 (11.4) 3.3	11.7 (11.5) 2.8	11.9 (11.8) 2	11.9 (11.8) 1.7	11.9 (11.9) 1.5	11.9 (11.9) 1.3
55	10.9 (10.4) 5.1	11.4 (11.3) 3.8	11.6 (11.5) 3.1	11.7 (11.6) 2.7	11.9 (11.8) 1.9	11.9 (11.9) 1.6	11.9 (11.9) 1.4	11.9 (11.9) 1.3
60	11 (10.5) 5	11.5 (11.3) 3.5	11.7 (11.6) 3	11.8 (11.7) 2.6	11.9 (11.8) 1.9	11.9 (11.9) 1.6	11.9 (11.9) 1.4	12 (11.9) 1.2
65	11 (10.6) 4.8	11.5 (11.3) 3.5	11.7 (11.6) 2.8	11.7 (11.7) 2.5	11.9 (11.8) 1.8	11.9 (11.9) 1.5	11.9 (11.9) 1.3	11.9 (11.9) 1.2
70	11.2 (10.8) 4.6	11.5 (11.3) 3.3	11.7 (11.6) 2.8	11.8 (11.7) 2.4	11.9 (11.8) 1.7	11.9 (11.9) 1.4	11.9 (11.9) 1.2	12 (12) 1.1
75	11.2 (11) 4.5	11.6 (11.4) 3.2	11.7 (11.6) 2.7	11.8 (11.8) 2.3	11.9 (11.9) 1.7	11.9 (11.9) 1.4	11.9 (11.9) 1.2	12 (12) 1.1
80	11.3 (11) 4.4	11.6 (11.5) 3.2	11.7 (11.6) 2.6	11.8 (11.8) 2.3	11.9 (11.9) 1.6	11.9 (11.9) 1.3	12 (11.9) 1.2	12 (12) 1.1
85	11.3 (11) 4.2	11.6 (11.5) 3.1	11.8 (11.6) 2.5	11.8 (11.7) 2.2	11.9 (11.9) 1.6	11.9 (11.9) 1.3	12 (11.9) 1.1	12 (11.9) 1
90	11.3 (11) 4.1	11.6 (11.5) 2.9	11.8 (11.7) 2.5	11.8 (11.8) 2.1	11.9 (11.9) 1.5	11.9 (11.9) 1.3	12 (11.9) 1.1	11.9 (11.9) 1
95	11.4 (11.2) 4	11.7 (11.6) 2.9	11.8 (11.7) 2.4	11.8 (11.8) 2	11.9 (11.9) 1.5	11.9 (11.9) 1.2	11.9 (11.9) 1.1	12 (12) 1
100	11.4 (11.2) 3.9	11.7 (11.6) 2.8	11.8 (11.7) 2.3	11.8 (11.8) 2	11.9 (11.9) 1.5	11.9 (11.9) 1.2	11.9 (11.9) 1	12 (12) 0.9

Table 33: Simulation 2 length results for the Chi-Square method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2819	0.4029	0.4826	0.5340	0.6407	0.6889	0.7116	0.7249
10	0.4242	0.5534	0.6242	0.6714	0.7529	0.7914	0.8084	0.8270
15	0.4957	0.6235	0.6868	0.7319	0.8005	0.8362	0.8377	0.8485
20	0.5610	0.6745	0.7311	0.7651	0.8325	0.8565	0.8591	0.8726
25	0.5936	0.7202	0.7661	0.7905	0.8514	0.8758	0.8669	0.8797
30	0.6378	0.7416	0.7896	0.8107	0.8532	0.8814	0.8880	0.8869
35	0.6519	0.7571	0.8050	0.8231	0.8684	0.8884	0.8936	0.8981
40	0.6786	0.7840	0.8187	0.8371	0.8800	0.8925	0.8993	0.8956
45	0.6911	0.7899	0.8266	0.8462	0.8853	0.8931	0.9001	0.9034
50	0.7050	0.7972	0.8364	0.8565	0.8895	0.8965	0.9052	0.9051
55	0.7329	0.8092	0.8486	0.8666	0.8937	0.9041	0.9032	0.9100
60	0.7417	0.8204	0.8495	0.8677	0.8941	0.9037	0.9059	0.9114
65	0.7407	0.8247	0.8582	0.8665	0.9012	0.9104	0.9186	0.9080
70	0.7644	0.8316	0.8583	0.8796	0.8963	0.9069	0.9173	0.9163
75	0.7692	0.8423	0.8595	0.8848	0.8991	0.9094	0.9172	0.9157
80	0.7790	0.8428	0.8699	0.8839	0.9090	0.9133	0.9164	0.9143
85	0.7799	0.8463	0.8750	0.8875	0.9061	0.9181	0.9197	0.9240
90	0.7868	0.8563	0.8773	0.8869	0.9092	0.9187	0.9223	0.9208
95	0.8011	0.8556	0.8787	0.8945	0.9132	0.9171	0.9168	0.9213
100	0.8109	0.8589	0.8826	0.8975	0.9059	0.9205	0.9212	0.9221

Table 34: Simulation 2 results for the Gamma method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	33.2 (9) 60.5	24 (8.5) 40.1	19.1 (8.5) 27.6	17.2 (8.9) 22.9	12.9 (8.5) 13.9	11.2 (8.3) 10.5	10.1 (7.8) 8.6	9.3 (7.6) 7.3
10	22.4 (7.5) 38	16.2 (8) 22	14.2 (8.6) 16.6	13.4 (9) 14.2	10.5 (8.4) 8.5	9.1 (7.7) 6.1	8.2 (7.1) 5.1	7.5 (6.7) 4.2
15	17.9 (7.4) 26.6	14.4 (8.5) 17.1	12.5 (8.7) 12.4	11.6 (8.8) 10.2	9.3 (7.9) 6	7.9 (7.1) 4.4	7 (6.4) 3.5	6.4 (5.9) 3
20	16.2 (7.9) 21.9	13 (8.9) 13.4	11.7 (8.8) 10.2	10.7 (8.8) 8.2	8.3 (7.4) 4.8	7.1 (6.4) 3.4	6.3 (5.8) 2.8	5.7 (5.3) 2.3
25	14.7 (7.9) 18.7	12.4 (9) 11.5	11 (8.8) 8.7	10 (8.4) 7	7.7 (6.9) 3.9	6.5 (6) 2.9	5.7 (5.3) 2.3	5.2 (4.9) 1.9
30	14.1 (8.5) 16.4	11.7 (8.9) 10	10.3 (8.5) 7.5	9.3 (8.1) 6	7 (6.5) 3.3	6 (5.6) 2.4	5.3 (5) 1.9	4.8 (4.6) 1.6
35	13.4 (8.6) 14.7	11.2 (8.9) 9.3	9.8 (8.3) 6.6	8.8 (7.8) 5.3	6.7 (6.1) 3	5.6 (5.3) 2.1	4.9 (4.7) 1.7	4.5 (4.3) 1.4
40	13.1 (8.8) 13.4	10.8 (8.9) 8.1	9.4 (8.2) 5.9	8.3 (7.4) 4.7	6.3 (5.9) 2.6	5.3 (5) 1.9	4.6 (4.4) 1.5	4.2 (4) 1.3
45	12.7 (8.7) 12.7	10.3 (8.5) 7.4	8.9 (7.9) 5.2	8 (7.2) 4.4	6 (5.7) 2.4	5 (4.8) 1.7	4.4 (4.2) 1.3	4 (3.8) 1.1
50	12.3 (8.9) 11.6	9.9 (8.4) 6.8	8.6 (7.7) 5	7.7 (7) 3.8	5.8 (5.4) 2.2	4.8 (4.6) 1.6	4.2 (4) 1.2	3.8 (3.7) 1
55	12 (8.9) 10.8	9.6 (8.2) 6.3	8.4 (7.5) 4.6	7.4 (6.8) 3.5	5.5 (5.2) 2	4.6 (4.4) 1.4	4 (3.9) 1.1	3.6 (3.5) 0.9
60	11.7 (8.9) 10.3	9.3 (8.1) 5.8	8.1 (7.3) 4.2	7.2 (6.6) 3.4	5.3 (5.1) 1.8	4.4 (4.3) 1.3	3.8 (3.7) 1	3.5 (3.4) 0.9
65	11.3 (8.7) 9.6	9 (7.8) 5.5	7.8 (7.1) 3.9	6.9 (6.4) 3.1	5.1 (4.9) 1.7	4.2 (4.1) 1.2	3.7 (3.6) 1	3.3 (3.3) 0.8
70	11.2 (8.9) 9	8.8 (7.8) 5.2	7.5 (6.9) 3.7	6.7 (6.2) 2.9	4.9 (4.7) 1.6	4.1 (4) 1.1	3.6 (3.5) 0.9	3.2 (3.2) 0.7
75	11.1 (8.9) 8.6	8.6 (7.7) 4.8	7.3 (6.7) 3.5	6.5 (6.1) 2.7	4.8 (4.6) 1.5	4 (3.9) 1.1	3.5 (3.4) 0.8	3.1 (3.1) 0.7
80	10.8 (8.8) 8.2	8.5 (7.6) 4.7	7.2 (6.6) 3.3	6.3 (6) 2.6	4.7 (4.5) 1.4	3.9 (3.8) 1	3.4 (3.3) 0.8	3 (3) 0.7
85	10.6 (8.7) 7.8	8.2 (7.4) 4.4	7 (6.5) 3.1	6.2 (5.8) 2.5	4.5 (4.4) 1.3	3.7 (3.7) 0.9	3.3 (3.2) 0.7	2.9 (2.9) 0.6
90	10.3 (8.6) 7.4	8 (7.2) 4.2	6.8 (6.3) 3	6 (5.7) 2.3	4.4 (4.3) 1.3	3.7 (3.6) 0.9	3.2 (3.1) 0.7	2.9 (2.8) 0.6
95	10.2 (8.6) 7.1	7.8 (7.1) 4	6.7 (6.3) 2.9	5.9 (5.6) 2.2	4.3 (4.2) 1.2	3.6 (3.5) 0.9	3.1 (3) 0.7	2.8 (2.7) 0.6
100	10 (8.5) 6.8	7.7 (7) 3.9	6.5 (6.1) 2.7	5.8 (5.5) 2.1	4.2 (4.1) 1.2	3.5 (3.4) 0.8	3 (3) 0.6	2.7 (2.7) 0.5

Table 35: Simulation 2 length results for the Gamma method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.3020	0.4492	0.5493	0.6077	0.7301	0.7834	0.8077	0.8289
10	0.4571	0.6044	0.6775	0.7197	0.7818	0.8071	0.8140	0.8234
15	0.5442	0.6682	0.7249	0.7564	0.7969	0.8130	0.7994	0.7961
20	0.6397	0.7104	0.7526	0.7752	0.8007	0.8024	0.7858	0.7842
25	0.6319	0.7391	0.7664	0.7808	0.7954	0.7883	0.7561	0.7524
30	0.6688	0.7497	0.7765	0.7841	0.7769	0.7725	0.7463	0.7245
35	0.6797	0.7572	0.7810	0.7835	0.7761	0.7553	0.7224	0.7015
40	0.6953	0.7719	0.7836	0.7844	0.7668	0.7356	0.7003	0.6720
45	0.7065	0.7694	0.7821	0.7821	0.7561	0.7139	0.6824	0.6596
50	0.7145	0.7741	0.7836	0.7837	0.7427	0.7007	0.6594	0.6309
55	0.7351	0.7759	0.7849	0.7783	0.7285	0.6882	0.6408	0.6258
60	0.7398	0.7797	0.7828	0.7672	0.7062	0.6598	0.6328	0.6073
65	0.7346	0.7736	0.7801	0.7548	0.7025	0.6558	0.6230	0.5805
70	0.7545	0.7793	0.7671	0.7560	0.6861	0.6350	0.6019	0.5752
75	0.7573	0.7841	0.7551	0.7521	0.6726	0.6163	0.5899	0.5534
80	0.7610	0.7769	0.7682	0.7411	0.6696	0.6163	0.5713	0.5359
85	0.7572	0.7739	0.7615	0.7346	0.6603	0.6027	0.5583	0.5273
90	0.7616	0.7731	0.7532	0.7283	0.6438	0.5885	0.5513	0.5144
95	0.7696	0.7684	0.7503	0.7277	0.6439	0.5676	0.5338	0.5088
100	0.7744	0.7633	0.7472	0.7184	0.6236	0.5734	0.5280	0.5000

Table 36: Simulation 2 results for the Unbounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	39.3 (20) 39.5	35.9 (19.3) 36.9	33.6 (18.7) 34.3	31.9 (18.5) 32.5	26 (16.3) 26.7	22.4 (14.7) 23	20.1 (13.1) 21.3	19 (12.3) 21.1
10	32.5 (15.4) 35.8	28 (15.2) 30.9	25.2 (15) 27	23.1 (14.5) 24.3	15.7 (11.4) 15.2	12.2 (9.5) 10.4	10.3 (8.2) 8.6	8.9 (7.2) 7.3
15	28.4 (14) 32.1	23.7 (13.8) 25.8	19.9 (13) 20.5	17.1 (12.1) 16.6	11.4 (8.9) 9.1	8.6 (7.2) 5.9	7.1 (6.1) 4.5	6.1 (5.3) 3.7
20	26.7 (16) 28.9	20.3 (13.5) 21	16.8 (11.8) 16.1	14.3 (10.7) 12.5	9 (7.4) 6.3	6.7 (5.8) 4	5.5 (4.8) 3.1	4.7 (4.2) 2.4
25	23.5 (13.2) 26.6	18 (12.3) 18	14.4 (10.7) 13	12.1 (9.4) 10.1	7.4 (6.3) 4.7	5.6 (4.9) 3.2	4.5 (4) 2.3	3.8 (3.5) 1.8
30	22.1 (13.2) 24.3	16.1 (11.5) 15.2	12.7 (9.6) 10.9	10.4 (8.3) 8	6.3 (5.5) 3.7	4.7 (4.1) 2.4	3.8 (3.4) 1.8	3.2 (3) 1.4
35	20.4 (12.5) 21.9	14.7 (10.6) 13.7	11.3 (8.9) 9.1	9.3 (7.5) 6.8	5.5 (4.8) 3.1	4.1 (3.7) 2	3.3 (3) 1.4	2.8 (2.6) 1.1
40	19.3 (12.3) 20.4	13.5 (10.2) 11.8	10.2 (8.2) 7.7	8.2 (6.8) 5.7	4.9 (4.4) 2.6	3.6 (3.3) 1.7	2.9 (2.7) 1.2	2.5 (2.3) 0.9
45	18.2 (11.7) 19.2	12.3 (9.4) 10.5	9.2 (7.6) 6.6	7.6 (6.3) 5.2	4.5 (4) 2.3	3.2 (2.9) 1.4	2.6 (2.4) 1	2.2 (2.1) 0.8
50	17.1 (11.7) 17.5	11.4 (8.8) 9.4	8.6 (7.1) 6.1	6.8 (5.9) 4.3	4 (3.6) 2	2.9 (2.7) 1.2	2.3 (2.2) 0.9	2 (1.9) 0.7
55	16.3 (11.2) 16.1	10.6 (8.3) 8.3	8 (6.6) 5.5	6.3 (5.5) 3.8	3.7 (3.3) 1.7	2.7 (2.5) 1.1	2.1 (2) 0.8	1.8 (1.7) 0.6
60	15.5 (10.9) 15.1	9.9 (8) 7.5	7.4 (6.2) 4.9	5.9 (5.1) 3.5	3.4 (3.1) 1.5	2.5 (2.3) 1	2 (1.8) 0.7	1.7 (1.6) 0.5
65	14.5 (10.2) 14	9.2 (7.4) 6.9	6.9 (5.9) 4.3	5.5 (4.8) 3.2	3.2 (2.9) 1.4	2.3 (2.1) 0.8	1.8 (1.7) 0.6	1.5 (1.5) 0.5
70	14.2 (10.3) 13.1	8.8 (7.2) 6.4	6.5 (5.5) 4	5.2 (4.5) 2.9	2.9 (2.7) 1.3	2.1 (2) 0.8	1.7 (1.6) 0.5	1.4 (1.4) 0.4
75	13.6 (10.1) 12.2	8.3 (6.9) 5.8	6 (5.2) 3.7	4.8 (4.2) 2.6	2.8 (2.5) 1.2	2 (1.9) 0.7	1.6 (1.5) 0.5	1.3 (1.3) 0.4
80	13 (9.7) 11.5	8 (6.6) 5.6	5.8 (5) 3.5	4.6 (4.1) 2.4	2.6 (2.4) 1.1	1.9 (1.8) 0.6	1.5 (1.4) 0.5	1.2 (1.2) 0.3
85	12.5 (9.4) 10.9	7.5 (6.3) 5.2	5.5 (4.8) 3.2	4.3 (3.9) 2.3	2.4 (2.3) 1	1.8 (1.7) 0.6	1.4 (1.3) 0.4	1.2 (1.1) 0.3
90	11.8 (9.1) 10.1	7.1 (5.9) 4.8	5.2 (4.6) 3	4.1 (3.7) 2	2.3 (2.2) 0.9	1.7 (1.6) 0.5	1.3 (1.3) 0.4	1.1 (1.1) 0.3
95	11.5 (9) 9.5	6.8 (5.8) 4.3	5 (4.4) 2.8	3.9 (3.5) 2	2.2 (2.1) 0.8	1.6 (1.5) 0.5	1.3 (1.2) 0.4	1.1 (1) 0.3
100	11.1 (8.7) 9.2	6.6 (5.6) 4.2	4.7 (4.2) 2.5	3.8 (3.4) 1.8	2.1 (2) 0.8	1.5 (1.4) 0.5	1.2 (1.1) 0.3	1 (1) 0.2

Table 37: Simulation 2 length results for the Unbounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.3324	0.4765	0.5767	0.6392	0.7735	0.8441	0.8766	0.8999
10	0.4769	0.6308	0.7129	0.7697	0.8685	0.9131	0.9377	0.9511
15	0.5684	0.7146	0.7881	0.8348	0.9109	0.9392	0.9540	0.9664
20	0.6324	0.7654	0.8318	0.8653	0.9333	0.9563	0.9688	0.9753
25	0.6732	0.8115	0.8578	0.8909	0.9447	0.9621	0.9716	0.9793
30	0.7187	0.8326	0.8834	0.9069	0.9519	0.9724	0.9791	0.9833
35	0.7349	0.8522	0.8971	0.9206	0.9608	0.9766	0.9834	0.9880
40	0.7633	0.8726	0.9131	0.9271	0.9652	0.9761	0.9845	0.9874
45	0.7817	0.8761	0.9135	0.9320	0.9697	0.9776	0.9837	0.9879
50	0.7930	0.8895	0.9256	0.9424	0.9739	0.9808	0.9853	0.9915
55	0.8130	0.8969	0.9299	0.9494	0.9737	0.9840	0.9883	0.9900
60	0.8288	0.9085	0.9358	0.9542	0.9748	0.9828	0.9872	0.9893
65	0.8350	0.9116	0.9447	0.9536	0.9794	0.9853	0.9882	0.9907
70	0.8523	0.9166	0.9467	0.9588	0.9757	0.9847	0.9897	0.9903
75	0.8564	0.9310	0.9482	0.9646	0.9786	0.9877	0.9880	0.9906
80	0.8679	0.9268	0.9511	0.9647	0.9818	0.9867	0.9919	0.9898
85	0.8709	0.9307	0.9536	0.9653	0.9812	0.9877	0.9917	0.9925
90	0.8802	0.9377	0.9549	0.9672	0.9828	0.9884	0.9923	0.9933
95	0.8863	0.9371	0.9586	0.9702	0.9870	0.9891	0.9912	0.9922
100	0.8970	0.9430	0.9613	0.9722	0.9861	0.9892	0.9921	0.9938

Table 38: Simulation 2 results for the Bounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	83.4 (22.6) 150.4	60.6 (21.9) 99.6	48.4 (21.9) 68.3	43.7 (23.1) 56.7	33.3 (21.9) 34.3	29.2 (22.1) 25.7	26.8 (21.2) 21.2	25 (20.8) 18
10	53.4 (17.5) 90.2	38.2 (19.2) 51.7	33.4 (20.2) 38.9	31.3 (20.9) 33.1	24.5 (19.5) 19.6	21.3 (18.1) 14.1	19.4 (16.7) 11.6	17.8 (15.8) 9.6
15	41.6 (17.3) 61.7	33 (19.3) 39.1	28.4 (19.7) 28.3	26.1 (19.7) 22.9	20.6 (17.5) 13.5	17.6 (15.6) 9.8	15.7 (14.3) 7.8	14.4 (13.3) 6.6
20	36.9 (17.9) 50.3	29.1 (19.7) 30.4	25.9 (19.3) 22.9	23.6 (19) 18.2	18 (15.9) 10.5	15.2 (13.8) 7.4	13.6 (12.5) 6.1	12.3 (11.5) 5
25	33.3 (17.8) 42.5	27.3 (19.6) 25.7	23.9 (18.9) 19.2	21.5 (17.9) 15.4	16.2 (14.5) 8.5	13.6 (12.5) 6.2	12 (11.1) 4.9	10.9 (10.2) 4
30	31.5 (18.8) 37.1	25.4 (19.2) 22.1	22.1 (18) 16.5	19.7 (16.8) 13	14.7 (13.4) 7.1	12.3 (11.4) 5.1	10.9 (10.3) 4.1	10 (9.4) 3.4
35	29.7 (19) 32.9	24.2 (18.8) 20.4	20.7 (17.3) 14.3	18.5 (16) 11.4	13.7 (12.5) 6.3	11.5 (10.7) 4.5	10 (9.5) 3.5	9.1 (8.7) 2.9
40	28.6 (19.1) 29.9	23.1 (18.7) 17.7	19.7 (16.9) 12.7	17.2 (15.1) 10	12.8 (11.8) 5.5	10.7 (10.1) 4	9.3 (8.9) 3.1	8.5 (8.1) 2.6
45	27.7 (18.7) 28	21.7 (17.8) 16.1	18.4 (16.1) 11.2	16.5 (14.6) 9.3	12.1 (11.3) 5	10 (9.4) 3.5	8.8 (8.4) 2.7	7.9 (7.6) 2.3
50	26.6 (19.2) 25.5	20.9 (17.3) 14.8	17.8 (15.6) 10.6	15.6 (14) 8.1	11.4 (10.7) 4.5	9.5 (9) 3.2	8.3 (7.9) 2.5	7.5 (7.2) 2.1
55	25.8 (19) 23.8	20 (16.9) 13.5	17.1 (15.1) 9.9	14.9 (13.5) 7.4	10.9 (10.2) 4.2	9 (8.5) 2.9	7.9 (7.6) 2.3	7.1 (6.8) 1.9
60	25.1 (18.9) 22.4	19.2 (16.5) 12.4	16.4 (14.7) 9	14.4 (13.1) 7	10.4 (9.8) 3.8	8.6 (8.2) 2.7	7.5 (7.2) 2.1	6.8 (6.6) 1.7
65	24.1 (18.3) 21	18.5 (15.9) 11.7	15.8 (14.2) 8.2	13.8 (12.6) 6.5	10 (9.5) 3.5	8.2 (7.9) 2.4	7.2 (6.9) 1.9	6.5 (6.3) 1.6
70	23.9 (18.6) 19.6	18.1 (15.7) 11.1	15.2 (13.8) 7.7	13.3 (12.3) 6.1	9.6 (9.1) 3.3	7.9 (7.5) 2.3	6.9 (6.6) 1.8	6.2 (6.1) 1.5
75	23.3 (18.5) 18.6	17.6 (15.4) 10.3	14.5 (13.2) 7.3	12.8 (11.8) 5.6	9.3 (8.8) 3.2	7.6 (7.3) 2.2	6.7 (6.4) 1.7	6 (5.8) 1.4
80	22.8 (18.3) 17.7	17.2 (15.2) 10	14.3 (13) 7	12.5 (11.6) 5.3	9 (8.6) 2.9	7.3 (7.1) 2	6.4 (6.2) 1.6	5.8 (5.6) 1.3
85	22.2 (18.1) 17	16.6 (14.6) 9.4	13.9 (12.7) 6.5	12.1 (11.2) 5.1	8.6 (8.3) 2.7	7.1 (6.8) 1.9	6.2 (6) 1.5	5.6 (5.4) 1.2
90	21.5 (17.7) 15.9	16.1 (14.3) 8.9	13.5 (12.3) 6.3	11.7 (11) 4.7	8.4 (8) 2.6	6.9 (6.7) 1.8	6 (5.9) 1.4	5.4 (5.3) 1.1
95	21.3 (17.6) 15.2	15.7 (14.1) 8.3	13.2 (12.1) 6	11.5 (10.7) 4.6	8.2 (7.8) 2.4	6.7 (6.5) 1.7	5.8 (5.7) 1.3	5.3 (5.1) 1.1
100	20.8 (17.3) 14.7	15.4 (13.9) 8.1	12.7 (11.8) 5.5	11.2 (10.4) 4.4	7.9 (7.6) 2.4	6.5 (6.3) 1.6	5.6 (5.5) 1.2	5.1 (5) 1

Table 39: Simulation 2 length results for the Bounded Bernstein method with $\mu = 5$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2822	0.3945	0.4808	0.5335	0.6448	0.6986	0.7392	0.7633
10	0.4056	0.5321	0.5984	0.6525	0.7463	0.7863	0.8092	0.8283
15	0.4781	0.5981	0.6670	0.7102	0.7914	0.8318	0.8552	0.8622
20	0.5350	0.6476	0.7054	0.7486	0.8060	0.8437	0.8653	0.8793
25	0.5677	0.6780	0.7445	0.7653	0.8389	0.8686	0.8836	0.8931
30	0.6141	0.7044	0.7652	0.7930	0.8538	0.8816	0.8858	0.8947
35	0.6309	0.7348	0.7736	0.8121	0.8638	0.8862	0.8928	0.9036
40	0.6530	0.7484	0.7916	0.8235	0.8750	0.8908	0.9084	0.9094
45	0.6719	0.7592	0.8042	0.8279	0.8786	0.8940	0.9096	0.9161
50	0.6918	0.7761	0.8203	0.8389	0.8845	0.8998	0.9074	0.9112
55	0.6970	0.7827	0.8302	0.8469	0.8870	0.9048	0.9131	0.9168
60	0.7127	0.7917	0.8265	0.8521	0.8892	0.9088	0.9145	0.9189
65	0.7225	0.8006	0.8363	0.8545	0.8905	0.9053	0.9184	0.9230
70	0.7379	0.8106	0.8381	0.8581	0.9018	0.9115	0.9189	0.9218
75	0.7443	0.8166	0.8553	0.8731	0.9017	0.9134	0.9201	0.9237
80	0.7530	0.8165	0.8561	0.8757	0.9072	0.9152	0.9245	0.9269
85	0.7541	0.8222	0.8561	0.8763	0.8973	0.9190	0.9259	0.9312
90	0.7606	0.8352	0.8651	0.8791	0.9103	0.9176	0.9276	0.9321
95	0.7691	0.8333	0.8661	0.8793	0.9098	0.9244	0.9260	0.9345
100	0.7739	0.8387	0.8578	0.8874	0.9160	0.9205	0.9239	0.9345

Table 40: Simulation 2 results for the Wald method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	67.9 (16.1) 138	45.6 (15) 76.4	39.1 (16.8) 58	34.4 (17.5) 46.7	27.5 (18.6) 29	24 (17.8) 21.7	22.3 (17.6) 17.8	20.6 (17) 14.8
10	43.8 (14) 73.9	34.5 (16.4) 48.1	29.4 (17.4) 34.5	27.5 (18.5) 28.6	22.2 (18) 16.9	19.2 (16.4) 12.3	17.3 (15.2) 10.1	15.9 (14.2) 8.6
15	36.6 (14.9) 55.9	29.7 (17.4) 35.1	26.1 (18.2) 25.9	24 (18.3) 21	19.3 (16.5) 12.3	16.6 (14.8) 9	14.8 (13.5) 7.1	13.5 (12.5) 5.9
20	33.8 (16.1) 47	26.4 (18.2) 27.2	24.1 (18.4) 20.8	21.9 (17.9) 16.4	16.9 (15) 9.5	14.7 (13.3) 7.2	13 (12.1) 5.5	11.8 (11.2) 4.6
25	30.5 (16.5) 38.9	25.4 (18.3) 23.9	22.7 (18.3) 17.7	20.2 (16.8) 14	15.9 (14.4) 8.2	13.4 (12.4) 5.7	11.8 (11.1) 4.4	10.7 (10.2) 3.7
30	29.8 (18) 35.3	24.5 (18.5) 21.4	21.4 (17.6) 15.6	19.3 (16.6) 12.4	14.7 (13.6) 6.9	12.4 (11.6) 4.9	10.8 (10.3) 3.8	9.9 (9.5) 3.2
35	28.5 (18) 32.1	23.3 (18.3) 18.8	20.1 (17) 13.5	18.2 (16) 10.7	13.6 (12.5) 6	11.5 (10.8) 4.2	10.1 (9.7) 3.3	9.2 (8.8) 2.8
40	27.3 (18.4) 28.3	22 (18) 16.4	19.1 (16.6) 12.1	17.3 (15.4) 9.6	13 (12.1) 5.4	10.8 (10.3) 3.8	9.5 (9.2) 2.9	8.6 (8.3) 2.4
45	26.5 (18.6) 26.2	21.1 (17.5) 15.2	18.4 (16.1) 11	16.4 (14.7) 8.6	12.3 (11.6) 4.8	10.2 (9.7) 3.4	9.1 (8.8) 2.6	8.2 (7.9) 2.2
50	25.7 (18.8) 24.2	20.3 (17.3) 13.7	17.7 (15.8) 10	15.6 (14.2) 7.8	11.8 (11.2) 4.3	9.8 (9.4) 3.1	8.6 (8.3) 2.4	7.8 (7.6) 2
55	24.7 (18.7) 22.1	19.8 (17) 13	17.2 (15.4) 9.3	15.2 (13.8) 7.3	11.3 (10.7) 3.9	9.4 (9) 2.8	8.2 (8) 2.2	7.5 (7.3) 1.8
60	24.2 (18.6) 20.5	19 (16.6) 11.8	16.4 (14.7) 8.7	14.6 (13.4) 6.8	10.8 (10.3) 3.7	9 (8.7) 2.6	7.9 (7.7) 2	7.1 (6.9) 1.7
65	23.6 (18.3) 19.7	18.5 (16.2) 11	15.8 (14.4) 8	14.1 (13) 6.3	10.4 (9.9) 3.4	8.7 (8.4) 2.4	7.6 (7.4) 1.9	6.9 (6.7) 1.5
70	23.4 (18.4) 19	18.1 (15.8) 10.7	15.4 (14.1) 7.5	13.6 (12.7) 5.8	10.1 (9.6) 3.2	8.4 (8.1) 2.2	7.3 (7.1) 1.7	6.6 (6.5) 1.4
75	22.5 (18.2) 17.4	17.7 (15.7) 10.1	15.1 (13.7) 7.1	13.3 (12.4) 5.5	9.8 (9.4) 3	8.1 (7.9) 2.1	7.1 (6.9) 1.7	6.4 (6.3) 1.3
80	22.5 (18.2) 17	17.3 (15.4) 9.8	14.6 (13.5) 6.7	13 (12.2) 5.3	9.5 (9.2) 2.8	7.9 (7.7) 2	6.9 (6.7) 1.5	6.2 (6.1) 1.3
85	21.3 (17.6) 15.5	16.7 (15) 8.9	14.2 (13.1) 6.4	12.5 (11.7) 4.9	9.2 (8.9) 2.7	7.7 (7.5) 1.9	6.7 (6.6) 1.5	6.1 (5.9) 1.2
90	21.1 (17.5) 15.1	16.3 (14.8) 8.5	14 (13) 6.1	12.2 (11.5) 4.7	9 (8.7) 2.5	7.4 (7.2) 1.8	6.6 (6.4) 1.4	5.9 (5.8) 1.1
95	20.8 (17.5) 14.5	16 (14.5) 8.2	13.6 (12.6) 5.8	12 (11.4) 4.5	8.8 (8.5) 2.5	7.3 (7.1) 1.7	6.3 (6.2) 1.3	5.7 (5.6) 1.1
100	20.5 (17.4) 14	15.7 (14.4) 7.8	13.1 (12.3) 5.5	11.7 (11.1) 4.2	8.6 (8.3) 2.3	7.1 (6.9) 1.6	6.2 (6.1) 1.2	5.6 (5.5) 1

Table 41: Simulation 2 length results for the Wald method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.1646	0.2641	0.3496	0.4153	0.5886	0.6902	0.7619	0.8127
10	0.2670	0.4074	0.5129	0.5809	0.7673	0.8551	0.9108	0.9352
15	0.3437	0.5036	0.6144	0.6932	0.8568	0.9289	0.9605	0.9809
20	0.4063	0.5967	0.6881	0.7711	0.9073	0.9608	0.9827	0.9913
25	0.4630	0.6443	0.7480	0.8214	0.9465	0.9808	0.9929	0.9980
30	0.5209	0.6872	0.8020	0.8594	0.9657	0.9889	0.9956	0.9990
35	0.5512	0.7298	0.8351	0.8931	0.9761	0.9934	0.9990	0.9996
40	0.5898	0.7786	0.8632	0.9111	0.9834	0.9968	0.9995	1.0000
45	0.6250	0.7979	0.8831	0.9355	0.9890	0.9980	0.9999	1.0000
50	0.6586	0.8251	0.9025	0.9466	0.9936	0.9989	0.9999	1.0000
55	0.6693	0.8453	0.9179	0.9562	0.9947	0.9993	1.0000	1.0000
60	0.6972	0.8686	0.9297	0.9667	0.9969	0.9999	0.9999	1.0000
65	0.7131	0.8826	0.9467	0.9707	0.9981	0.9999	0.9999	1.0000
70	0.7379	0.8904	0.9557	0.9793	0.9991	1.0000	1.0000	1.0000
75	0.7565	0.9012	0.9594	0.9823	0.9991	1.0000	1.0000	1.0000
80	0.7707	0.9113	0.9671	0.9867	0.9993	1.0000	1.0000	1.0000
85	0.7920	0.9256	0.9723	0.9886	0.9995	1.0000	1.0000	1.0000
90	0.7960	0.9288	0.9747	0.9921	0.9997	1.0000	1.0000	1.0000
95	0.8157	0.9383	0.9794	0.9917	1.0000	1.0000	1.0000	1.0000
100	0.8230	0.9469	0.9822	0.9935	0.9998	1.0000	1.0000	1.0000

Table 42: Simulation 2 results for the Chi-Square method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	16.9 (11.2) 16.6	15.2 (11.2) 12.9	15 (12.3) 11.3	14.9 (13) 10.2	15.3 (14.5) 8.1	15.7 (15.2) 7	16.2 (15.7) 6.2	16.4 (16.1) 5.6
10	14.6 (10.8) 12.9	14.7 (12.5) 10.6	14.9 (13.4) 9	15.3 (14.4) 8.2	16.2 (15.7) 6.1	16.5 (16.2) 5.1	16.6 (16.4) 4.5	16.8 (16.6) 4.1
15	14.2 (11.3) 11.5	14.9 (13.4) 9.2	15.3 (14.5) 7.8	15.7 (15.2) 6.9	16.5 (16.2) 5.1	16.8 (16.6) 4.2	16.9 (16.8) 3.7	17 (16.9) 3.3
20	14.4 (12.3) 10.6	15.1 (14.2) 8	15.7 (15.2) 6.9	16.1 (15.7) 6	16.5 (16.4) 4.4	16.8 (16.6) 3.7	16.9 (16.8) 3.2	17 (16.9) 2.9
25	14.4 (12.7) 9.8	15.5 (14.7) 7.5	16.1 (15.7) 6.3	16.2 (16) 5.5	16.8 (16.7) 4	16.9 (16.8) 3.3	17 (16.9) 2.8	17 (17) 2.6
30	14.9 (13.6) 9.2	15.8 (15.2) 7	16.3 (16) 5.8	16.5 (16.3) 5.1	16.9 (16.8) 3.7	17 (16.9) 3	17 (17) 2.6	17.1 (17) 2.4
35	15 (13.9) 8.7	16 (15.6) 6.5	16.3 (16.1) 5.4	16.6 (16.4) 4.7	16.9 (16.8) 3.4	17 (16.9) 2.8	17.1 (17) 2.4	17.1 (17) 2.2
40	15.2 (14.3) 8.3	16.1 (15.7) 6	16.4 (16.2) 5	16.7 (16.5) 4.4	17 (16.9) 3.2	17 (17) 2.6	17.1 (17) 2.2	17.1 (17.1) 2
45	15.4 (14.7) 7.8	16.2 (15.9) 5.8	16.6 (16.4) 4.8	16.7 (16.5) 4.2	17 (17) 3	17 (17) 2.4	17.1 (17.1) 2.1	17.1 (17.1) 1.9
50	15.6 (14.9) 7.4	16.3 (16) 5.4	16.7 (16.5) 4.6	16.7 (16.5) 3.9	17.1 (17) 2.8	17 (17) 2.3	17.1 (17.1) 2	17.1 (17.1) 1.8
55	15.6 (15.1) 7.2	16.4 (16.2) 5.3	16.7 (16.6) 4.3	16.9 (16.7) 3.8	17 (16.9) 2.7	17.1 (17.1) 2.2	17.1 (17) 1.9	17.2 (17.1) 1.7
60	15.8 (15.3) 6.9	16.4 (16.2) 5	16.7 (16.5) 4.2	16.9 (16.8) 3.6	17 (17) 2.6	17.1 (17.1) 2.1	17.1 (17.1) 1.9	17.2 (17.1) 1.7
65	15.9 (15.4) 6.7	16.5 (16.3) 4.8	16.8 (16.6) 4	16.9 (16.8) 3.5	17 (17) 2.5	17.1 (17.1) 2.1	17.2 (17.2) 1.8	17.1 (17.1) 1.6
70	16.1 (15.6) 6.5	16.6 (16.3) 4.7	16.8 (16.6) 3.8	16.9 (16.8) 3.4	17.1 (17) 2.4	17.1 (17.1) 2	17.1 (17.1) 1.7	17.1 (17.1) 1.5
75	16.1 (15.7) 6.2	16.7 (16.5) 4.6	16.9 (16.8) 3.7	17 (16.8) 3.2	17.1 (17) 2.3	17.1 (17.1) 1.9	17.1 (17.1) 1.7	17.2 (17.2) 1.5
80	16.2 (15.8) 6.1	16.7 (16.5) 4.4	16.9 (16.8) 3.6	17 (16.9) 3.1	17.1 (17.1) 2.2	17.2 (17.1) 1.9	17.2 (17.1) 1.6	17.2 (17.1) 1.4
85	16.1 (15.7) 5.8	16.6 (16.4) 4.2	16.9 (16.7) 3.5	16.9 (16.8) 3	17.1 (17.1) 2.2	17.2 (17.2) 1.8	17.2 (17.2) 1.5	17.2 (17.2) 1.4
90	16.2 (15.9) 5.7	16.7 (16.6) 4.2	17 (16.9) 3.4	17 (16.9) 2.9	17.1 (17.1) 2.1	17.1 (17.1) 1.7	17.2 (17.2) 1.5	17.2 (17.2) 1.3
95	16.3 (16) 5.6	16.7 (16.6) 4	16.9 (16.8) 3.3	17 (17) 2.9	17.1 (17.1) 2.1	17.2 (17.1) 1.7	17.2 (17.2) 1.5	17.2 (17.2) 1.3
100	16.4 (16) 5.5	16.8 (16.7) 3.9	16.9 (16.8) 3.2	17 (17) 2.8	17.1 (17.1) 2	17.1 (17.1) 1.6	17.2 (17.1) 1.4	17.2 (17.2) 1.3

Table 43: Simulation 2 length results for the Chi-Square method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2913	0.4099	0.4955	0.5511	0.6604	0.7092	0.7428	0.7596
10	0.4220	0.5562	0.6285	0.6759	0.7663	0.8005	0.8185	0.8303
15	0.4998	0.6272	0.6937	0.7333	0.8081	0.8443	0.8609	0.8639
20	0.5595	0.6795	0.7337	0.7717	0.8231	0.8554	0.8706	0.8782
25	0.5961	0.7096	0.7678	0.7875	0.8527	0.8770	0.8884	0.8918
30	0.6435	0.7285	0.7916	0.8180	0.8677	0.8858	0.8901	0.8952
35	0.6614	0.7631	0.7988	0.8322	0.8753	0.8925	0.8946	0.9039
40	0.6825	0.7762	0.8174	0.8424	0.8870	0.8955	0.9111	0.9103
45	0.7032	0.7884	0.8263	0.8474	0.8866	0.9010	0.9122	0.9150
50	0.7237	0.8055	0.8413	0.8582	0.8957	0.9033	0.9062	0.9099
55	0.7282	0.8096	0.8542	0.8656	0.8964	0.9111	0.9163	0.9137
60	0.7425	0.8189	0.8491	0.8702	0.8970	0.9134	0.9158	0.9171
65	0.7525	0.8279	0.8562	0.8702	0.8993	0.9082	0.9184	0.9206
70	0.7669	0.8341	0.8619	0.8746	0.9058	0.9133	0.9157	0.9209
75	0.7752	0.8368	0.8711	0.8873	0.9087	0.9160	0.9193	0.9222
80	0.7823	0.8394	0.8737	0.8873	0.9118	0.9148	0.9243	0.9235
85	0.7856	0.8477	0.8728	0.8896	0.9059	0.9198	0.9262	0.9288
90	0.7887	0.8537	0.8799	0.8952	0.9165	0.9200	0.9261	0.9285
95	0.7997	0.8561	0.8783	0.8918	0.9158	0.9257	0.9259	0.9317
100	0.8018	0.8592	0.8775	0.9014	0.9232	0.9232	0.9234	0.9311

Table 44: Simulation 2 results for the Gamma method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	63.3 (14.9) 129.4	42.5 (13.8) 71.9	36.5 (15.6) 54.7	32.1 (16.3) 44.1	25.7 (17.3) 27.6	22.5 (16.6) 20.8	20.9 (16.4) 17.1	19.2 (15.9) 14.4
10	41.1 (13) 69.6	32.5 (15.4) 45.5	27.9 (16.4) 32.8	26.1 (17.5) 27.2	21.1 (17.2) 16.3	18.3 (15.7) 12	16.5 (14.5) 9.8	15.1 (13.5) 8.4
15	34.5 (13.9) 52.8	28.2 (16.5) 33.4	24.9 (17.3) 24.7	23 (17.6) 20.1	18.6 (16) 12	16 (14.3) 8.8	14.3 (13) 7	12.9 (12) 5.9
20	32 (15.2) 44.5	25.2 (17.4) 26	23.1 (17.6) 19.9	21.1 (17.3) 15.8	16.4 (14.5) 9.3	14.2 (12.9) 7	12.5 (11.7) 5.5	11.4 (10.7) 4.5
25	29 (15.6) 37	24.4 (17.6) 22.9	21.9 (17.7) 17.1	19.5 (16.3) 13.5	15.4 (14) 8	13 (12) 5.7	11.4 (10.8) 4.4	10.4 (9.8) 3.7
30	28.4 (17.2) 33.6	23.5 (17.8) 20.6	20.7 (17.1) 15.1	18.8 (16.1) 12.1	14.4 (13.3) 6.8	12.1 (11.3) 4.8	10.5 (10) 3.8	9.6 (9.1) 3.2
35	27.2 (17.3) 30.6	22.5 (17.7) 18.1	19.5 (16.5) 13.1	17.7 (15.6) 10.4	13.3 (12.2) 5.9	11.2 (10.5) 4.2	9.9 (9.4) 3.3	8.9 (8.5) 2.8
40	26.1 (17.7) 27.1	21.3 (17.4) 15.8	18.6 (16.2) 11.7	16.9 (15) 9.4	12.7 (11.9) 5.3	10.6 (10) 3.7	9.3 (8.9) 2.9	8.4 (8.1) 2.4
45	25.4 (17.9) 25.1	20.5 (17.1) 14.7	17.9 (15.8) 10.7	16 (14.4) 8.4	12.1 (11.4) 4.8	10 (9.5) 3.3	8.8 (8.5) 2.6	7.9 (7.7) 2.2
50	24.7 (18.1) 23.2	19.7 (16.8) 13.3	17.3 (15.4) 9.8	15.3 (13.9) 7.6	11.6 (11) 4.3	9.6 (9.2) 3.1	8.4 (8.1) 2.4	7.5 (7.3) 2
55	23.8 (18) 21.2	19.2 (16.6) 12.6	16.8 (15.1) 9.1	14.9 (13.6) 7.2	11 (10.5) 3.9	9.2 (8.8) 2.8	8 (7.8) 2.2	7.2 (7) 1.8
60	23.3 (18) 19.7	18.5 (16.2) 11.5	16 (14.4) 8.5	14.3 (13.2) 6.7	10.6 (10.1) 3.6	8.8 (8.5) 2.5	7.7 (7.5) 2	6.9 (6.7) 1.7
65	22.8 (17.8) 18.9	18 (15.8) 10.7	15.5 (14.1) 7.8	13.8 (12.8) 6.2	10.2 (9.8) 3.4	8.5 (8.2) 2.4	7.5 (7.3) 1.9	6.7 (6.5) 1.6
70	22.6 (17.8) 18.3	17.7 (15.5) 10.4	15.1 (13.9) 7.3	13.4 (12.4) 5.7	9.9 (9.5) 3.2	8.2 (8) 2.2	7.2 (6.9) 1.7	6.4 (6.3) 1.4
75	21.8 (17.8) 16.8	17.3 (15.4) 9.8	14.8 (13.4) 7	13.1 (12.2) 5.4	9.6 (9.2) 3	8 (7.7) 2.1	6.9 (6.8) 1.7	6.2 (6.1) 1.4
80	21.8 (17.7) 16.4	17 (15.1) 9.5	14.4 (13.3) 6.6	12.8 (12) 5.2	9.4 (9) 2.8	7.7 (7.5) 2	6.7 (6.6) 1.5	6.1 (5.9) 1.3
85	20.7 (17.1) 15	16.3 (14.7) 8.7	14 (12.9) 6.3	12.3 (11.6) 4.8	9.1 (8.8) 2.7	7.5 (7.3) 1.9	6.6 (6.4) 1.5	5.9 (5.8) 1.2
90	20.4 (17.1) 14.6	16 (14.5) 8.4	13.8 (12.8) 6	12.1 (11.4) 4.6	8.8 (8.5) 2.5	7.3 (7.1) 1.8	6.4 (6.3) 1.4	5.7 (5.6) 1.1
95	20.3 (17.1) 14.1	15.7 (14.2) 8	13.4 (12.5) 5.7	11.8 (11.2) 4.4	8.7 (8.3) 2.5	7.1 (7) 1.7	6.2 (6.1) 1.3	5.6 (5.5) 1.1
100	20 (16.9) 13.6	15.4 (14.1) 7.6	12.9 (12.1) 5.4	11.5 (10.9) 4.2	8.4 (8.2) 2.3	7 (6.8) 1.6	6 (5.9) 1.3	5.5 (5.4) 1

Table 45: Simulation 2 length results for the Gamma method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2850	0.4307	0.5354	0.6053	0.7321	0.7760	0.8039	0.8177
10	0.4376	0.5922	0.6741	0.7195	0.7861	0.8048	0.8115	0.8160
15	0.5205	0.6605	0.7204	0.7522	0.7958	0.8148	0.8180	0.8031
20	0.5860	0.7045	0.7476	0.7722	0.7899	0.7932	0.7897	0.7740
25	0.6250	0.7209	0.7682	0.7770	0.7962	0.7880	0.7754	0.7544
30	0.6627	0.7368	0.7790	0.7869	0.7854	0.7753	0.7401	0.7220
35	0.6782	0.7591	0.7762	0.7901	0.7753	0.7481	0.7203	0.7020
40	0.6962	0.7666	0.7843	0.7933	0.7651	0.7293	0.7067	0.6812
45	0.7135	0.7678	0.7844	0.7829	0.7545	0.7109	0.6868	0.6638
50	0.7276	0.7753	0.7846	0.7786	0.7429	0.6934	0.6691	0.6244
55	0.7278	0.7767	0.7906	0.7741	0.7304	0.6872	0.6502	0.6186
60	0.7398	0.7775	0.7756	0.7689	0.7147	0.6635	0.6268	0.5878
65	0.7452	0.7771	0.7746	0.7619	0.6937	0.6432	0.6243	0.5867
70	0.7579	0.7754	0.7695	0.7526	0.6897	0.6380	0.6031	0.5732
75	0.7594	0.7749	0.7769	0.7573	0.6772	0.6325	0.5815	0.5638
80	0.7640	0.7775	0.7683	0.7470	0.6675	0.6135	0.5736	0.5384
85	0.7626	0.7747	0.7559	0.7408	0.6529	0.6007	0.5625	0.5282
90	0.7617	0.7722	0.7561	0.7339	0.6540	0.5900	0.5602	0.5269
95	0.7670	0.7682	0.7452	0.7197	0.6390	0.5749	0.5351	0.5197
100	0.7688	0.7683	0.7315	0.7205	0.6265	0.5721	0.5216	0.4983

Table 46: Simulation 2 results for the Unbounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	48.1 (34.6) 41.6	45.4 (30.4) 40.1	45.7 (33.3) 39	44.8 (33) 37.6	41.3 (31) 33.4	36.6 (27.2) 29.6	33.7 (25.3) 26.8	30.6 (23.5) 24.2
10	43 (26.5) 40.2	42.1 (28.3) 37.4	39.9 (28.1) 34.5	38 (27.5) 32.2	29.4 (22.7) 23.5	23.3 (18.7) 17.6	19.7 (16.2) 14.1	17 (14.2) 11.6
15	40.7 (25.9) 38.3	38.7 (27.1) 34.1	35.1 (25.5) 30.1	31.9 (23.7) 26.7	22.3 (17.8) 16.7	17.1 (14.3) 11.6	14.1 (12.1) 8.6	11.9 (10.5) 6.9
20	40.2 (26.7) 36.5	35.1 (25.3) 30.2	31.3 (23.3) 26.4	27 (20.9) 21.7	17.4 (14.3) 12.1	13.3 (11.4) 8.3	10.8 (9.4) 5.9	9.1 (8.2) 4.6
25	37.8 (25.1) 34.9	33 (23.6) 28.4	28 (21.3) 22.9	23.2 (18.1) 18.3	14.9 (12.5) 9.7	11 (9.5) 6	8.8 (7.9) 4.3	7.5 (6.8) 3.5
30	37.5 (25.7) 33.6	30.8 (22.7) 26.3	24.9 (19.1) 20	20.9 (16.6) 15.9	12.6 (10.9) 7.5	9.3 (8.3) 4.8	7.4 (6.7) 3.4	6.3 (5.8) 2.7
35	35.9 (25.1) 32.1	28.4 (21.4) 23.6	22.3 (17.6) 17.3	18.5 (15.1) 13.1	10.9 (9.5) 6.2	8 (7.2) 3.9	6.4 (5.9) 2.7	5.4 (5) 2.2
40	34.8 (24.7) 30.7	26 (20) 21	20.2 (16.3) 15.2	16.7 (13.7) 11.4	9.8 (8.7) 5.3	7.1 (6.4) 3.3	5.7 (5.2) 2.3	4.8 (4.4) 1.8
45	33.5 (24.1) 29.3	23.9 (18.5) 19.3	18.5 (15.1) 13.3	15 (12.6) 9.9	8.8 (7.9) 4.5	6.3 (5.8) 2.8	5.1 (4.7) 2	4.3 (4) 1.5
50	32.2 (23.6) 27.7	22.3 (17.6) 17.2	17.1 (14.2) 11.8	13.6 (11.6) 8.6	8 (7.2) 3.9	5.8 (5.3) 2.4	4.6 (4.3) 1.7	3.8 (3.6) 1.3
55	30.7 (22.6) 26.3	21 (16.8) 16.2	16 (13.4) 10.7	12.7 (10.9) 7.8	7.3 (6.6) 3.3	5.3 (4.9) 2.1	4.2 (3.9) 1.5	3.5 (3.3) 1.1
60	29.6 (21.9) 25.2	19.6 (15.8) 14.5	14.7 (12.2) 9.9	11.8 (10.1) 7	6.7 (6.1) 3.1	4.8 (4.5) 1.8	3.9 (3.6) 1.3	3.2 (3.1) 1
65	28.4 (21) 24.1	18.4 (15.1) 13.2	13.7 (11.5) 8.7	10.9 (9.5) 6.2	6.2 (5.7) 2.7	4.5 (4.2) 1.7	3.6 (3.4) 1.2	3 (2.9) 0.9
70	27.6 (20.6) 23.2	17.6 (14.3) 12.8	12.9 (11.1) 8	10.2 (9) 5.6	5.9 (5.4) 2.5	4.2 (3.9) 1.5	3.3 (3.1) 1.1	2.8 (2.7) 0.8
75	26.2 (20.1) 21.8	16.7 (13.7) 11.6	12.3 (10.4) 7.6	9.7 (8.5) 5.2	5.5 (5) 2.2	3.9 (3.7) 1.4	3.1 (3) 1	2.6 (2.5) 0.7
80	25.7 (19.6) 21	16 (13.1) 11.4	11.6 (10) 6.8	9.2 (8.1) 4.9	5.2 (4.8) 2.1	3.7 (3.5) 1.3	2.9 (2.8) 0.8	2.5 (2.4) 0.7
85	23.9 (18.4) 19.2	15 (12.6) 10.1	11 (9.5) 6.4	8.6 (7.7) 4.4	4.9 (4.5) 1.9	3.5 (3.3) 1.1	2.8 (2.6) 0.8	2.3 (2.2) 0.6
90	23.2 (18) 18.7	14.2 (12) 9.4	10.5 (9.1) 6	8.2 (7.3) 4.2	4.6 (4.3) 1.7	3.3 (3.1) 1.1	2.6 (2.5) 0.7	2.2 (2.1) 0.5
95	22.7 (17.8) 18.1	13.7 (11.6) 8.9	9.9 (8.7) 5.5	7.8 (7) 3.9	4.4 (4.1) 1.7	3.1 (3) 1	2.5 (2.4) 0.7	2.1 (2) 0.5
100	21.9 (17.1) 17.3	13.2 (11.2) 8.3	9.3 (8.2) 5.1	7.4 (6.7) 3.6	4.2 (3.9) 1.5	3 (2.8) 0.9	2.3 (2.2) 0.6	2 (1.9) 0.5

Table 47: Simulation 2 length results for the Unbounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.3366	0.4825	0.5780	0.6424	0.7789	0.8424	0.8850	0.9083
10	0.4834	0.6341	0.7238	0.7736	0.8767	0.9149	0.9381	0.9472
15	0.5654	0.7145	0.7878	0.8262	0.9121	0.9443	0.9557	0.9659
20	0.6279	0.7695	0.8284	0.8711	0.9276	0.9511	0.9631	0.9735
25	0.6691	0.8020	0.8573	0.8872	0.9479	0.9639	0.9739	0.9787
30	0.7193	0.8238	0.8853	0.9060	0.9560	0.9707	0.9771	0.9807
35	0.7414	0.8506	0.8899	0.9252	0.9598	0.9721	0.9795	0.9846
40	0.7651	0.8709	0.9050	0.9304	0.9679	0.9762	0.9852	0.9878
45	0.7898	0.8746	0.9165	0.9382	0.9668	0.9816	0.9835	0.9889
50	0.8125	0.8890	0.9277	0.9434	0.9713	0.9837	0.9850	0.9885
55	0.8114	0.8958	0.9357	0.9515	0.9768	0.9827	0.9881	0.9888
60	0.8265	0.9051	0.9362	0.9509	0.9783	0.9826	0.9886	0.9898
65	0.8375	0.9152	0.9448	0.9527	0.9770	0.9861	0.9907	0.9921
70	0.8528	0.9163	0.9446	0.9576	0.9807	0.9856	0.9888	0.9925
75	0.8583	0.9231	0.9505	0.9625	0.9825	0.9858	0.9882	0.9902
80	0.8682	0.9245	0.9533	0.9671	0.9818	0.9884	0.9913	0.9925
85	0.8718	0.9317	0.9559	0.9653	0.9829	0.9876	0.9911	0.9939
90	0.8753	0.9366	0.9572	0.9708	0.9842	0.9885	0.9913	0.9926
95	0.8850	0.9360	0.9573	0.9695	0.9855	0.9890	0.9920	0.9942
100	0.8885	0.9394	0.9590	0.9720	0.9847	0.9891	0.9903	0.9940

Table 48: Simulation 2 results for the Bounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. These coverage probabilities include the proportion of data sets that contain all zeros.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	158.3 (37.8) 322	106.5 (34.7) 178.4	91.5 (39.5) 135.7	80.5 (41.3) 109.4	64.9 (43.7) 68.4	57.1 (42.6) 51.3	53.6 (42.5) 42.3	49.8 (41.5) 35.3
10	97.5 (30.7) 164.9	76.3 (36) 107.1	65 (38.4) 76.8	60.6 (40.4) 63.5	48.5 (39.2) 37.7	42 (35.7) 27.4	37.9 (33.1) 22.5	34.9 (30.9) 19.1
15	79.8 (32.3) 122.6	64.2 (37.5) 76.6	56.2 (38.8) 56.3	51.5 (39.2) 45.6	40.8 (34.6) 26.8	34.9 (30.9) 19.4	31.1 (28.2) 15.5	28.3 (26) 13
20	72.9 (34.8) 102.3	56.3 (38.4) 58.8	50.9 (38.4) 44.7	45.9 (37.2) 35.1	34.9 (30.6) 20.4	30 (27) 15.4	26.5 (24.5) 11.8	24.1 (22.5) 9.8
25	65.2 (34.9) 83.8	53.6 (37.9) 51.3	47.4 (37.7) 37.7	41.8 (34.5) 29.7	32.3 (28.7) 17.4	27 (24.7) 12.1	23.6 (22) 9.3	21.5 (20.2) 7.9
30	63.3 (37.8) 75.8	51.1 (38.2) 45.4	44.2 (35.8) 33	39.6 (33.5) 26.3	29.5 (26.9) 14.5	24.6 (22.8) 10.3	21.3 (20.1) 7.9	19.4 (18.4) 6.8
35	60.1 (38) 68.7	48.3 (37.4) 39.8	41.1 (34.2) 28.5	36.8 (32) 22.4	27 (24.5) 12.6	22.4 (21) 8.8	19.7 (18.7) 6.8	17.7 (16.8) 5.8
40	57.2 (38.1) 60.2	45.2 (36.5) 34.4	38.8 (33.2) 25.3	34.8 (30.5) 20.2	25.6 (23.5) 11.2	21 (19.7) 7.8	18.4 (17.4) 6	16.5 (15.8) 5.1
45	55.3 (38.4) 55.5	43.1 (35.2) 32	37 (32.1) 22.9	32.7 (29.1) 18	24 (22.3) 10	19.7 (18.5) 7	17.3 (16.6) 5.4	15.5 (14.9) 4.4
50	53.4 (38.8) 51.1	41.3 (34.6) 28.7	35.5 (31.1) 20.8	30.9 (27.8) 16.1	22.8 (21.4) 8.9	18.6 (17.6) 6.4	16.2 (15.5) 4.9	14.6 (14) 4
55	51 (38.3) 46.5	40 (33.9) 27	34.2 (30.4) 19.3	29.9 (26.9) 15.2	21.6 (20.2) 8.1	17.7 (16.8) 5.7	15.4 (14.8) 4.4	13.9 (13.4) 3.7
60	49.7 (37.8) 43.1	38.3 (32.8) 24.5	32.5 (28.7) 18.1	28.6 (26) 14	20.6 (19.4) 7.5	16.9 (16.1) 5.2	14.7 (14.2) 4.1	13.2 (12.7) 3.5
65	48.4 (37) 41.2	37 (32.1) 22.9	31.2 (27.9) 16.5	27.4 (25) 12.9	19.7 (18.6) 7	16.2 (15.5) 4.9	14.1 (13.6) 3.8	12.7 (12.3) 3.1
70	47.8 (37.1) 39.8	36.1 (31.1) 22.2	30.2 (27.3) 15.4	26.3 (24.2) 11.9	19 (18) 6.6	15.5 (14.9) 4.5	13.5 (12.9) 3.5	12.1 (11.8) 2.9
75	46 (36.8) 36.6	35.1 (30.8) 20.8	29.4 (26.4) 14.7	25.7 (23.7) 11.3	18.3 (17.4) 6.1	15 (14.4) 4.3	13 (12.6) 3.4	11.7 (11.4) 2.7
80	45.7 (36.6) 35.4	34.4 (29.9) 20.2	28.5 (25.9) 13.8	25 (23.1) 10.8	17.8 (17) 5.8	14.5 (14) 4	12.5 (12.2) 3	11.3 (11) 2.5
85	43.2 (35.2) 32.3	32.9 (29.3) 18.4	27.6 (25.2) 13.1	24 (22.2) 10	17.2 (16.4) 5.5	14.1 (13.5) 3.8	12.2 (11.8) 2.9	10.9 (10.6) 2.4
90	42.6 (35) 31.5	32.1 (28.6) 17.6	27 (24.8) 12.5	23.3 (21.7) 9.6	16.6 (16) 5.1	13.6 (13.1) 3.6	11.8 (11.5) 2.8	10.5 (10.3) 2.2
95	42 (35) 30.3	31.3 (28.1) 16.8	26.1 (24) 11.8	22.8 (21.4) 9.2	16.2 (15.4) 5	13.2 (12.8) 3.3	11.4 (11.1) 2.6	10.2 (10) 2.1
100	41.2 (34.2) 29.1	30.8 (27.7) 16	25.1 (23.2) 11.2	22.1 (20.7) 8.6	15.7 (15.1) 4.6	12.8 (12.4) 3.2	11.1 (10.8) 2.5	10 (9.7) 2

Table 49: Simulation 2 length results for the Bounded Bernstein method with $\mu = 10$ with coverage probabilities estimated from 10,000 simulated data sets at the corresponding values of n and θ. All values are of the form $\bar{X}(m)SD$, where $\bar{X}$ is the average length, m is the median length, and SD is the standard deviation of length. For reference, these statistics are computed only for confidence intervals generated from non-zero data sets.

n/θ	0.1	1	10	10000
10	0.0174	0.0000	0.0000	0.0000
20	0.0006	0.0000	0.0000	0.0000
30	0.0000	0.0000	0.0000	0.0000
40	0.0000	0.0000	0.0000	0.0000
50	0.0000	0.0000	0.0000	0.0000
60	0.0000	0.0000	0.0000	0.0000
70	0.0000	0.0000	0.0000	0.0000
80	0.0000	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000	0.0000

Table 50: Proportion of all–zero data sets in Simulation 1 with $\mu = 5$.

n/θ	0.1	1	10	10000
10	0.0100	0.0000	0.0000	0.0000
20	0.0000	0.0000	0.0000	0.0000
30	0.0000	0.0000	0.0000	0.0000
40	0.0000	0.0000	0.0000	0.0000
50	0.0000	0.0000	0.0000	0.0000
60	0.0000	0.0000	0.0000	0.0000
70	0.0000	0.0000	0.0000	0.0000
80	0.0000	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000	0.0000

Table 51: Proportion of all–zero data sets in Simulation 1 with $\mu = 10$.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.5144	0.3165	0.2032	0.1438	0.0380	0.0115	0.0053	0.0026
10	0.2663	0.0983	0.0437	0.0198	0.0017	0.0001	0.0000	0.0000
15	0.1381	0.0320	0.0084	0.0027	0.0000	0.0000	0.0000	0.0000
20	0.0694	0.0110	0.0021	0.0005	0.0000	0.0000	0.0000	0.0000
25	0.0347	0.0036	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000
30	0.0182	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
35	0.0108	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
40	0.0047	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
45	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
50	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
55	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
65	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
75	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 52: Proportion of all–zero data sets in Simulation 2 with $\mu = 5$.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.4749	0.2651	0.1572	0.0987	0.0191	0.0049	0.0015	0.0003
10	0.2165	0.0727	0.0236	0.0096	0.0005	0.0000	0.0000	0.0000
15	0.1051	0.0191	0.0043	0.0010	0.0000	0.0000	0.0000	0.0000
20	0.0506	0.0059	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000
25	0.0241	0.0011	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
30	0.0110	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
35	0.0055	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
40	0.0018	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
45	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
50	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
55	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 53: Proportion of all–zero data sets in Simulation 2 with $\mu = 10$.

n/θ	0.1	1	10	10000
10	0.0104	0.0060	0.2634	0.5542
20	0.0018	0.0000	0.1534	0.5374
30	0.0001	0.0000	0.0978	0.5382
40	0.0000	0.0000	0.0606	0.5330
50	0.0000	0.0000	0.0393	0.5233
60	0.0000	0.0000	0.0228	0.5160
70	0.0000	0.0000	0.0173	0.5209
80	0.0000	0.0000	0.0092	0.5243
90	0.0000	0.0000	0.0096	0.5263
100	0.0000	0.0000	0.0052	0.5279

Table 54: Simulation 1 percentages of θ estimates below zero produced by the Method of Moments.

n/θ	0.1	1	10	10000
10	2.3e+12 (7.2e+13)	2.1e+12 (2.1e+14)	-1.8e+12 (1.5e+15)	2e+13 (1.4e+15)
20	-4.5e+10 (2.3e+12)	1.4 (0.86)	3.2e+12 (3.9e+14)	5.4e+12 (5.3e+14)
30	0.17 (0.079)	1.2 (0.53)	8.3e+12 (2.9e+14)	9.6e+12 (3.7e+14)
40	0.15 (0.064)	1.2 (0.43)	-3.6e+11 (3.6e+13)	-9.9 (710)
50	0.14 (0.056)	1.1 (0.36)	2e+12 (2e+14)	7.3e+11 (5.3e+13)
60	0.14 (0.051)	1.1 (0.33)	20 (300)	-2.3e+12 (1.2e+14)
70	0.13 (0.048)	1.1 (0.3)	14 (120)	4.1e+12 (1.9e+14)
80	0.13 (0.044)	1.1 (0.28)	16 (110)	1.6e+12 (2.2e+14)
90	0.13 (0.042)	1.1 (0.26)	13 (210)	1.3e+12 (2.3e+14)
100	0.12 (0.04)	1.1 (0.25)	8.3 (390)	-33 (2100)

Table 55: Simulation 1 summary statistics for the Method of Moments estimator of θ. All values are of the form $Ave(SD)$, where Ave is the average value and SD is the standard deviation of the θ estimates.

n/θ	0.1	1	10	10000
10	0.2312	1.3151	5.4878	-8.0003
20	0.1746	1.1883	8.0431	-10.0066
30	0.1523	1.1283	8.9494	-11.9916
40	0.1430	1.0908	9.6184	-13.4307
50	0.1352	1.0783	9.9105	-13.9219
60	0.1299	1.0773	10.1617	-14.4553
70	0.1280	1.0605	10.1439	-16.0855
80	0.1246	1.0536	10.1272	-17.6132
90	0.1216	1.0460	10.1659	-18.6565
100	0.1208	1.0477	10.3327	-20.6484

Table 56: Simulation 1 median values for the Method of Moments estimator of θ.

n/θ	0.1	1	10	10000
10	0.0047	0.0003	0.1293	0.5609
20	0.0005	0.0000	0.0356	0.5569
30	0.0001	0.0000	0.0125	0.5324
40	0.0000	0.0000	0.0043	0.5366
50	0.0000	0.0000	0.0022	0.5278
60	0.0000	0.0000	0.0002	0.5254
70	0.0000	0.0000	0.0006	0.5199
80	0.0000	0.0000	0.0001	0.5186
90	0.0000	0.0000	0.0000	0.5168
100	0.0000	0.0000	0.0000	0.5152

Table 57: Simulation 1 percentages of θ estimates below zero produced by the Method of Moments.

n/θ	0.1	1	10	10000
10	6.9e+11 (3.9e+13)	1.6 (2.8)	3.8e+13 (1.1e+15)	7.9e+13 (1.7e+15)
20	0.19 (0.14)	1.3 (0.59)	-1.6e+12 (1.3e+14)	-2.3e+13 (1.3e+15)
30	0.16 (0.098)	1.2 (0.45)	4.8e+12 (3.1e+14)	2.2e+13 (7.1e+14)
40	0.15 (0.061)	1.1 (0.37)	13 (92)	1.8e+13 (9.4e+14)
50	0.14 (0.054)	1.1 (0.32)	16 (350)	-1.1e+13 (8e+14)
60	0.14 (0.049)	1.1 (0.29)	13 (56)	-1.9e+12 (3.6e+14)
70	0.13 (0.047)	1.1 (0.26)	11 (32)	9.8e+12 (7.4e+14)
80	0.13 (0.044)	1.1 (0.25)	12 (20)	4.5e+12 (4.5e+14)
90	0.13 (0.041)	1.1 (0.23)	11 (5)	-5.1e+12 (4.4e+14)
100	0.12 (0.039)	1.1 (0.22)	11 (4.3)	3e+11 (4e+14)

Table 58: Simulation 1 summary statistics for the Method of Moments estimator of θ. All values are of the form $Ave(SD)$, where Ave is the average value and SD is the standard deviation of the θ estimates.

n/θ	0.1	1	10	10000
10	0.2273	1.2747	8.8711	−16.5086
20	0.1724	1.1514	10.2803	−21.7824
30	0.1522	1.1133	10.5649	−23.3937
40	0.1408	1.0839	10.4243	−27.3827
50	0.1348	1.0675	10.2670	−28.2496
60	0.1301	1.0586	10.3075	−30.4283
70	0.1260	1.0565	10.1856	−31.8551
80	0.1234	1.0463	10.2378	−33.5901
90	0.1227	1.0446	10.2806	−34.6954
100	0.1208	1.0359	10.1383	−35.5266

Table 59: Simulation 1 median values for the Method of Moments estimator of θ.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.0058	0.0149	0.0169	0.0202	0.0272	0.0297	0.0337	0.0411
10	0.0105	0.0138	0.0150	0.0119	0.0057	0.0025	0.0023	0.0024
15	0.0108	0.0092	0.0079	0.0043	0.0010	0.0001	0.0002	0.0001
20	0.0513	0.0128	0.0053	0.0027	0.0000	0.0000	0.0000	0.0000
25	0.0085	0.0032	0.0010	0.0004	0.0000	0.0000	0.0000	0.0000
30	0.0066	0.0021	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
35	0.0133	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
40	0.0038	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
45	0.0030	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
50	0.0012	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
55	0.0011	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
65	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
70	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
75	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
80	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 60: Simulation 2 percentages of θ estimates below zero produced by the Method of Moments.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	1.2e14 (3.1e14)	1.1e14 (2.9e14)	7.9e13 (2.6e14)	6.9e13 (2.8e14)	2.9e13 (3.6e14)	9e12 (2.9e14)	-1.2e13 (4.5e14)	-2e13 (5.6e14)
10	3.3e11 (2.7e13)	5.3e11 (3.4e13)	1.2e12 (5.2e13)	2.3e12 (7.3e13)	2.7e12 (7.8e13)	2e12 (6.8e13)	-1.4e11 (4.2e13)	2.3e11 (2.3e13)
15	1.7e13 (6.9e13)	7.2e12 (4.6e13)	3.1e12 (3e13)	1.3e12 (1.9e13)	3e10 (3e12)	0.5 (0.36)	0.63 (0.59)	0.77 (0.6)
20	-4.3e12 (2.2e13)	-8.5e11 (9.8e12)	-1.8e11 (4.5e12)	-1.1e11 (3.6e12)	0.32 (0.17)	0.45 (0.45)	0.57 (0.56)	0.7 (0.42)
25	4.1e12 (2.7e13)	7.6e11 (1.2e13)	7.2e10 (3.6e12)	1.8e10 (1.8e12)	0.3 (0.14)	0.42 (0.19)	0.54 (0.24)	0.65 (0.29)
30	1.9e12 (1.7e13)	2.1e11 (5.6e12)	0.14 (0.067)	0.17 (0.085)	0.28 (0.12)	0.4 (0.17)	0.51 (0.21)	0.63 (0.25)
35	-2.3e12 (2.4e13)	-1.3e11 (5.8e12)	0.13 (0.061)	0.16 (0.071)	0.27 (0.11)	0.39 (0.15)	0.5 (0.19)	0.61 (0.24)
40	0.056 (0.12)	0.091 (0.063)	0.12 (0.064)	0.15 (0.064)	0.26 (0.1)	0.38 (0.14)	0.49 (0.18)	0.6 (0.21)
45	0.053 (0.11)	0.089 (0.043)	0.12 (0.05)	0.15 (0.06)	0.26 (0.095)	0.37 (0.13)	0.48 (0.17)	0.59 (0.2)
50	0.054 (0.069)	0.085 (0.042)	0.11 (0.046)	0.14 (0.057)	0.25 (0.091)	0.36 (0.12)	0.47 (0.16)	0.58 (0.19)
55	9e10 (2.7e12)	0.083 (0.038)	0.11 (0.045)	0.14 (0.054)	0.25 (0.087)	0.36 (0.12)	0.46 (0.14)	0.57 (0.18)
60	4.5e10 (1.8e12)	0.081 (0.037)	0.11 (0.043)	0.14 (0.051)	0.25 (0.082)	0.35 (0.11)	0.46 (0.14)	0.57 (0.17)
65	-5.5e10 (2.8e12)	0.079 (0.032)	0.11 (0.041)	0.13 (0.05)	0.24 (0.079)	0.35 (0.11)	0.45 (0.13)	0.56 (0.16)
70	1.9e10 (1.1e12)	0.076 (0.033)	0.1 (0.04)	0.13 (0.048)	0.24 (0.077)	0.35 (0.11)	0.46 (0.13)	0.56 (0.16)
75	1.2e10 (8.5e11)	0.075 (0.03)	0.1 (0.038)	0.13 (0.046)	0.24 (0.073)	0.34 (0.1)	0.45 (0.13)	0.55 (0.15)
80	-1.1e10 (1.1e12)	0.073 (0.028)	0.1 (0.037)	0.13 (0.045)	0.24 (0.071)	0.34 (0.098)	0.45 (0.12)	0.55 (0.15)
85	0.044 (0.019)	0.073 (0.028)	0.1 (0.036)	0.13 (0.043)	0.23 (0.07)	0.34 (0.094)	0.44 (0.12)	0.55 (0.14)
90	0.043 (0.018)	0.072 (0.027)	0.1 (0.036)	0.13 (0.042)	0.23 (0.067)	0.34 (0.091)	0.44 (0.11)	0.55 (0.14)
95	0.043 (0.018)	0.071 (0.027)	0.098 (0.034)	0.12 (0.041)	0.23 (0.065)	0.33 (0.089)	0.44 (0.11)	0.54 (0.13)
100	0.042 (0.017)	0.07 (0.025)	0.097 (0.033)	0.12 (0.04)	0.23 (0.064)	0.33 (0.085)	0.44 (0.11)	0.54 (0.13)

Table 61: Simulation 2 summary statistics for the Method of Moments estimator of θ. All values are of the form $Ave(SD)$, where Ave is the average value and SD is the standard deviation of the θ estimates.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2400	0.2739	0.3000	0.3380	0.4840	0.6079	0.7200	0.8196
10	0.1250	0.1582	0.2000	0.2255	0.3591	0.4813	0.5958	0.7212
15	0.0996	0.1343	0.1607	0.1956	0.3102	0.4273	0.5428	0.6541
20	0.0767	0.1127	0.1425	0.1723	0.2909	0.4049	0.5141	0.6301
25	0.0733	0.1030	0.1331	0.1609	0.2734	0.3840	0.4961	0.5992
30	0.0660	0.0954	0.1243	0.1535	0.2647	0.3756	0.4814	0.5874
35	0.0598	0.0901	0.1188	0.1474	0.2587	0.3652	0.4736	0.5753
40	0.0556	0.0856	0.1143	0.1443	0.2509	0.3600	0.4650	0.5709
45	0.0529	0.0827	0.1114	0.1384	0.2460	0.3556	0.4583	0.5625
50	0.0503	0.0799	0.1083	0.1364	0.2431	0.3501	0.4530	0.5588
55	0.0486	0.0783	0.1054	0.1315	0.2413	0.3453	0.4481	0.5552
60	0.0468	0.0763	0.1033	0.1305	0.2368	0.3436	0.4482	0.5511
65	0.0464	0.0751	0.1015	0.1287	0.2325	0.3404	0.4434	0.5466
70	0.0446	0.0725	0.1004	0.1266	0.2327	0.3408	0.4448	0.5451
75	0.0433	0.0720	0.0991	0.1274	0.2318	0.3330	0.4372	0.5381
80	0.0425	0.0697	0.0974	0.1242	0.2297	0.3343	0.4374	0.5413
85	0.0416	0.0702	0.0965	0.1237	0.2285	0.3311	0.4337	0.5371
90	0.0407	0.0690	0.0965	0.1231	0.2265	0.3277	0.4305	0.5348
95	0.0401	0.0678	0.0952	0.1213	0.2264	0.3279	0.4309	0.5311
100	0.0396	0.0674	0.0941	0.1206	0.2255	0.3293	0.4307	0.5342

Table 62: Simulation 2 median values for the Method of Moments estimator of θ.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.0086	0.0091	0.0122	0.0135	0.0163	0.0134	0.0137	0.0142
10	0.0087	0.0104	0.0078	0.0066	0.0017	0.0004	0.0004	0.0004
15	0.0076	0.0063	0.0029	0.0017	0.0002	0.0001	0.0000	0.0000
20	0.0299	0.0058	0.0029	0.0006	0.0000	0.0000	0.0000	0.0000
25	0.0057	0.0015	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
30	0.0032	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
35	0.0082	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
40	0.0030	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
45	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
50	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
55	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
60	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
65	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
70	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
75	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
95	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 63: Simulation 2 percentages of θ estimates below zero produced by the Method of Moments.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	1e14 (2.8e14)	8.1e13 (2.6e14)	6.4e13 (2.3e14)	5e13 (2.1e14)	1.4e13 (1.9e14)	-2.2e12 (3.2e14)	-2.5e12 (2.1e14)	-3.9e12 (2.2e14)
10	0.14 (0.22)	2.5e11 (2.4e13)	1.4e12 (5.6e13)	1.1e12 (5.1e13)	6.8e11 (3.9e13)	2.3e11 (2.3e13)	2.3e11 (2.3e13)	0.85 (1.1)
15	1.3e13 (6.1e13)	4.2e12 (3.5e13)	1.3e12 (2e13)	4.2e11 (1.1e13)	0.35 (0.19)	0.47 (0.31)	0.59 (0.31)	0.73 (0.38)
20	-2.5e12 (1.7e13)	-3.3e11 (6.1e12)	-1.2e11 (3.7e12)	-3.4e10 (2e12)	0.31 (0.15)	0.43 (0.2)	0.55 (0.25)	0.67 (0.31)
25	2.5e12 (2.1e13)	4e11 (8.4e12)	0.14 (0.075)	0.18 (0.084)	0.29 (0.13)	0.41 (0.17)	0.52 (0.21)	0.64 (0.26)
30	1.4e12 (1.4e13)	3e10 (2.1e12)	1.5e10 (1.5e12)	0.16 (0.072)	0.28 (0.11)	0.39 (0.16)	0.51 (0.19)	0.62 (0.23)
35	-1.7e12 (2.1e13)	-2.6e10 (2.6e12)	0.13 (0.055)	0.16 (0.066)	0.27 (0.11)	0.39 (0.14)	0.49 (0.18)	0.6 (0.21)
40	0.056 (0.11)	0.091 (0.043)	0.12 (0.051)	0.15 (0.061)	0.26 (0.098)	0.37 (0.13)	0.48 (0.16)	0.59 (0.2)
45	0.055 (0.073)	0.088 (0.038)	0.12 (0.048)	0.14 (0.058)	0.26 (0.093)	0.37 (0.12)	0.47 (0.16)	0.58 (0.18)
50	0.054 (0.046)	0.085 (0.036)	0.11 (0.045)	0.14 (0.055)	0.25 (0.087)	0.36 (0.12)	0.47 (0.15)	0.57 (0.17)
55	5.7e10 (2.2e12)	0.082 (0.035)	0.11 (0.044)	0.14 (0.053)	0.25 (0.082)	0.36 (0.11)	0.46 (0.14)	0.57 (0.17)
60	7.5e09 (7.5e11)	0.08 (0.033)	0.11 (0.042)	0.14 (0.05)	0.24 (0.08)	0.35 (0.11)	0.46 (0.13)	0.56 (0.16)
65	-1.4e10 (1.4e12)	0.078 (0.031)	0.11 (0.04)	0.13 (0.048)	0.24 (0.077)	0.35 (0.1)	0.45 (0.13)	0.55 (0.15)
70	0.047 (0.026)	0.077 (0.031)	0.1 (0.038)	0.13 (0.046)	0.24 (0.074)	0.34 (0.099)	0.45 (0.13)	0.55 (0.15)
75	0.046 (0.024)	0.075 (0.029)	0.1 (0.038)	0.13 (0.044)	0.24 (0.072)	0.34 (0.096)	0.45 (0.12)	0.55 (0.14)
80	0.045 (0.019)	0.073 (0.028)	0.1 (0.036)	0.13 (0.044)	0.23 (0.07)	0.34 (0.093)	0.44 (0.11)	0.55 (0.14)
85	0.044 (0.018)	0.072 (0.028)	0.1 (0.036)	0.13 (0.042)	0.23 (0.067)	0.34 (0.09)	0.44 (0.11)	0.54 (0.13)
90	0.043 (0.018)	0.072 (0.027)	0.099 (0.034)	0.13 (0.041)	0.23 (0.065)	0.34 (0.088)	0.44 (0.11)	0.54 (0.13)
95	0.042 (0.018)	0.07 (0.026)	0.098 (0.034)	0.12 (0.041)	0.23 (0.065)	0.33 (0.085)	0.44 (0.11)	0.54 (0.13)
100	0.042 (0.017)	0.069 (0.025)	0.097 (0.033)	0.12 (0.039)	0.23 (0.063)	0.33 (0.083)	0.44 (0.1)	0.54 (0.12)

Table 64: Simulation 2 summary statistics for the Method of Moments estimator of θ. All values are of the form $Ave(SD)$, where Ave is the average value and SD is the standard deviation of the θ estimates.

n/θ	0.025	0.05	0.075	0.1	0.2	0.3	0.4	0.5
5	0.2286	0.2667	0.2907	0.3216	0.4658	0.5956	0.7111	0.8339
10	0.1210	0.1578	0.1986	0.2278	0.3513	0.4734	0.5855	0.7125
15	0.0966	0.1315	0.1609	0.1902	0.3072	0.4223	0.5363	0.6509
20	0.0763	0.1118	0.1424	0.1731	0.2861	0.3998	0.5068	0.6203
25	0.0723	0.1005	0.1317	0.1627	0.2734	0.3790	0.4871	0.5942
30	0.0647	0.0940	0.1243	0.1518	0.2636	0.3679	0.4787	0.5814
35	0.0588	0.0883	0.1181	0.1459	0.2590	0.3670	0.4690	0.5760
40	0.0555	0.0846	0.1132	0.1415	0.2508	0.3557	0.4617	0.5697
45	0.0524	0.0826	0.1101	0.1374	0.2451	0.3521	0.4551	0.5564
50	0.0500	0.0793	0.1074	0.1356	0.2425	0.3477	0.4515	0.5573
55	0.0488	0.0773	0.1035	0.1318	0.2378	0.3450	0.4486	0.5525
60	0.0471	0.0754	0.1036	0.1304	0.2373	0.3409	0.4462	0.5487
65	0.0456	0.0735	0.1025	0.1287	0.2350	0.3389	0.4425	0.5409
70	0.0439	0.0728	0.0995	0.1277	0.2324	0.3358	0.4410	0.5407
75	0.0432	0.0719	0.0999	0.1253	0.2318	0.3330	0.4355	0.5377
80	0.0422	0.0705	0.0969	0.1243	0.2286	0.3324	0.4345	0.5350
85	0.0415	0.0692	0.0960	0.1226	0.2271	0.3305	0.4334	0.5330
90	0.0410	0.0693	0.0958	0.1216	0.2258	0.3303	0.4317	0.5360
95	0.0402	0.0676	0.0951	0.1215	0.2245	0.3275	0.4324	0.5337
100	0.0400	0.0670	0.0945	0.1213	0.2237	0.3235	0.4296	0.5319

Table 65: Simulation 2 median values for the Method of Moments estimator of θ.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.6975	0.8052	0.8547	0.8843	0.8977	0.9126	0.9218	0.9260	0.9335	0.9419
5	0.7286	0.8218	0.8664	0.8934	0.9051	0.9186	0.9262	0.9315	0.9370	0.9458
10	0.7770	0.8438	0.8793	0.9033	0.9131	0.9249	0.9309	0.9361	0.9411	0.9488
15	0.8460	0.8680	0.8943	0.9143	0.9200	0.9317	0.9347	0.9396	0.9441	0.9529
20	0.9795	0.8923	0.9086	0.9247	0.9284	0.9376	0.9405	0.9440	0.9477	0.9563
25	1.0000	0.9198	0.9207	0.9358	0.9357	0.9422	0.9454	0.9491	0.9518	0.9603
30	1.0000	0.9482	0.9361	0.9441	0.9432	0.9488	0.9517	0.9558	0.9561	0.9633
35	1.0000	0.9777	0.9514	0.9547	0.9507	0.9554	0.9570	0.9602	0.9599	0.9671
40	1.0000	0.9987	0.9653	0.9637	0.9574	0.9617	0.9609	0.9640	0.9638	0.9695
45	1.0000	1.0000	0.9784	0.9724	0.9637	0.9672	0.9657	0.9687	0.9671	0.9730
50	1.0000	1.0000	0.9901	0.9798	0.9695	0.9730	0.9695	0.9728	0.9710	0.9762
55	1.0000	1.0000	0.9974	0.9860	0.9768	0.9783	0.9736	0.9766	0.9753	0.9785
60	1.0000	1.0000	0.9999	0.9928	0.9821	0.9811	0.9767	0.9807	0.9779	0.9802
65	1.0000	1.0000	1.0000	0.9962	0.9874	0.9842	0.9810	0.9836	0.9813	0.9829
70	1.0000	1.0000	1.0000	0.9984	0.9921	0.9892	0.9847	0.9859	0.9846	0.9852
75	1.0000	1.0000	1.0000	0.9998	0.9947	0.9930	0.9878	0.9877	0.9865	0.9867
80	1.0000	1.0000	1.0000	1.0000	0.9969	0.9948	0.9907	0.9903	0.9885	0.9885
85	1.0000	1.0000	1.0000	1.0000	0.9987	0.9967	0.9932	0.9920	0.9906	0.9901
90	1.0000	1.0000	1.0000	1.0000	0.9997	0.9980	0.9951	0.9939	0.9923	0.9921
95	1.0000	1.0000	1.0000	1.0000	0.9999	0.9990	0.9970	0.9952	0.9942	0.9944
100	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9979	0.9968	0.9954	0.9954
105	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9986	0.9978	0.9966	0.9968
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991	0.9985	0.9978	0.9978
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9989	0.9982	0.9981
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9995	0.9988	0.9984
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9991	0.9992
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9995
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9995
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 66: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 5$ and $\theta = 0.1$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.9320	0.9567	0.9697	0.9750	0.9772	0.9830	0.9821	0.9833	0.9862	0.9856
5	0.9658	0.9730	0.9815	0.9818	0.9846	0.9881	0.9874	0.9871	0.9893	0.9890
10	0.9917	0.9874	0.9902	0.9889	0.9903	0.9918	0.9909	0.9909	0.9924	0.9913
15	0.9994	0.9965	0.9960	0.9930	0.9940	0.9950	0.9938	0.9935	0.9948	0.9936
20	1.0000	0.9993	0.9987	0.9960	0.9958	0.9972	0.9960	0.9962	0.9964	0.9952
25	1.0000	1.0000	0.9996	0.9979	0.9981	0.9987	0.9979	0.9976	0.9974	0.9969
30	1.0000	1.0000	0.9999	0.9990	0.9988	0.9993	0.9993	0.9987	0.9982	0.9981
35	1.0000	1.0000	1.0000	0.9994	0.9995	0.9997	0.9996	0.9995	0.9992	0.9989
40	1.0000	1.0000	1.0000	0.9998	0.9998	0.9998	0.9998	0.9998	0.9997	0.9994
45	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9999	1.0000	0.9998	0.9997
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	0.9997
55	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	0.9998
60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	0.9998
65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
95	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
105	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 67: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 5$ and $\theta = 1$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.9696	0.9818	0.9858	0.9877	0.9883	0.9909	0.9903	0.9905	0.9903	0.9899
5	0.9963	0.9960	0.9955	0.9956	0.9956	0.9964	0.9955	0.9963	0.9952	0.9954
10	1.0000	0.9998	0.9991	0.9990	0.9989	0.9986	0.9979	0.9986	0.9979	0.9975
15	1.0000	1.0000	1.0000	0.9996	0.9997	0.9997	0.9993	0.9996	0.9993	0.9988
20	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9998	0.9998	0.9995
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
55	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
95	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
105	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 68: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 5$ and $\theta = 10$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.9740	0.9860	0.9889	0.9909	0.9878	0.9910	0.9908	0.9902	0.9921	0.9924
5	0.9985	0.9979	0.9972	0.9978	0.9968	0.9969	0.9975	0.9964	0.9966	0.9972
10	1.0000	0.9999	0.9997	0.9996	0.9992	0.9996	0.9989	0.9987	0.9992	0.9989
15	1.0000	1.0000	1.0000	0.9999	0.9999	1.0000	0.9999	0.9998	0.9997	1.0000
20	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	0.9998	1.0000
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
55	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
95	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
105	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 69: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 5$ and $\theta = 10000$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.7017	0.8070	0.8575	0.8791	0.9011	0.9176	0.9250	0.9278	0.9309	0.9442
5	0.7159	0.8160	0.8649	0.8839	0.9048	0.9209	0.9268	0.9298	0.9332	0.9453
10	0.7343	0.8254	0.8702	0.8882	0.9082	0.9236	0.9285	0.9323	0.9359	0.9464
15	0.7544	0.8344	0.8775	0.8923	0.9117	0.9264	0.9304	0.9348	0.9382	0.9473
20	0.7805	0.8457	0.8844	0.8970	0.9158	0.9293	0.9332	0.9373	0.9396	0.9496
25	0.8080	0.8571	0.8916	0.9014	0.9197	0.9333	0.9358	0.9406	0.9417	0.9511
30	0.8491	0.8678	0.8979	0.9072	0.9236	0.9368	0.9382	0.9424	0.9435	0.9530
35	0.9014	0.8807	0.9034	0.9135	0.9274	0.9392	0.9407	0.9447	0.9460	0.9547
40	0.9802	0.8937	0.9095	0.9183	0.9319	0.9426	0.9430	0.9468	0.9479	0.9567
45	1.0000	0.9080	0.9164	0.9233	0.9371	0.9453	0.9461	0.9494	0.9504	0.9583
50	1.0000	0.9212	0.9241	0.9277	0.9396	0.9476	0.9485	0.9512	0.9519	0.9595
55	1.0000	0.9344	0.9306	0.9334	0.9432	0.9503	0.9516	0.9540	0.9540	0.9610
60	1.0000	0.9497	0.9376	0.9375	0.9470	0.9526	0.9543	0.9566	0.9555	0.9631
65	1.0000	0.9641	0.9456	0.9424	0.9508	0.9556	0.9574	0.9592	0.9584	0.9644
70	1.0000	0.9783	0.9522	0.9475	0.9542	0.9575	0.9601	0.9615	0.9603	0.9658
75	1.0000	0.9923	0.9578	0.9527	0.9571	0.9606	0.9627	0.9639	0.9620	0.9675
80	1.0000	0.9990	0.9646	0.9581	0.9610	0.9630	0.9657	0.9664	0.9640	0.9689
85	1.0000	1.0000	0.9719	0.9644	0.9647	0.9661	0.9684	0.9683	0.9659	0.9708
90	1.0000	1.0000	0.9779	0.9699	0.9696	0.9697	0.9700	0.9698	0.9677	0.9720
95	1.0000	1.0000	0.9850	0.9740	0.9720	0.9720	0.9720	0.9716	0.9692	0.9730
100	1.0000	1.0000	0.9906	0.9784	0.9753	0.9755	0.9744	0.9735	0.9712	0.9746
105	1.0000	1.0000	0.9944	0.9819	0.9782	0.9781	0.9765	0.9758	0.9740	0.9760
110	1.0000	1.0000	0.9979	0.9857	0.9799	0.9799	0.9790	0.9777	0.9760	0.9778
115	1.0000	1.0000	0.9994	0.9886	0.9827	0.9819	0.9810	0.9792	0.9778	0.9792
120	1.0000	1.0000	1.0000	0.9916	0.9851	0.9835	0.9837	0.9805	0.9803	0.9801
125	1.0000	1.0000	1.0000	0.9933	0.9874	0.9847	0.9852	0.9821	0.9829	0.9812
130	1.0000	1.0000	1.0000	0.9953	0.9899	0.9863	0.9867	0.9835	0.9842	0.9821
135	1.0000	1.0000	1.0000	0.9974	0.9920	0.9884	0.9881	0.9847	0.9851	0.9835
140	1.0000	1.0000	1.0000	0.9983	0.9936	0.9899	0.9896	0.9863	0.9857	0.9851
145	1.0000	1.0000	1.0000	0.9996	0.9948	0.9918	0.9907	0.9874	0.9869	0.9859
150	1.0000	1.0000	1.0000	0.9998	0.9963	0.9928	0.9917	0.9886	0.9887	0.9869
155	1.0000	1.0000	1.0000	1.0000	0.9973	0.9939	0.9925	0.9894	0.9903	0.9883
160	1.0000	1.0000	1.0000	1.0000	0.9979	0.9948	0.9938	0.9911	0.9910	0.9893
165	1.0000	1.0000	1.0000	1.0000	0.9983	0.9960	0.9948	0.9919	0.9916	0.9907
170	1.0000	1.0000	1.0000	1.0000	0.9992	0.9967	0.9959	0.9929	0.9932	0.9916
175	1.0000	1.0000	1.0000	1.0000	0.9994	0.9979	0.9967	0.9939	0.9936	0.9923
180	1.0000	1.0000	1.0000	1.0000	0.9996	0.9983	0.9971	0.9950	0.9945	0.9928
185	1.0000	1.0000	1.0000	1.0000	1.0000	0.9988	0.9977	0.9953	0.9948	0.9936
190	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9986	0.9960	0.9951	0.9945
195	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9988	0.9965	0.9957	0.9948
200	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9989	0.9973	0.9962	0.9954

Table 70: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 10$ and $\theta = 0.1$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.9279	0.9622	0.9703	0.9746	0.9789	0.9814	0.9832	0.9840	0.9839	0.9866
5	0.9466	0.9707	0.9766	0.9792	0.9829	0.9852	0.9862	0.9862	0.9862	0.9886
10	0.9653	0.9781	0.9813	0.9844	0.9860	0.9876	0.9889	0.9888	0.9880	0.9902
15	0.9822	0.9849	0.9856	0.9871	0.9891	0.9902	0.9910	0.9909	0.9900	0.9919
20	0.9924	0.9903	0.9896	0.9905	0.9914	0.9919	0.9924	0.9921	0.9919	0.9929
25	0.9972	0.9946	0.9929	0.9923	0.9933	0.9942	0.9943	0.9943	0.9929	0.9937
30	0.9998	0.9970	0.9957	0.9942	0.9945	0.9957	0.9956	0.9955	0.9943	0.9946
35	1.0000	0.9985	0.9974	0.9965	0.9960	0.9971	0.9968	0.9961	0.9950	0.9954
40	1.0000	0.9995	0.9985	0.9977	0.9972	0.9980	0.9975	0.9970	0.9957	0.9968
45	1.0000	0.9997	0.9994	0.9984	0.9978	0.9987	0.9983	0.9977	0.9970	0.9974
50	1.0000	0.9999	0.9994	0.9992	0.9986	0.9990	0.9989	0.9984	0.9976	0.9983
55	1.0000	1.0000	0.9997	0.9995	0.9992	0.9993	0.9993	0.9986	0.9982	0.9991
60	1.0000	1.0000	0.9999	0.9998	0.9993	0.9997	0.9994	0.9987	0.9986	0.9992
65	1.0000	1.0000	1.0000	0.9998	0.9995	0.9999	0.9995	0.9994	0.9986	0.9994
70	1.0000	1.0000	1.0000	0.9999	0.9996	0.9999	0.9997	0.9997	0.9993	0.9994
75	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997	0.9998	0.9996	0.9997
80	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9997	0.9998	0.9998	0.9997
85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9999	0.9998	0.9999
90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9999
95	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000
105	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 71: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 10$ and $\theta = 1$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.9699	0.9844	0.9862	0.9890	0.9900	0.9878	0.9908	0.9911	0.9914	0.9913
5	0.9907	0.9937	0.9933	0.9941	0.9946	0.9928	0.9954	0.9942	0.9949	0.9941
10	0.9987	0.9976	0.9969	0.9971	0.9967	0.9953	0.9976	0.9966	0.9966	0.9961
15	1.0000	0.9994	0.9989	0.9987	0.9979	0.9981	0.9989	0.9982	0.9983	0.9979
20	1.0000	1.0000	1.0000	0.9994	0.9990	0.9994	0.9993	0.9989	0.9991	0.9991
25	1.0000	1.0000	1.0000	0.9998	0.9998	0.9998	0.9999	0.9993	0.9994	0.9995
30	1.0000	1.0000	1.0000	0.9998	1.0000	0.9998	1.0000	0.9997	0.9997	0.9995
35	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9998	0.9999	0.9996
40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
55	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
95	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
105	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 72: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 10$ and $\theta = 10$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.

b/n	10	20	30	40	50	60	70	80	90	100
0	0.9745	0.9871	0.9900	0.9895	0.9874	0.9902	0.9921	0.9936	0.9912	0.9927
5	0.9960	0.9965	0.9964	0.9958	0.9947	0.9956	0.9962	0.9970	0.9952	0.9965
10	0.9996	0.9995	0.9989	0.9983	0.9983	0.9981	0.9982	0.9985	0.9976	0.9983
15	1.0000	0.9999	0.9999	0.9994	0.9997	0.9997	0.9995	0.9995	0.9992	0.9994
20	1.0000	0.9999	1.0000	0.9997	1.0000	1.0000	0.9997	1.0000	0.9996	0.9999
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000	0.9998	1.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
55	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
65	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
75	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
85	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
95	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
105	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
110	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
120	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
125	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
130	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
140	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
145	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
155	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
160	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
165	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
170	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
175	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
180	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
185	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
190	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
195	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 73: Simulation coverage probabilities of the 95% Bounded Bernstein confidence interval when $\mu = 10$ and $\theta = 10000$. The values of the upper bound b and the sample size n are given in the rows and columns of the table, respectively.