COALESCING SYSTEMS OF BROWNIAN PARTICLES ON THE SIERPINSKI GASKET AND STABLE PARTICLES ON THE LINE OR CIRCLE

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Abstract. A well-known result of Arratia shows that one can make rigorous the notion of starting an independent Brownian motion at every point of an arbitrary closed subset of the real line and then building a set-valued process by requiring particles to coalesce when they collide. Arratia noted that the value of this process will be almost surely a locally finite set at all positive times, and a finite set almost surely if the initial value is compact: the key to both of these facts is the observation that, because of the topology of the real line and the continuity of Brownian sample paths, at the time when two particles collide one or the other of them must have already collided with each particle that was initially between them. We investigate whether such instantaneous coalescence still occurs for coalescing systems of particles where either the state space of the individual particles is not locally homeomorphic to an interval or the sample paths of the individual particles are discontinuous. We show that Arratia's conclusion is valid for Brownian motions on the Sierpinski gasket and for stable processes on the real line with stable index greater than one.

1. Introduction

A construction due to Richard Arratia [Arr79, Arr81] shows that it is possible to make rigorous sense of the informal notion of starting an independent Brownian motion at each point of the real line and letting particles coalesce when they collide.

Arratia proved that the set of particles remaining at any positive time is locally finite almost surely. Arratia's argument is based on the simple observation that at the time two particles collide, one or the other must have already collided with each particle that was initially between them. The same argument shows that if we start an independent circular Brownian motion at each point of the circle and let particles coalesce when they collide, then, almost surely, there are only finitely many particles remaining at any positive time.

Arratia established something even stronger: it is possible to construct a flow of random maps $(F_{s,t})_{s<t}$ from the real line to itself in such a way that for each fixed s the process $(F_{s,s+u})_{u\geq 0}$ is given by the above particle system. Arratia's flow has since been studied by several authors such as [TW98, STW00, SW02, LJR04, Tsi04, FINR04, HW09] for purposes as diverse as giving a rigorous definition of a one-dimensional self-repelling Brownian motion to providing examples of noises

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that are, in some sense, completely "orthogonal" to those produced by Poisson processes or Brownian motions.

Coalescing systems of more general Markov processes have been investigated because of their appearance as the duals of models in genetics of the stepping stone type, see, for example, [Kle96, EF96, Eva97, DEF$^+$00, Zho03, XZ05, HT05, MRTZ06, Zho08].

Arratia's "topological" argument for instantaneous coalescence to a locally finite set fails when one considers Markov processes on the line or circle with discontinuous sample paths or Markov processes with state spaces that are not locally like the real line. We show, however, that analogous conclusions holds for coalescing Brownian motions on the "finite" and "infinite" (that is, compact and non-compact) Sierpinski gaskets and stable processes on the circle and line – provided, of course, that the stable index is greater than 1, so that an independent pair of such motions collides with positive probability. Similar methods will apply to Markov processes on more general state spaces when the process and the state space have suitable local self-similarity properties, but we have not pursued a result with more encompassing hypotheses. As well as providing an interesting test case of a process with continuous sample paths on a state space that is not locally one-dimensional but is such that two independent copies of the process will collide with positive probability, the Brownian motion on the Sierpinski gasket was introduced as a model for diffusion in disordered media and it has since attracted a considerable amount of attention. The reader can get a feeling for this literature by consulting some of the earlier works such as [BP88, Lin90, Bar98] and more recent papers such as [HK03, KS05] and the references therein.

We give a sketch of the argument for the simplest case of stable processes on the circle in order to motivate some of the estimates that we must develop before we can present proofs in each of the various settings.

Observe from the scaling property of stable processes on the line that if X' and X'' are independent copies of a stable processes of index $\alpha > 1$ on the unit circle $\mathbb{T}$ starting from two distinct points that are $2\pi\varepsilon$ apart, then there exist positive constants β and p such that the probability the processes will collide by time $\beta\varepsilon^\alpha$ is bounded below by p.

Suppose we start with $n+1$ stable particles in some configuration on $\mathbb{T}$. By the pigeonhole principle, there will be (at least) one pair of particles that are distance at most $2\pi/n$ apart. Therefore, with probability at least p, these two particles in isolation would collide with each other by time $\beta n^{-\alpha}$. Hence, in the coalescing system the probability that there is at least one collision between some pair of particles within the time interval $[0, \beta n^{-\alpha}]$ is certainly at least p (either the two distinguished particles collide with each other and no others or some other particle(s) collides with one or both of the distinguished particles). Moreover, if there is no collision between any pair of particles after time $\beta n^{-\alpha}$, then we can again find at time $\beta n^{-\alpha}$ a possibly different pair of particles that are within distance $2\pi/n$ from each other, and the probability that this pair of particles will collide within the time interval $[\beta n^{-\alpha}, 2\beta n^{-\alpha}]$ is again at least p. By repeating this argument and using the Markov property, we see that if we let τ_n^{n+1} be the first time there are n surviving particles starting from $(n+1)$ particles, then, regardless of the particular initial configuration of the $n+1$ particles,

$$\mathbb{P}\left\{\tau_n^{n+1} \geq k\beta n^{-\alpha}\right\} \leq (1-p)^k.$$

In particular, the expected time needed to reduce the number of particles from $n+1$ to n is bounded above by $Cn^{-\alpha}$ for a suitable constant C.

Thus, if we start with N particles somewhere on $\mathbb{T}$, then the probability that after some positive time t the number of particles remaining is greater than m is, by Markov's inequality, bounded above by

$$\frac{1}{t}\sum_{n=m}^{N-1}\mathbb{E}\left[\tau_n^{n+1}\right] \le \frac{C}{t}\sum_{n=m}^{N-1} n^{-\alpha} \le \frac{C'}{t}m^{1-\alpha}$$

for some constant C'. Letting $N \to \infty$ and then letting $m \to \infty$, we conclude that by time t there are only finitely many particles almost surely.

The above reasoning uses the compactness of $\mathbb{T}$ in a crucial way and it cannot be applied as it stands to deal with, say, coalescing stable processes on the real line. The primary difficulty is that the argument bounds the time to coalesce from some number of particles to a smaller number by considering a particular sequence of coalescent events, and while waiting for such an event to occur the particles might spread out to such an extent that the pigeon hole argument can no longer be applied. We overcome this problem by using a somewhat more sophisticated pigeonhole argument to assign the bulk of the particles to a collection of suitable disjoint pairs (rather than just selecting a single suitable pair) and then employing a simple large deviation bound to ensure that with high probability at least a certain fixed proportion of the pairs will have collided over an appropriate time interval.

2. Countable systems of coalescing Feller processes

In this section we develop some general properties of coalescing systems of Markov processes that we will apply later to Brownian motions on the Sierpinski gasket and stable processes on the line or circle.

2.1. **Vector-valued coalescing process.**

Fix $N \in \mathbb{N} \cup \{\infty\}$, where, as usual, $\mathbb{N}$ is the set of positive integers. Write $[N]$ for the set $\{1, 2, \ldots, N\}$ when N is finite and for the set $\mathbb{N}$ when $N = \infty$.

Fix a locally compact, second-countable, Hausdorff space E. Note that E is metrizable. Let d be a metric giving the topology on E. Denote by $D := D(\mathbb{R}_+, E)$ the usual Skorokhod space of E-valued càdlàg paths. Fix a bijection $\sigma : N \to N$. We will call σ a *ranking* of $[N]$. Define a mapping $\Lambda_\sigma : D^N \to D^N$ by setting $\Lambda_\sigma \boldsymbol{\xi} = \boldsymbol{\zeta}$ for $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots) \in D^N$, where $\boldsymbol{\zeta}$ is defined inductively as follows. Set $\zeta_{\sigma(1)} \equiv \xi_{\sigma(1)}$. For $i > 1$, set

$$\tau_i := \inf\left\{t \ge 0 : \xi_{\sigma(i)}(t) \in \{\zeta_{\sigma(1)}(t), \zeta_{\sigma(2)}(t), \ldots, \zeta_{\sigma(i-1)}(t)\}\right\},$$

with the usual convention that $\inf \emptyset = \infty$. Put

$$J_i := \min\left\{j \in \{1, 2, \ldots, i-1\} : \xi_{\sigma(i)}(\tau_i) = \zeta_{\sigma(j)}(\tau_i)\right\} \quad \text{if } \tau_i < \infty.$$

For $t \ge 0$, define

$$\zeta_{\sigma(i)}(t) := \begin{cases} \xi_{\sigma(i)}(t) & \text{if } t < \tau_i \\ \zeta_{\sigma(J_i)}(t) & \text{if } t \ge \tau_i. \end{cases}$$

We call the map Λ_σ a *collision rule*. It produces a vector of "coalescing" paths from of a vector of "free" paths: after the free paths labeled i and j collide, the corresponding coalescing paths both subsequently follow either the path labeled i

or the path labeled j, according to whether $\sigma(i) < \sigma(j)$ or $\sigma(i) > \sigma(j)$. Note for each $n < N$ that the value of $(\zeta_{\sigma(i)})_{1\le i\le n}$ is unaffected by the value of $(\xi_{\sigma(j)})_{j>n}$.

Suppose from now on that the paths $\xi_1, \xi_2, \dots$ are realizations of independent copies of a Feller Markov process X with state space E.

A priori, the distribution of the finite or countable coalescing system $\boldsymbol{\zeta} = \Lambda_\sigma \boldsymbol{\xi}$ depends on the ranking σ. However, we have the following result, which is a consequence of the strong Markov property of $\boldsymbol{\xi}$ and the observation that if we are given a bijection $\pi : [N] \to [N]$ and define a map $\Sigma_\pi : D^N \to D^N$ by $(\Sigma_\pi \boldsymbol{\xi})_i = \xi_{\pi(i)}$, $i \in [N]$, then $\Sigma_\pi \Lambda_\sigma = \Lambda_{\sigma\pi^{-1}} \Sigma_\pi$.

Lemma 2.1 ([Arr79, Arr81])**.** *The distribution of $\boldsymbol{\zeta} = \Lambda_\sigma \boldsymbol{\xi}$ is the same for all bijections $\sigma : [N] \to [N]$.*

From now on, we will, unless we explicitly state otherwise, take $\sigma = \mathrm{id}$, where $\mathrm{id} : [N] \to [N]$ is the identity bijection. To simplify notation, we will write Λ for the collision rule Λ_{id}.

It is intuitively clear that the coalescing system $\boldsymbol{\zeta}$ is Markov. For the sake of completeness, we establish this formally in the next lemma, the proof of which is essentially an argument from [Arr79, Arr81].

Define the right-continuous filtration $(\mathcal{F}_t)_{t\ge 0}$ by

$$\mathcal{F}_t := \bigcap_{\varepsilon>0} \sigma\{\xi_i(s) : s \le t+\varepsilon, i \ge 1\}.$$

Lemma 2.2. *The stochastic process $\boldsymbol{\zeta} = \Lambda\boldsymbol{\xi}$ is strong Markov with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$.*

Proof. Define maps $m : \{1,2,\dots,N\} \times E^N \to \{1,2,\dots,N\}$ and $\Pi : E^N \times E^N \to E^N$ by setting $m(i, \mathbf{x}) := \min\{j : x_j = x_i\}$ and $\Pi(\mathbf{x}, \mathbf{y})_i := y_{m(i,\mathbf{x})}$. Note that

$$\Pi(\Lambda\boldsymbol{\eta}(t), \boldsymbol{\eta}(t)) := \Lambda\boldsymbol{\eta}(t), \quad \boldsymbol{\eta} \in D^N,\ t \ge 0.$$

Define a map $\widetilde{\Pi} : E^N \times D^N \to D^N$ by

$$\widetilde{\Pi}(\mathbf{x}, \boldsymbol{\eta})(t) = \Pi(\mathbf{x}, \boldsymbol{\eta}(t)), \quad \mathbf{x} \in E^N,\ \boldsymbol{\eta} \in D^N,\ t \ge 0.$$

Writing $\{\theta_s\}_{t\ge 0}$ for the usual family of shift operators on D^N, that is, $(\theta_s\boldsymbol{\eta})(t) = \boldsymbol{\eta}(s+t)$, we have

$$\theta_s \Lambda\boldsymbol{\eta} = \Lambda\widetilde{\Pi}(\Lambda\boldsymbol{\eta}(s), \theta_s\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in D^N,\ s \ge 0.$$

Fix a bounded measurable function on $f : D^N \to \mathbb{R}$ and set

$$g(\mathbf{x}, \mathbf{y}) = \mathbb{E}^{\mathbf{y}}\left[f\left(\Lambda\widetilde{\Pi}(\mathbf{x}, \boldsymbol{\xi})\right)\right].$$

Note that since the components of $\boldsymbol{\xi}$ are independent, if $\Pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$, then $g(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{x})$. Thus, for a finite $(\mathcal{F}_t)_{t\ge 0}$ stopping time S we have from the strong Markov property of $\boldsymbol{\xi}$ that

$$\begin{aligned}
\mathbb{E}^{\mathbf{x}}\left[f(\theta_S \Lambda\boldsymbol{\xi}) \,\middle|\, \mathcal{F}_S\right] &= \mathbb{E}^{\mathbf{x}}\left[f\left(\Lambda\widetilde{\Pi}(\Lambda\boldsymbol{\xi}(S), (\theta_S\boldsymbol{\xi}))\right) \,\middle|\, \mathcal{F}_S\right] \\
&= g(\Lambda\boldsymbol{\xi}(S), \boldsymbol{\xi}(S)) \\
&= g(\Lambda\boldsymbol{\xi}(S), \Lambda\boldsymbol{\xi}(S)) \\
&= \mathbb{E}^{\Lambda\boldsymbol{\xi}(S)}[f(\Lambda\boldsymbol{\xi})],
\end{aligned}$$

as required. ∎

2.2. **Set-valued coalescing process.** Write $\mathcal{K} = \mathcal{K}(E)$ for the set of nonempty compact subsets of E equipped with the usual Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \inf\{\varepsilon > 0 : K_1^\varepsilon \supseteq K_2 \text{ and } K_2^\varepsilon \supseteq K_1\},$$

where $K^\varepsilon := \{y \in E : \exists x \in K, \, d(y,x) < \varepsilon\}$. The metric space $(\mathcal{K}, d_H)$ is complete. It is compact if E is.

If the locally compact space E is not compact, write $\mathcal{C} = \mathcal{C}(E)$ for the set of nonempty closed subsets of E. Identify the elements of $\mathcal{C}$ with their closures in the one-point compactification $\bar{E}$ of E. Write d_C for the metric on $\mathcal{C}$ that arises from the Hausdorff metric on the compact subsets of $\bar{E}$ corresponding to some metric on $\bar{E}$ that induces the topology of $\bar{E}$.

Let $\Xi_t \subseteq E$ denote the closure of the set $\{\zeta_i(t) : i = 1, 2, \ldots\}$ in E, where $\boldsymbol{\zeta} = \Lambda \boldsymbol{\xi}$.

The following result is an almost immediate consequence of Lemma 2.1.

Lemma 2.3. *If $\mathbf{x}', \mathbf{x}'' \in E^N$ are such that the sets $\{x_i' : i \in [N]\}$ and $\{x_i'' : i \in [N]\}$ are equal, then the distributions of the process Ξ under $\mathbb{P}^{\mathbf{x}'}$ and $\mathbb{P}^{\mathbf{x}''}$ are also equal.*

For the remainder of this section, we will make the following assumption.

Assumption 2.4. The Feller process X is such that if X' and X'' are two independent copies of X, then, for all $t_0 > 0$ and $x' \in E$,

$$\lim_{x'' \to x'} \mathbb{P}^{x', x''}\big\{X_t' = X_t'' \text{ for some } t \in [0, t_0]\big\} = 1.$$

Proposition 2.5. *Let $\mathbf{x}', \mathbf{x}'' \in E^N$ be such that the sets $\{x_i' : i \in [N]\}$ and $\{x_i'' : i \in [N]\}$ have the same closure. Then, the process Ξ has the same distribution under $\mathbb{P}^{\mathbf{x}'}$ and $\mathbb{P}^{\mathbf{x}''}$.*

Proof. We will consider the case where E is compact. The non-compact case is essentially the same, and we leave the details to the reader.

We need to show for any finite set of times $0 < t_1 < \ldots < t_k$ that the distribution of $(\Xi_{t_1}, \ldots, \Xi_{t_k})$ is the same under $\mathbb{P}^{\mathbf{x}'}$ and $\mathbb{P}^{x''}$.

We may suppose without loss of generality that $x_1', x_2', \ldots$ (resp. $x_1'', x_2'', \ldots$) are distinct.

Fix $n \in [N]$ and $\delta > 0$. Given $\varepsilon > 0$ that will be specified later, choose $y_1'', y_2'', \ldots, y_n'' \in \{x_i'' : i \in [N]\}$ such that $d(x_i', y_i'') \le \varepsilon$ for $1 \le i \le n$. Let $\boldsymbol{\eta}'$ (resp. $\boldsymbol{\eta}''$) be an E^n-valued process with coordinates that are independent copies of X started at $(x_1', \ldots, x_n')$ (resp. $(y_1'', y_2'', \ldots, y_n'')$).

By the Feller property, there is a time $0 < t_0 \le t_1$ that depends on $x_1', \ldots, x_n'$ such that for all ε sufficiently small

$$\mathbb{P}\{\eta_i''(t) = \eta_j''(t) \text{ for some } 1 \le i \ne j \le n \text{ and } 0 < t \le t_0\} \le \frac{\delta}{2}.$$

By our standing Assumption 2.4, if we take ε sufficiently small, then

$$\mathbb{P}\{\eta_i'(t) \ne \eta_i''(t) \text{ for all } 0 < t \le t_0\} \le \frac{\delta}{2n}, \quad 1 \le i \le n.$$

Write Ξ' (resp. Ξ'', $\hat{\Xi}$, $\check{\Xi}$) for the set-valued processes constructed from $\boldsymbol{\eta}'$ (resp. $\boldsymbol{\eta}''$, $(\boldsymbol{\eta}', \boldsymbol{\eta}'')$, $(\boldsymbol{\eta}'', \boldsymbol{\eta}')$) in the same manner that Ξ is constructed from $\boldsymbol{\xi}$. We have

$$\mathbb{P}\{\check{\Xi}_t = \Xi_t'' \text{ for all } t \ge t_0\} \ge 1 - \delta,$$

$$\Xi_t' \subseteq \hat{\Xi}_t, \quad \text{for all } t \ge 0,$$

and, by Lemma 2.3,

$$\hat{\Xi} \stackrel{d}{=} \check{\Xi}.$$

For each $z \in E$, define a continuous function $\phi_z : \mathcal{K} \to \mathbb{R}_+$ by

$$\phi_z(K) := \inf\{d(z,w) : w \in K\}.$$

Note that $K' \subseteq K''$ implies that $\phi_z(K') \geq \phi_z(K'')$ for any $z \in E$. It follows that for points $z_{\ell p} \in E$, $1 \leq p \leq q_\ell$, $1 \leq \ell \leq k$,

$$\begin{aligned}
\mathbb{E}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi'_{t_\ell})\right] &\geq \mathbb{E}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\hat{\Xi}_{t_\ell})\right] \\
&= \mathbb{E}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\check{\Xi}_{t_\ell})\right] \\
&\geq \mathbb{E}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi''_{t_\ell})\right] - \delta\left(\sup\{d(z,w) : z, w \in E\}\right)^{\sum_\ell q_\ell}
\end{aligned}$$

Observe that

$$\mathbb{E}^{\mathbf{x}'}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi_{t_\ell})\right] = \lim_{n\to\infty} \mathbb{E}^{\mathbf{x}'}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi'_{t_\ell})\right]$$

and

$$\mathbb{E}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi''_{t_\ell})\right] \geq \mathbb{E}^{\mathbf{x}''}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi_{t_\ell})\right].$$

Since δ is arbitrary,

$$\mathbb{E}^{\mathbf{x}'}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi_{t_\ell})\right] \geq \mathbb{E}^{\mathbf{x}''}\left[\prod_{\ell=1}^{k}\prod_{p=1}^{q_\ell} \phi_{z_{\ell p}}(\Xi_{t_\ell})\right].$$

Moreover, we see from interchanging the roles of $\mathbf{x}'$ and $\mathbf{x}''$ that the last inequality is actually an equality.

It remains to observe from the Stone-Weierstrass theorem that the algebra of continuous functions generated by the constants and the set $\{\phi_z : z \in E\}$ is uniformly dense in the space of continuous functions on E. $\square$

With Proposition 2.5 in hand, it makes sense to talk about the distribution of the process Ξ for a given initial state Ξ_0. The following result follows immediately from Dynkin's criterion for a function of Markov process to be also Markov.

Corollary 2.6. *The process $(\Xi_t)_{t\geq 0}$ is strong Markov with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.*

3. Brownian motion on the Sierpinski gasket

3.1. **Definition and properties of the gasket.** Let

$$G_0 := \{(0,0), (1,0), (1/2, \sqrt{3}/2)\}$$

be the vertices of the unit triangle in $\mathbb{R}^2$ and denote by H_0 the closed convex hull of G_0. The *Sierpinski gasket*, which we also call the *finite gasket*, is a fractal subset of the plane that can be constructed via the following Cantor-like cut-out procedure. Let $\{b_0, b_1, b_2\}$ be the midpoints of three sides of H_0 and let A be the interior of the triangle with vertices $\{b_0, b_1, b_2\}$. Define $H_1 := H_0 \setminus A$ so that H_1 is the union of 3 closed upward facing triangles of side length 2^{-1}. Now repeat this operation on each of the smaller triangles to obtain a set H_2, consisting of 9 upward facing closed triangles, each of side 2^{-2}. Continuing this fashion, we have a decreasing sequence of closed non-empty sets $\{H_n\}_{n=0}^{\infty}$ and we define the Sierpinski gasket as

$$G := \bigcap_{n=0}^{\infty} H_n.$$

We call each of the 3^n triangles of side 2^{-n} that make up H_n an *n-triangle* of G. Denote by $\mathcal{T}_n$ the collection of all n-triangles of G. Let $\mathcal{V}_n$ be the set of vertices of the n-triangles.

We call the unbounded set

$$\widetilde{G} := \bigcup_{n=0}^{\infty} 2^n G$$

the *infinite gasket* (where, as usual, we write $cB := \{cx : x \in B\}$ for $c \in \mathbb{R}$ and $B \subseteq \mathbb{R}^2$) . The concept of n-triangle, where n may now be a negative integer, extends in the obvious way to the infinite gasket. Denote the set of all n-triangles of $\widetilde{G}$ by $\widetilde{\mathcal{T}}_n$. Let $\widetilde{\mathcal{V}}_n$ be the vertices of $\widetilde{\mathcal{T}}_n$.

Given a pathwise connected subset $A \in \mathbb{R}^2$, let ρ_A be the *shortest-path metric* on A given by

$$\rho_A(x, y) := \inf\{|\gamma| : \gamma \text{ is a path between } x \text{ and } y \text{ and } \gamma \subseteq A\},$$

where $|\gamma|$ denote the length (that is, the 1-dimensional Hausdorff measure) of γ. For the finite gasket G, ρ_G is comparable to the usual Euclidean metric $|\cdot|$ (see, for example, [Bar98, Lemma 2.12]) with the relation,

$$|x - y| \le \rho_G(x, y) \le c|x - y|, \quad \forall x, y \in G,$$

for a suitable constant $1 < c < \infty$. It is obvious that the same is also true for the metric $\rho_{\widetilde{G}}$ on the infinite gasket.

Let μ denote the d_f-dimensional Hausdorff measure on $\widetilde{G}$ where $d_f := \log 3/ \log 2$ is the *fractal* or *mass dimension* of the gasket. For the finite gasket G we have $0 < \mu(G) < \infty$ and, with a slight abuse of notation, we will also use the notation μ to denote the restriction of this measure to G. Moreover, we have the following estimate on the volume growth of μ

$$\mu(B(x, r)) \le C r^{d_f} \quad \text{for } x \in \widetilde{G},\ 0 < r < 1, \tag{3.1}$$

where $B(x, r) \subseteq \widetilde{G}$ is the open ball with center x and radius r in the Euclidean metric and C is a suitable constant (see [BP88]).

3.2. **Brownian motions.** We construct a graph G_n (respectively, $\widetilde{G}_n$) embedded in the plane with vertices $\mathcal{V}_n$ (resp. $\widetilde{\mathcal{V}}_n$) by adding edges between pairs of vertices that are distance 2^{-n} apart from each other. Let X^n (resp. $\widetilde{X}^n$) be the natural random walk on G_n (resp. $\widetilde{G}_n$); that is, the discrete time Markov chain that at each step chooses uniformly at random from one of the neighbors of the current state. It is known (see [BP88, Bar98]) that the sequence $(X^n_{\lfloor 5^n t \rfloor})_{t \ge 0}$ (resp. $(\widetilde{X}^n_{\lfloor 5^n t \rfloor})_{t \ge 0}$) converges in distribution as $n \to \infty$ to a limiting process $(X_t)_{t \ge 0}$ (resp. $(\widetilde{X}_t)_{t \ge 0}$) that is a G-valued (resp. $\widetilde{G}$-valued) strong Markov process (indeed, a Feller process) with continuous sample paths. The processes X and $\widetilde{X}$ are called, for obvious reasons, the Brownian motion on the finite and infinite gaskets, respectively. The Brownian motion on the *infinite* gasket has the following scaling property:

$$(2\widetilde{X}_t)_{t\geq 0} \text{ under } \mathbb{P}^x \text{ has same law as } (\widetilde{X}_{5t})_{t\geq 0} \text{ under } \mathbb{P}^{2x}. \tag{3.2}$$

The process $\widetilde{X}$ has a family $\widetilde{p}(t,x,y)$, $x,y \in \widetilde{G}$, $t>0$, of transition densities with respect to the measure μ that is jointly continuous on $(0,\infty)\times\widetilde{G}\times\widetilde{G}$. Moreover, $\widetilde{p}(t,x,y)=\widetilde{p}(t,y,x)$ for all $x,y\in\widetilde{G}$ and $t>0$, so that the process $\widetilde{X}$ is symmetric with respect to μ.

Let $d_w := \log 5/\log 2$ denote the *walk dimension* of the gasket. The following crucial "heat kernel bound" is established in [BP88]

$$\widetilde{p}(t,x,y) \leq c_1 t^{-d_f/d_w} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right). \tag{3.3}$$

Because the infinite gasket $\widetilde{G}$ and the associated Brownian motion $\widetilde{X}$ both have re-scaling invariances that G and X do not, it will be convenient to work with $\widetilde{X}$ and then use the following observation to transfer our results to X.

Lemma 3.1 (Folding lemma)**.** *There exists a continuous mapping $\psi:\widetilde{G}\to G$ such that ψ restricted to G is the identity, ψ restricted to any* 0*-triangle is an isometry, and $|\psi(x)-\psi(y)|\leq|x-y|$ for arbitrary $x,y\in\widetilde{G}$. Moreover, if the $\widetilde{G}$-valued process $\widetilde{X}$ is started at an arbitrary $x\in\widetilde{G}$, then the G-valued process $\psi\circ\widetilde{X}$ has the same distribution the process X started at $\psi(x)$.*

Proof. Let L be the subset of the plane formed by the set of points of the form $n_1(1,0)+n_2(1/2,\sqrt{3}/2)$, where n_1,n_2 are non-negative integers, and the line segments that join such points that are distance 1 apart. It is easy to see that there is a unique labeling of the vertices of L by $\{1,\omega,\omega^2\}$ that has the following properties.

- Label $(0,0)$ with 1.
- If vertex v is labeled $\mathfrak{a}\in\{1,\omega,\omega^2\}$, then the vertex $v+(1,0)$ are labeled with $\mathfrak{a}\omega$.
- If we think of the labels as referring to elements of the cyclic group of order 3, then if vertex v is labeled $\mathfrak{a}\in\{1,\omega,\omega^2\}$, then vertex $v+(1/2,\sqrt{3}/2)$ is labeled with $\mathfrak{a}\omega^2$.

Indeed, the label of the vertex $n_1(1,0)+n_2(1/2,\sqrt{3}/2)$ is $\omega^{n_1+2n_2}$.

Given a vertex $v\in L$, let $\iota(v)$ be the unique vertex in $\{(0,0),(1,0),(1/2,\sqrt{3}/2)\}$ that has the same label as v. If the vertices $v_1,v_2,v_3\in L$ are the vertices of a triangle with side length 1, then $\iota(v_1),\iota(v_2),\iota(v_3)$ are all distinct.

With the above preparation, let us now define the map ψ. Given $x\in\widetilde{G}$, let $\Delta\in\widetilde{\mathcal{T}}_0$ be a triangle with vertices v_1,v_2,v_3 that contains x (if x belongs to $\widetilde{\mathcal{V}}_n$, then there may be more than one such triangle, but the choice will not matter). We may write x as a unique convex combination of the vertices v_1,v_2,v_3,

$$x=\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3, \quad \sum_{i=1}^3 \lambda_i = 1, \lambda_i\geq 0.$$

The triple $(\lambda_1,\lambda_2,\lambda_3)$ is the vector of *barycentric* coordinates of x. We define $\psi(x)$ by

$$\psi(x) := \lambda_1\iota(v_1)+\lambda_2\iota(v_2)+\lambda_3\iota(v_3).$$

It is clear that $\psi:\widetilde{G}\to G$ is well-defined and has the stated properties.

Recall that $\widetilde{X}^{(n)}$ be the natural random walk on $\widetilde{G}_n$. It can be verified easily that the projected process $\psi \circ \widetilde{X}^{(n)}$ is the natural random walk on G_n. The result follows by taking the limit as $n \to \infty$ and using the continuoity of ψ. $\square$

Lemma 3.2 (Maximal inequality). *(a) Let $\widetilde{X}^i$, $1 \le i \le n$, be n independent Brownian motions on the infinite gasket $\widetilde{G}$ starting from the initial states x^i, $1 \le i \le n$. For any $t > 0$,*

$$\mathbb{P}\left\{\sup_{0\le s\le t} |\widetilde{X}^i_s - x^i| > r, \text{ for some } 1 \le i \le n\right\} \le 2nc_1 \exp\Big(- c_2(r^{d_w}/t)^{1/(d_w-1)}\Big),$$

where $c_1, c_2 > 0$ are constants and $d_w = \log 5/\log 2$ is the walk dimension of the gasket.
(b) The same estimate holds for the case of n independent Brownian motions X^i, $1 \le i \le n$, on the finite gasket G starting from the initial states x^i, $1 \le i \le n$.

Proof. (a) Let $\widetilde{X} = (\widetilde{X}_t)_{t\ge 0}$ be a Brownian motion on $\widetilde{G}$. Then for $x \in \widetilde{G}$, $t > 0$, and $r > 0$,

$$\begin{aligned}\mathbb{P}^x\left\{\sup_{0\le s\le t} |\widetilde{X}_s - x| > r\right\} &\le \mathbb{P}^x\{|\widetilde{X}_t - x| > r/2\} \\ &\quad + \mathbb{P}^x\big\{|\widetilde{X}_t - x| \le r/2, \sup_{0\le s\le t} |\widetilde{X}_s - x| > r\big\}.\end{aligned}$$

Writing $S := \inf\{s > 0 : |\widetilde{X}_s - x| > r\}$, the second term above equals

$$\mathbb{E}^x\left[1_{\{S<t\}}\mathbb{P}^{\widetilde{X}_S}\{|\widetilde{X}_{t-S} - x| \le r/2\}\right] \le \sup_{y\in\partial B(x,r)} \sup_{s\le t} \mathbb{P}^y\{|\widetilde{X}_{t-s} - y| > r/2\},$$

so that

$$\begin{aligned}\mathbb{P}^x\left\{\sup_{0\le s\le t} |\widetilde{X}_s - x| > r\right\} &\le 2 \sup_{y\in\widetilde{G}} \sup_{s\le t} \mathbb{P}^y\{|\widetilde{X}_s - y| > r/2\} \\ &\le 2c_1 \exp\Big(- c_2(r^{d_w}/t)^{1/(d_w-1)}\Big),\end{aligned}$$

where the last estimate is taken from [Bar98, Theorem 2.23(e)]. The lemma now follows by a union bound.

(b) This is immediate from part (a) and Lemma 3.1. $\square$

3.3. **Collision time estimates.** We first show that two independent copies of $\widetilde{X}$ collide with positive probability.

Proposition 3.3. *Let $\widetilde{X}'$ and $\widetilde{X}''$ be two independent copies of $\widetilde{X}$. Then,*

$$\mathbb{P}^{(x',x'')}\{\exists t > 0 : \widetilde{X}'_t = \widetilde{X}''_t\} > 0$$

for all $(x', x'') \in \widetilde{G} \times \widetilde{G}$.

Proof. Note that $\widetilde{\mathbf{X}} = (\widetilde{X}', \widetilde{X}'')$ is a Feller process on the locally compact separable metric space $\widetilde{G} \times \widetilde{G}$ that is symmetric with respect to the Radon measure $\mu \otimes \mu$ and has transition densities $\widetilde{p}(t, x', y') \times \widetilde{p}(t, x'', y'')$. The corresponding α-potential density is

$$u_\alpha(\mathbf{x}, \mathbf{y}) := \int_0^\infty e^{-\alpha t}\widetilde{p}(t, x_1, y_1) \times \widetilde{p}(t, x_2, y_2)\, dt \quad \text{for } \alpha > 0,$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. A standard potential theoretic result says that a compact set $B \subseteq \widetilde{G} \times \widetilde{G}$ is non-polar if there exists a non-zero finite measure ν that is supported on B and has finite energy, that is,

$$\int\int u^\alpha(\mathbf{x}, \mathbf{y})\, \nu(d\mathbf{x})\, \nu(d\mathbf{y}) < \infty.$$

Take $B = \{(x', x'') \in G \times G : x' = x''\}$ and ν to be the 'lifting' of the Hausdorff measure μ on the finite gasket onto B. We want to show that

$$\int_G \int_G \int_0^\infty e^{-\alpha t} \widetilde{p}^2(t, x, y)\, dt\, \mu(dx)\mu(dy) < \infty.$$

It will be enough to show that

$$\int_G \int_G \int_0^\infty \widetilde{p}^2(t, x, y)\, dt\, \mu(dx)\, \mu(dy) < \infty.$$

It follows from the transition density estimate (3.3) and Lemma 3.4 below that

$$\int_0^\infty \widetilde{p}^2(t, x, y)\, dt \le C|x - y|^{-\gamma}$$

for some constant C, where $\gamma := 2d_f - d_w$. Thus,

$$\begin{aligned}
&\int_G \int_G \int_0^\infty \widetilde{p}^2(t, x, y)\, dt\, \mu(dx)\, \mu(dy) \\
&\le C \int_G \int_G |x - y|^{-\gamma}\, \mu(dx)\, \mu(dy) \\
&\le C \int_G \int_0^\infty \mu\{x \in G : |x - y|^{-\gamma} > s\}\, ds\, \mu(dy) \\
&\le C \int_G \int_0^\infty \mu\{x \in G : |x - y| < s^{-1/\gamma}\}\, ds\, \mu(dy) \\
&\le C + C \int_G \int_1^\infty \mu\{x \in G : |x - y| < s^{-1/\gamma}\}\, ds\, \mu(dy) \\
&\le C + C_1 \int_G \int_1^\infty s^{-d_f/\gamma}\, ds\, \mu(dy) \quad [\text{ By (3.1)}] \\
&\le C + C_2 \int_1^\infty s^{-d_f/\gamma}\, ds.
\end{aligned}$$

It remains to note that $\gamma - d_f = (2\log 3/\log 2 - \log 5/\log 2) - (\log 3/\log 2) = (\log 3 - \log 5)/\log 2 < 0$, and so $d_f/\gamma < 1$.

This shows that $\mathbb{P}^{(x', x'')}\{\widetilde{\mathbf{X}} \text{ hits the diagonal}\} > 0$ for some $(x', x'') \in \widetilde{G} \times \widetilde{G}$. Because $\widetilde{p}^2(t, x, y) > 0$ for all $x, y \in \widetilde{G}$ and $t > 0$, we even have $\mathbb{P}^{(x', x'')}\{\widetilde{\mathbf{X}} \text{ hits the diagonal}\} > 0$ for all $(x', x'') \in \widetilde{G} \times \widetilde{G}$. □

We needed the following elementary result in the proof of Proposition 3.3.

Lemma 3.4. *For $\alpha > 1, \beta > 0$ and $A > 0$,*

$$\int_0^\infty t^{-\alpha} \exp(-A/t^\beta) dt = \Gamma\left(\frac{\alpha - 1}{\beta}\right) \beta^{-1} A^{-\frac{\alpha-1}{\beta}} < \infty.$$

Proof. Make the change of variables $u = At^{-\beta}$ in the integration. □

We next establish a uniform lower bound on the collision probability of a pair of independent Brownian motions on the infinite gasket as long as the distance between their starting points remains bounded.

Theorem 3.5. *There exist constants $\beta > 0$ and $\underline{p} > 0$ such that if $\widetilde{X}'$ and $\widetilde{X}''$ are two independent Brownian motions on $\widetilde{G}$ starting from any two points x, y belonging to the same n-triangle of $\widetilde{G}$, then*

$$\mathbb{P}^{(x,y)}\{\widetilde{X}'_t = \widetilde{X}''_t \text{ for some } t \in (0, \beta 5^{-n})\} \geq \underline{p}.$$

This result will require a certain amount of work, so we first note that it leads easily to an analogous result for the finite gasket.

Corollary 3.6. *If X' and X'' are two independent Brownian motions on G starting from any two points x, y belonging to the same n-triangle of G, then*

$$\mathbb{P}^{(x,y)}\{X'_t = X''_t \text{ for some } t \in (0, \beta 5^{-n})\} \geq \underline{p},$$

where $\beta > 0$ and $\underline{p} > 0$ are the constants given in Theorem 3.5.

Proof. The proof follows immediately from Lemma 3.1, because if $\widetilde{X}'_t = \widetilde{X}''_t$ for some t, then it is certainly the case that $\psi \circ \widetilde{X}'_t = \psi \circ \widetilde{X}''_t$. □

Definition 3.7 (Extended triangles for the infinite gasket)**.** Recall that $\widetilde{\mathcal{T}}_n$ is the set of all n-triangles of $\widetilde{G}$. Given $\Delta \in \widetilde{\mathcal{T}}_0$ such that Δ does not have the origin as one its vertices, we define the corresponding *extended triangle* $\Delta^e \subset \widetilde{G}$ as the interior of the union of the original 0-triangle Δ with the three neighboring 1-triangles in $\widetilde{G}$ which share one vertex with Δ and are not contained in Δ. Note that for the (unique) triangle Δ in $\widetilde{\mathcal{T}}_n$ having the origin as one of its vertices, there are two neighboring 1-triangles in $\widetilde{G}$ that share one vertex with it which are not contained in Δ. In this case, by Δ^e, we mean the interior of the union of Δ and these two triangles.

Fix some $\Delta \in \widetilde{\mathcal{T}}_0$. Let $\widetilde{Z}$ be the Brownian motion on Δ^e killed when it exits Δ^e. It follows from arguments similar to those on [Doo01, page 590], that $\widetilde{Z}$ has transition densities $\widetilde{p}_K(t, x, y)$, $t > 0$, $x, y \in \Delta^e$, with respect to the restriction of μ to Δ^e, and these densities have the following properties:

- $\widetilde{p}_K(t, x, y) = \widetilde{p}_K(t, y, x)$ for all $t > 0$, $x, y \in \Delta^e$.
- $\widetilde{p}_K(t, x, y) \leq \widetilde{p}(t, x, y)$, for all $t > 0$, $x, y \in \Delta^e$.
- $y \mapsto \widetilde{p}_K(t, x, y)$ is continuous for all $t > 0$, $x \in \Delta^e$, and $x \mapsto \widetilde{p}_K(t, x, y)$ is continuous for all $t > 0$, $y \in \Delta^e$.

It follows that the process $\widetilde{Z}$ is Feller and symmetric with respect to the measure μ.

Lemma 3.8. *Let $\widetilde{Z}', \widetilde{Z}''$ be two independent copies of the killed Brownian motion $\widetilde{Z}$. Given any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that the set of $(x, y) \in \Delta^e \times \Delta^e$ for which*

$$\mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (\delta, \epsilon)\} > 0$$

has positive $\mu \otimes \mu$ mass.

Proof. An argument similar to that in the proof of Proposition 3.3 shows that

$$\mathbb{P}^{(x_0,y_0)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t > 0\} > 0$$

for some $(x_0, y_0) \in \Delta^e \times \Delta^e$.

Thus, for any $\epsilon > 0$, we can partition the interval $(0, \infty)$ into the subintervals $(0, \epsilon)$, $[i\epsilon, (i+1)\epsilon)$, $i \geq 0$ and use the Markov property to deduce that there exists a point $(x_1, y_1) \in \Delta^e \times \Delta^e$ such that

$$\mathbb{P}^{(x_1,y_1)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (0, \epsilon)\} > 0. \tag{3.4}$$

By continuity of probability, we can find $0 < \eta < \epsilon < \infty$ such that

$$\mathbb{P}^{(x_1,y_1)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (\eta, \epsilon)\} > 0.$$

By the Markov property,

$$\begin{aligned} 0 &< \mathbb{P}^{(x_1,y_1)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (\eta, \epsilon)\} \\ &= \int_{\Delta^e}\int_{\Delta^e} \widetilde{p}_K(\eta/2, x_1, x)\widetilde{p}_K(\eta/2, y_1, y) \\ &\quad \times \mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t, \text{ for some } t \in (\eta/2, \epsilon - \eta/2)\}\, \mu(dx)\, \mu(dy). \end{aligned}$$

Therefore, the initial points $(x, y) \in \Delta^e \times \Delta^e$ for which the probability

$$\mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (\eta/2, \epsilon - \eta/2)\}$$

is positive form a set with positive $\mu \otimes \mu$ measure. The proof now follows by taking $\delta = \eta/2$. $\square$

We record the following result for the reader's ease of reference.

Lemma 3.9 (Lemma 3.35 of [Bar98])**.** *There exists a constant $c_1 > 1$ such that if $x, y \in \Delta^e$, $r = |x - y|$, then*

$$\mathbb{P}^x\{\widetilde{X}_t = y \ \text{ for some } \ t \in (0, r^{d_w}) \ \text{ and } \ |\widetilde{X}_t - x| \leq c_1 r \ \text{ for all } \ t \leq r^{d_w}\} > 0.$$

Lemma 3.10. *There exists a constant $c > 0$ such that for each point $x \in \Delta$, each open subset $U \subset \Delta^e$, and each time $0 < t \leq c$*

$$\mathbb{P}^x\{\widetilde{Z}_t \in U\} > 0.$$

In particular, $\widetilde{p}_K(t, x, y) > 0$ for all $x, y \in \Delta^e$ and $0 < t \leq c$.

Proof. The following three steps combined with the strong Markov property establish the lemma.

Step 1. There exists a constant $c > 0$ such that starting from $x \in \Delta^e$, the unkilled Brownian motion on the infinite gasket $\widetilde{X}$ will stay within Δ^e up to time c with positive probability.

Step 2. Fix $y \in U$. For all sufficiently small $\eta > 0$,

$$\mathbb{P}^y\{\widetilde{X} \text{ does not exit } U \text{ before time } \eta\} > 0.$$

Step 3. For any $\delta > 0$, $z, y \in \Delta^e$

$$\mathbb{P}^z\{\widetilde{Z} \text{ hits } y \text{ before } \delta\} > 0.$$

Consider Step 1. Note that if $x \in \widetilde{G}$, then (see [Bar98, Equation 3.11]) there exists a constant $c > 0$ such that for the unkilled process $\widetilde{X}$, we have,

$$\mathbb{P}^x\{|\widetilde{X}_t - x| \leq 1/4 \ \text{ for } t \in [0, c]\} > 0.$$

But if $x \in \Delta^e$, then

$$\mathbb{P}^x\{\widetilde{X}_t \in \Delta^e \ \text{ for } t \in [0, c]\} \geq \mathbb{P}^x\{|\widetilde{X}_t - x| \leq 1/4 \ \text{ for } t \in [0, c]\},$$

and the claim follows.

Step 2 is obvious from the right continuity of the paths of the killed Brownian motion $\widetilde{Z}$ at time 0.

Consider Step 3. Fix $z, y \in \Delta^e$ and $0 < \delta \le |z - y|$. Let $\mathcal{S}_n$ be the n-th approximating graph of $\widetilde{G}$ with the set of vertices $\mathcal{V}_n$. Choose n large enough so that we can find points z_0 and y_0 in $\mathcal{V}_n$ close to z and y respectively so that

$$|z - z_0| \le \frac{\delta}{3}, \quad |y - y_0| \le \frac{\delta}{3}$$

and

$$B(z, c_1|z - z_0|) \subseteq \Delta^e, \quad B(y_0, c_1|y - y_0|) \subseteq \Delta^e$$

where c_1 is as in Lemma 3.9 and the notation $B(u, r)$ denotes the intersection with the infinite gasket $\widetilde{G}$ of the closed ball in the plane of radius r around the point u.

The length of a shortest path γ lying $\mathcal{S}_n$ between z_0 and y_0 is the same as their distance in the original metric $\rho_{\widetilde{G}}(z_0, y_0)$. Moreover, for any two points p and p' on γ, the length of the segment of γ between p and p' is the same as their distance in the original metric $\rho_{\widetilde{G}}(p, p')$.

Thus, we can choose $m + 1$ equally spaced points $z_0, z_1, \ldots, z_m = y_0$ on γ such that

$$\rho_{\widetilde{G}}(z_{i+1}, z_i) = \frac{1}{m} \rho_{\widetilde{G}}(z_0, y_0) \quad \text{for each } i.$$

Since γ is compact, $\operatorname{dist}(\gamma, \partial\Delta^e) > 0$. Thus we can choose m large so that

$$B(z_i, c_1|z_{i+1} - z_i|) \subseteq \Delta^e \quad \text{for each } i.$$

By repeated application of Lemma 3.9 and the strong Markov property, we conclude that the probability that $\widetilde{Z}$ hits y starting from z before the time

$$T_m := |z - z_0|^{d_w} + |y - y_0|^{d_w} + \sum_{i=0}^{m-1} |z_{i+1} - z_i|^{d_w}$$

is strictly positive. Step 3 follows immediately since

$$T_m \le \left(\frac{\delta}{3}\right)^{d_w} + \left(\frac{\delta}{3}\right)^{d_w} + \text{constant} \times m \times \frac{1}{m^{d_w}} |z_0 - y_0|^{d_w} \le \delta$$

for m sufficiently large, because $d_w > 1$. □

Lemma 3.11. *Let $\widetilde{Z}'$ and $\widetilde{Z}''$ be two independent copies of the killed Brownian motion $\widetilde{Z}$. For any $0 < \delta < \beta$, the map*

$$(x, y) \mapsto \mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \quad \textit{for some } t \in (\delta, \beta)\}$$

is continuous on $\Delta^e \times \Delta^e$.

Proof. We have

$$\begin{aligned}
&\mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \quad \text{for some } t \in (\delta, \beta)\} \\
&= \int_{\Delta^e} \int_{\Delta^e} \widetilde{p}_K(\delta, x, x')\, \widetilde{p}_K(\delta, y, y') \\
&\quad \times \mathbb{P}^{(x',y')}\{\widetilde{Z}'_t = \widetilde{Z}''_t \quad \text{for some } t \in (0, \beta - \delta)\}\, \mu(dx')\mu(dy'),
\end{aligned}$$

and the result follows from the continuity of $z \mapsto \widetilde{p}_K(\delta, z, z')$ for each $z' \in \Delta^e$. □

Proof of Theorem 3.5. For any $x, y \in \Delta$,

$$\begin{aligned} &\mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (\delta, \beta)\} \\ &= \int_{\Delta^e}\int_{\Delta^e} \widetilde{p}_K(\delta/2, x, x')\widetilde{p}_K(\delta/2, y, y') \\ &\quad \times \mathbb{P}^{(x',y')}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (\delta/2, \beta - \delta/2)\}\, \mu(dx')\, \mu(dy') > 0, \end{aligned} \tag{3.5}$$

by Lemmas 3.8, 3.10 and 3.11.

Applying Lemma 3.11 and equation (3.5) and the fact that a continuous function achieves its minimum on a compact set, we have for any $\Delta \in \widetilde{\mathcal{T}}_0$ that

$$\underline{q}(\Delta) := \inf_{x,y\in\Delta} \mathbb{P}^{(x,y)}\{\widetilde{Z}'_t = \widetilde{Z}''_t \text{ for some } t \in (0, \beta)\} > 0.$$

Note that for any two $\Delta_1, \Delta_2 \in \widetilde{\mathcal{T}}_0$ which do not contain the origin, there exists a *local isometry* between the corresponding extended triangles Δ_1^e, Δ_2^e. Since the unkilled Brownian motion $\widetilde{X}$ in $\widetilde{G}$ is invariant with respect to local isometries,

$$\underline{q}(\Delta_1) = \underline{q}(\Delta_2).$$

Given two independent copies $\widetilde{X}'$ and $\widetilde{X}''$ of $\widetilde{X}$, set

$$\underline{p} := \inf_{\Delta\in\widetilde{\mathcal{T}}_0} \inf_{x,y\in\Delta} \mathbb{P}^{(x,y)}\{\widetilde{X}'_t = \widetilde{X}''_t \text{ for some } t \in (0, \beta)\}.$$

The above observations enable us to conclude that $\underline{p} > 0$.

For the infinite gasket, if $\Delta \in \widetilde{\mathcal{T}}_n$, then $2^n\Delta \in \widetilde{\mathcal{T}}_0$ and the scaling property of Brownian motion on the infinite gasket gives us that for any $\Delta \in \widetilde{\mathcal{T}}_n$

$$\begin{aligned} &\inf_{x,y\in\Delta} \mathbb{P}^{(x,y)}\{\widetilde{X}'_t = \widetilde{X}''_t \text{ for some } t \in (0, 5^{-n}\beta)\} \\ &= \inf_{x,y\in 2^n\Delta} \mathbb{P}^{(x,y)}\{\widetilde{X}'_t = \widetilde{X}''_t \text{ for some } t \in (0, \beta)\}. \end{aligned}$$

Therefore, for any $\Delta \in \widetilde{\mathcal{T}}_n$ and any $x, y \in \Delta$,

$$\mathbb{P}^{(x,y)}\{\widetilde{X}'_t = \widetilde{X}''_t \text{ for some } t \in (0, 5^{-n}\beta)\} \geq \underline{p}. \tag{3.6}$$

□

Corollary 3.12. *The Brownian motions $\widetilde{X}$ and X on the infinite and finite gaskets both satisfy Assumption 2.4.*

Proof. By Theorem 3.5 and the Blumenthal zero-one law, we have for two independent Brownian motions $\widetilde{X}'$ and $\widetilde{X}''$ on $\widetilde{G}$ and any point $(x, x) \in \widetilde{G} \times \widetilde{G}$ that

$$\mathbb{P}^{(x,x)}\{\text{there exists } 0 < t < \epsilon \text{ such that } \widetilde{X}'_t = \widetilde{X}''_t \text{ for all } \epsilon > 0\} = 1.$$

Lemma 3.11 then gives the claim for $\widetilde{X}$. The proof for X is similar. □

4. Instantaneous coalescence on the gasket

We will establish the following three results in this section after obtaining some preliminary estimates.

Theorem 4.1 (Instantaneous Coalescence). *(a) Let Ξ be the set-valued coalescing Brownian motion process on $\widetilde{G}$ with Ξ_0 compact. Almost surely, Ξ_t is a finite set for all $t > 0$.*
(b) The conclusion of part (a) also holds for the set-valued coalescing Brownian motion process on G

Theorem 4.2 (Continuity at time zero). *(a) Let Ξ be the set-valued coalescing Brownian motion process on $\widetilde{G}$ with Ξ_0 compact. Almost surely, Ξ_t converges to Ξ_0 as $t \downarrow 0$.*
(b) The conclusion of part (a) also holds for the set-valued coalescing Brownian motion process on G.

Theorem 4.3 (Instantaneous local finiteness). *Let Ξ be the set-valued coalescing Brownian motion process on $\widetilde{G}$ with Ξ_0 a possibly unbounded closed set. Almost surely, Ξ_t is a locally finite set for all $t > 0$.*

Lemma 4.4 (Pigeon hole principle). *Place M balls in m boxes and allow any two balls to be paired off together if they belong to the same box. Then, the maximum number of disjoint pairs of balls possible is at least $(M - m)/2$.*

Proof. Note that in an optimal pairing there can be at most one unpaired ball per box. It follows that the number of paired balls is at least $M - m$ and hence the number of pairs is at least $(M - m)/2$. □

Define the ε-fattening of a set $A \subseteq \widetilde{G}$ to be the set $A^\varepsilon := \{y \in \widetilde{G} : \exists x \in A, |y - x| < \varepsilon\}$. Define the ε-fattening of a set $A \subseteq G$ in G similarly. Recall the constants $\underline{p}$ and β from Theorem 3.5. Set $\gamma := 1/(1 - \underline{p}/5) > 1$. Given a finite subset A of $\widetilde{G}$ or G and a time-interval $I \subseteq \mathbb{R}_+$, define the random variable $\mathcal{R}(A; I)$ to be the *range* of the set-valued coalescing process Ξ in the finite or the infinite gasket during time I with initial state A; that is,

$$\mathcal{R}(A; I) := \bigcup_{s \in I} \Xi_s.$$

Define a stopping time for the same process Ξ by $\tau_m^A := \inf\{t : \#\Xi_t \le m\}$.

Lemma 4.5. *(a) Let Ξ be the set-valued coalescing Brownian motion process in the infinite gasket with $\Xi_0 = A$, where $A \subset \widetilde{G}$ of cardinality n such that A^ε for some $\varepsilon > 0$ is contained in an extended triangle Δ^e of $\widetilde{G}$. Then, there exist constants C_1 and C_2 which may depend on ε but are independent of A such that*

$$\begin{aligned}(4.1)\qquad &\mathbb{P}\left\{\tau^A_{\lceil n\gamma^{-1}\rceil} > 25\beta n^{-\log_3 5}\text{or } \mathcal{R}(A, [0, \tau_{\lceil n\gamma^{-1}\rceil}]) \not\subseteq A^{\varepsilon n^{-(1/6)\log_3 5}}\right\} \\ &\le C_1 \exp(-C_2 n^{1/3}).\end{aligned}$$

(b) The same inequality holds for the set-valued coalescing coalescing Brownian motion process in the finite gasket.

Proof. (a) For any integer $b \ge 1$, the set A can be covered by at most 2×3^b b-triangles. Put

$$b_n := \max\{b : 2 \times 3^b \le n/2\},$$

or, equivalently,

$$b_n = \lfloor \log_3(n/4) \rfloor.$$

By Lemma 4.4, at time $t = 0$ it is possible to form at least $n/2 - n/4 = n/4$ disjoint pairs of particles, where two particles are deemed eligible to form a pair if they belong to the same b_n-triangle. Fix such an (incomplete) pairing of particles. Define a new "partial" coalescing system involving n particles, where a particle is only allowed to coalesce with the one it has been paired up with and after such a coalescence occurs the two partners in the pair both follow the path of the particle having the lower rank among the two. The number of surviving distinct particles in this partial coalescing system is at least as great as the number of surviving particles in the original coalescing system.

By Theorem 3.5, the probability that a pair in the partial coalescing system coalesces before time $t_n := \beta 5^{-b_n}$ is at least $\underline{p}$, independently of the other pairs. Thus, the number of coalescence by time t_n in the partial coalescing system stochastically dominates a random variable that is distributed as the number of successes in $n/4$ independent Bernoulli trials with common success probability $\underline{p}$. By Hoeffding's inequality, the probability that a random variable with the latter distribution takes a value $n\underline{p}/5$ or greater is at least $1 - e^{-C_1' n}$ for some constant $C_1' > 0$. Thus, the probability that the number of surviving particles in the original coalescing system drops below $\lceil (1 - \underline{p}/5)n \rceil = \lceil n\gamma^{-1} \rceil$ by time $t_n \le 25\beta n^{-\log_3 5}$ is at least $1 - e^{-C_1' n}$.

From Lemma 3.2(a) and the fact that during a fixed time interval the maximum displacement of particles in the coalescing system is always bounded by the maximum displacement of independent particles starting from the same initial configuration, the probability that over a time interval of length $25\beta n^{-\log_3 5}$ one of the coalescing particles has moved more than a distance $\varepsilon n^{-(1/6)\log_3 5}$ from its original position is bounded by

$$\begin{aligned}
&2nc_1 \exp\Big(- c_2((\varepsilon n^{-(1/6)\log_3 5})^{d_w} (25\beta n^{-\log_3 5})^{-1})^{1/(d_w - 1)}\Big) \\
&\le 2\exp\Big(\log n - C_2' (n^{(1/2)\log_3 5})^{1/(d_w-1)}\Big) \\
&\le C_1 \exp(-C_2 n^{(1/4)\log_3 5}) \\
&\le C_1 \exp(-C_2 n^{1/3}).
\end{aligned}$$

(b) The proof is identical to part (a). It uses Corollary 3.6 in place of Theorem 3.5 and Lemma 3.2(b) in place of Lemma 3.2(a). □

Lemma 4.6. *(a) Let Ξ be the set-valued coalescing Brownian motion process in the infinite gasket with $\Xi_0 = A$. Fix $\varepsilon > 0$. Set $\nu_i := \varepsilon\gamma^{-(1/6)\log_3 5 \times i}$ and $\eta_i = 25\beta\gamma^{-i\log_3 5}$ for $i \ge 1$. There are positive constants $C_1 = C_1(\varepsilon)$ and $C_2 = C_2(\varepsilon)$ such that*

$$\mathbb{P}\left\{ \tau^A_{\lceil \gamma^k \rceil} > \sum_{i=k+1}^m \eta_i \text{ or } \mathcal{R}(A; [0, \tau^A_{\lceil \gamma^k \rceil}]) \nsubseteq (A)^{\sum_{i=k+1}^m \nu_i} \right\} \le \sum_{i=k+1}^m C_1 \exp(-C_2 \gamma^{i/3}),$$

uniformly for all sets A of cardinality $\lceil \gamma^m \rceil$ such that the fattening $A^{\sum_{i=k+1}^m \nu_i}$ is contained in some extended triangle Δ^e of $\widetilde{G}$.
(b) The analogous inequality holds for the set-valued coalescing Brownian motion process in the finite gasket.

Proof. Fix an extended triangle Δ^e of the infinite gasket and a set A such that $\#A = \lceil \gamma^m \rceil$ and $A^{\sum_{i=k+1}^m \nu_i} \subseteq \Delta^e$. We will prove the bound by induction on

m. By the strong Markov property and Lemma 4.5, we have, using the notation $A_{\tau,m-1} := \Xi_{\tau^A_{\lceil \gamma^{m-1}\rceil}}$,

$$
\begin{aligned}
&\mathbb{P}\Big\{\tau^A_{\lceil \gamma^k\rceil} > \sum_{i=k+1}^{m} \eta_i \text{ or } \mathcal{R}(A;[0,\tau^A_{\lceil \gamma^k\rceil}]) \not\subseteq (A)^{\sum_{i=k+1}^m \nu_i}\Big\} \\
&\quad \le \mathbb{P}\Big\{\tau^A_{\lceil \gamma^{m-1}\rceil} > \eta_m \text{ or } \mathcal{R}(A;[0,\tau^A_{\lceil \gamma^{m-1}\rceil}]) \not\subseteq A^{\nu_m}\Big\} \\
&\qquad + \mathbb{E}\Big[1\Big\{A_{\tau,m-1} \subseteq A^{\nu_m}\Big\} \\
&\qquad\quad \times \mathbb{P}\Big\{\tau^{A_{\tau,m-1}}_{\lceil \gamma^k\rceil} > \sum_{i=k+1}^{m-1} \eta_i \text{ or } \mathcal{R}(A_{\tau,m-1};[0,\tau^{A_{\tau,m-1}}_{\lceil \gamma^k\rceil}]) \not\subseteq A^{\sum_{i=k+1}^{m-1}\nu_i}\Big\}\Big] \\
&\quad \le C_1 \exp(-C_2\gamma^{m/3}) \\
&\qquad + \sup_{A_1:|A_1|=\lfloor \gamma^{m-1}\rfloor, A_1\subseteq A^{\nu_m}} \mathbb{P}\left\{\tau^{A_1}_{\lceil \gamma^k\rceil} > \sum_{i=k+1}^{m-1} \eta_i \text{ or } \mathcal{R}(A_1;[0,\tau^{A_1}_{\lceil \gamma^k\rceil}]) \not\subseteq A_1^{\sum_{i=k+1}^{m-1}\nu_i}\right\}.
\end{aligned}
$$

Since $(A^{\nu_m})^{\nu_{m-1}} \subseteq A^{\nu_m+\nu_{m-1}} \subseteq \Delta^e$, the second term on the last expression can be bounded similarly as

$$
\begin{aligned}
&\sup_{A_1:|A_1|=\lfloor \gamma^{m-1}\rfloor, A_1\subseteq A^{\nu_m}} \mathbb{P}\left\{\tau^{A_1}_{\lceil \gamma^k\rceil} > \sum_{i=k+1}^{m-1} \eta_i \text{ or } \mathcal{R}(A_1;[0,\tau^{A_1}_{\lceil \gamma^k\rceil}]) \not\subseteq A_1^{\sum_{i=k+1}^{m-1}\nu_i}\right\} \\
&\le C_1 \exp(-C_2\gamma^{(m-1)/3}) \\
&+ \sup_{A_2:|A_2|=\lfloor \gamma^{m-2}\rfloor, A_2\subseteq A^{\nu_m+\nu_{m-1}}} \mathbb{P}\left\{\tau^{A_2}_{\lceil \gamma^k\rceil} > \sum_{i=k+1}^{m-2} \eta_i \text{ or } \mathcal{R}(A_2;[0,\tau^{A_2}_{\lceil \gamma^k\rceil}]) \not\subseteq A_2^{\sum_{i=k+1}^{m-2}\nu_i}\right\}.
\end{aligned}
$$

Iterating the above argument, the assertion follows.
(b) Same as part (a). □

Proof of Theorem 4.1. (a) We may assume that $Q := \Xi_0$ is infinite, since otherwise there is nothing to prove. By scaling, it is enough to prove the theorem when Q is contained in G. Let $Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q$ be a sequence of finite sets such that $\#Q_m = \lceil \gamma^m \rceil$ and Q is the closure $\bigcup_{m=1}^\infty Q_m$. By assigning suitable rankings to a system of independent particles starting from each point in $\bigcup_{m=1}^\infty Q_m$, we can obtain coupled set-valued coalescing processes $\Xi^1, \Xi^2, \ldots$ and Ξ with the property that $\Xi_0^m = Q_m$, $\Xi_0 = Q$, and for each $t > 0$,

$$\Xi_t^1 \subseteq \Xi_t^2 \subseteq \ldots \subseteq \Xi_t$$

and Ξ_t is the closure of $\bigcup_{m=1}^\infty \Xi_t^m$.

Fix $\varepsilon > 0$ so that $Q^{\varepsilon \sum_{i=0}^\infty \gamma^{-(1/6)\log_3 5 \times i}}$ is contained in the extended triangle corresponding to G. Set $\nu_i := \varepsilon\gamma^{-(1/6)\log_3 5\times i}$ and $\eta_i := 25\beta\gamma^{-i\log_3 5}$. Fix $t > 0$. Choose k_0 so that $\sum_{i=k_0+1}^\infty \eta_i \le t$. By Lemma 4.6 and the fact that $s \mapsto \#\Xi_s^m$ is non-increasing, we have, for each $k \ge k_0$,

$$\mathbb{P}\left\{\#\Xi_t^m \le \lceil \gamma^k \rceil\right\} \ge 1 - \sum_{i=k+1}^m C_1 \exp(-C_2\gamma^{i/3}).$$

By the coupling, the sequence of events $\{\#\Xi_t^m \le \lceil \gamma^k \rceil\}$ decreases to the event $\{\#\Xi_t \le \lceil \gamma^k \rceil\}$. Consequently, letting $m \to \infty$, we have, for each $k \ge k_0$,

$$\mathbb{P}\left\{\#\Xi_t \le \lceil \gamma^k \rceil\right\} \ge 1 - \sum_{i=k+1}^{\infty} C_1 \exp(-C_2 \gamma^{i/3}).$$

Finally letting $k \to \infty$, we conclude that

$$\mathbb{P}\{\#\Xi_t < \infty\} = 1.$$

(b) Same as part (a). $\square$

Proof of Theorem 4.2. (a) Assume without loss of generality that $Q := \Xi_0$ is infinite and contained in the 1-triangle that contains the origin. By Theorem 4.1, Ξ_t is almost surely finite and hence it can be considered as a random element in $(\mathcal{K}, d_H)$. It is enough to prove that $\lim_{t\downarrow 0} d_H(\Xi_t, \Xi_0) = 0$ almost surely.

Let $Q_1 \subseteq Q_2 \subseteq \cdots$ be a nested sequence of finite approximating sets of Q chosen as in the proof of Theorem 4.1, and let Ξ^m be the corresponding coupled sequence of set-valued processes.

Fix $\delta > 0$. Choose m sufficiently large that $Q \subseteq Q_m^{\delta/2}$. By the right-continuity of the finite coalescing process, we have

$$\lim_{t\downarrow 0} d_H(\Xi_t^m, Q_m) \to 0 \quad a.s.$$

Thus, with probability one, $(\Xi_t^m)^{\delta/2} \supseteq Q_m$ when t is sufficiently close to 0. But, by the choice of Q_m, with probability one,

$$(\Xi_t^m)^{\delta} \supseteq (Q_m)^{\delta/2} \supseteq Q \tag{4.2}$$

for t sufficiently close to 0.

Conversely, choose $\varepsilon > 0$ sufficiently small so that $\sum_{i=}^{\infty} \nu_i < \delta/2$ where ν_i is defined as in Lemma 4.6. Set $s_k := \sum_{i=k+1}^{\infty} \eta_i \sim C\gamma^{-k \log_3 5}$. From Lemma 4.6, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\mathcal{R}(Q_m; [0, s_k]) \not\subseteq (Q)^{\delta}\Big\} \\
&\le \mathbb{P}\Big\{\tau^{Q_m}_{\lceil \gamma^k \rceil} > \sum_{i=k+1}^{m} \eta_i \ \text{ or } \mathcal{R}(Q_m; [0, \tau^{Q_m}_{\lceil \gamma^k \rceil}]) \not\subseteq (Q)^{\delta/2}\Big\} \\
&+ \mathbb{P}\Big\{\text{max displacement of } \lceil \gamma^k \rceil \text{ independent particles in } [0, s_{k-1}] > \delta/2\Big\} \\
&\le \sum_{i=k+1}^{m} C_1 \exp(-C_2 \gamma^{i/3}) + C_1' \lceil \gamma^k \rceil \exp(-C_2' \gamma^k) \\
&\le C_3 \exp(-C_2 \gamma^{k/3}).
\end{aligned} \tag{4.3}$$

By Theorem 4.1 $\#\Xi_s < \infty$ almost surely, and hence $\Xi_s^m = \Xi_s$ for all m sufficiently large almost surely. Therefore, by letting $m \to \infty$ in (4.3), we obtain

$$\mathbb{P}\Big\{\mathcal{R}(Q; [0, s_k]) \not\subseteq (Q)^{\delta}\Big\} \le C_3 \exp(-C_2 \gamma^{k/3}).$$

Letting $k \to \infty$, we deduce that, with probability one,

$$\Xi_t \subseteq Q^{\delta}$$

for t sufficiently close to 0. Combined with (4.2), this gives the desired claim. $\square$

Proof of Theorem 4.3. By scaling, it suffices to show that almost surely, the set $\Xi_t \cap G$ is finite for all $t > 0$. Fix any $0 < t_1 < t_2$. We will show that almost surely, the set $\Xi_t \cap G$ is finite for all $t \in [t_1, t_2]$.

Set $J_{0,1} := G$. Now for $r \geq 1$, the set $2^r G \setminus 2^{r-1} G$ can be covered by exactly $2 \times 3^{r-1}$ many 0-triangles that we will denote by $J_{r,\ell}$ for $1 \leq \ell \leq 2 \times 3^{r-1}$. The collection $\{J_{r,\ell}\}$ forms a covering of the infinite gasket.

Put $Q := \Xi_0$ and let D be a countable dense subset of Q. Associate each point of D with one of the (at most two) 0-triangles to which it belongs. Denote by $D_{r,\ell}$ the subset of D consisting of particles associated with $J_{r,\ell}$. Construct a partial coalescing system starting from D such that two particles coalesce if and only if they collide and both of their initial positions belonged to the same set $D_{r,\ell}$. Let $(\Xi_t^{r,\ell})_{t \geq 0}$ denote the set-valued coalescing process consisting of the (possibly empty) subset of the particles associated with $J_{r,\ell}$. If we preserve the original ranking of the particles that started from $D_{r,\ell}$, then, for each $t > 0$,

$$\Xi_t \subseteq \bigcup_{(r,\ell)} \Xi_t^{r,\ell}$$

and it thus suffices to prove that almost surely, the set $G \cap \bigcup_{(r,\ell)} \Xi_t^{r,\ell}$ is finite for all $t \in [t_1, t_2]$.

Fix $\Delta = J_{r,\ell} \in \widetilde{\mathcal{T}}_0$. Recall the notation of Lemma 4.6. Find $\varepsilon > 0$ such that $\Delta^{\sum_{i=0}^{\infty} \nu_i} \subset \Delta^e$. Let $A_1 \subseteq A_2 \subseteq \ldots$ be an increasing sequence of sets such that $\bigcup_m A_m = D_{r,\ell}$. Construct coupled set-valued coalescing processes $\widetilde{\Xi}^1 \subseteq \widetilde{\Xi}^2 \subseteq \ldots \subseteq \Xi^{r,\ell}$ such that $\widetilde{\Xi}_0^m = A_m$. Note that by Lemma 4.6

$$
\begin{aligned}
&\mathbb{P}\Big\{\Xi_t^{r,\ell} \cap G \neq \emptyset \text{ for some } t \in [t_1, t_2]\Big\} \\
&\quad = \lim_{m\to\infty} \mathbb{P}\Big\{\widetilde{\Xi}_t^m \cap G \neq \emptyset \text{ for some } t \in [t_1, t_2]\Big\} \\
&\quad \leq \limsup_{m\to\infty} \mathbb{P}\Big\{\tau_{\lceil\gamma^r\rceil}^{A_m} > \sum_{i=r+1}^{\infty} \eta_i \text{ or } \Xi^m_{\tau^{A_m}_{\lceil\gamma^r\rceil}} \not\subseteq \Delta^e \text{ or max displacement} \\
&\qquad \text{of the remaining } \lceil\gamma^r\rceil \text{ coalescing particles in } [\tau^{A_m}_{\lceil\gamma^r\rceil}, t_2] > (r - 3/2)\Big\} \\
&\quad \leq \limsup_{m\to\infty} \mathbb{P}\Big\{\tau_{\lceil\gamma^r\rceil}^{A_m} > \sum_{i=r+1}^{\infty} \eta_i \text{ or } \Xi^m_{\tau^{A_m}_{\lceil\gamma^r\rceil}} \not\subseteq \Delta^e\Big\} \\
&\qquad + \mathbb{P}\Big\{\text{max displacement of } \lceil\gamma^r\rceil \text{ independent particles in } [0, t_2] > (r - 3/2)\Big\} \\
&\quad \leq C_1' \exp(-C_2 \gamma^{r/3}) + 2c_1 \lceil\gamma^r\rceil \exp\Big(-c_2((r-3/2)^{d_w}/t_2)^{1/(d_w-1)}\Big) \\
&\quad \leq C_3 \exp(-C_4 \gamma^{r/3})
\end{aligned}
$$

for some constants $C_3, C_4 > 0$ that may depend on t_2 but are independent of r and ℓ. The first of the above inequalities follows from the fact that

$$\inf_{x \in J_{r,\ell},\ y \in G} |x - y| \geq 2^{r-1} - 1 \geq r - 1,$$

which implies that Δ^e is at least at a distance $(r - 3/2)$ away from G.

Now by a union bound,

$$\mathbb{P}\Big\{\Xi_t^{r,\ell} \cap G \neq \emptyset \text{ for some } t \in [t_1, t_2] \text{ and for some } \ell\Big\} \leq 2 \times 3^{r-1} C_3 \exp(-C_4 \gamma^{r/3}).$$

By the Borel-Cantelli lemma, the events $\Xi_t^{r,\ell} \cap G \neq \emptyset$ for some $t \in [t_1, t_2]$ happen for only finitely many (r, ℓ) almost surely. This combined with the fact that $\#\Xi_t^{r,\ell} < \infty$ for all $t > 0$ almost surely gives that

$$\# \bigcup_{(r,\ell)} (G \cap \Xi_t^{r,\ell}) < \infty \text{ for all } t \in [t_1, t_2]$$

almost surely. □

5. Instantaneous coalescence of stable particles

5.1. Stable processes on the real line and unit circle. Let $X = (X_t)_{t \geq 0}$ be a (strictly) stable process with index $\alpha > 1$ on $\mathbb{R}$. The characteristic function of X_t can be expressed as $\exp(-\Psi(\lambda)t)$ where $\Psi(\cdot)$ is called the characteristic exponent and has the form

$$\Psi(\lambda) = c|\lambda|^\alpha \big(1 - i\upsilon \mathrm{sgn}(\lambda) \tan(\pi\alpha/2)\big), \quad \lambda \in (-\infty, \infty), i = \sqrt{-1}.$$

where $c > 0$ and $\upsilon \in [-1, 1]$. The Lévy measure of Π is absolutely continuous with respect to Lebesgue measure, with density

$$\Pi(dx) = \begin{cases} c^+ x^{-\alpha-1} dx & \text{if } x > 0, \\ c^- |x|^{-\alpha-1} dx & \text{if } x < 0, \end{cases}$$

where c^+, c^- are two nonnegative real numbers such that $\upsilon = (c^+ - c^-)/(c^+ + c^-)$. The process is symmetric if $c^+ = c^-$ or equivalently $\upsilon = 0$. The stable process has the scaling property

$$X \stackrel{d}{=} (c^{-1/\alpha} X_{ct})_{t \geq 0}$$

for any $c > 0$. If we put $Y_t := e^{2\pi i X_t}$, then the process $(Y_t)_{t \geq 0}$ is the stable process with index $\alpha > 1$ on the unit circle $\mathbb{T}$.

We define the distance between two points on $\mathbb{T}$ as the length of the shortest path between them and continue to use the same notation $|\cdot|$ as for the Euclidean metric on the real line.

Theorem 5.1 (Instantaneous Coalescence)**.** *(a) Let Ξ be the set-valued coalescing stable process on $\mathbb{R}$ with Ξ_0 compact. Almost surely, Ξ_t is a finite set for all $t > 0$.*
(b) The conclusion of part (a) holds for the set-valued coalescing stable process on $\mathbb{T}$.

Theorem 5.2 (Continuity at time zero)**.** *(a) Let Ξ be the set-valued coalescing stable process on $\mathbb{R}$ with Ξ_0 compact. Almost surely, Ξ_t converges to Ξ_0 as $t \downarrow 0$.*
(b) The conclusion of part (a) holds for the set-valued coalescing stable process on $\mathbb{T}$.

Theorem 5.3 (Instantaneous local finiteness)**.** *Let Ξ be the set-valued coalescing stable process on $\mathbb{R}$ with Ξ_0 a possibly unbounded closed set. Almost surely, Ξ_t is a locally finite set for all $t \geq 0$.*

We now proceed to establish hitting time estimates and maximal inequalities for stable processes that are analogous to those established for Brownian motions on the finite and infinite gaskets in Section 3. With these in hand, the proofs of Theorem 5.1 and Theorem 5.2 follow along similar, but simpler, lines to those in the proofs of the corresponding results for the gasket (Theorem 4.1 and Theorem 4.2), and so we omit them. However, the proof of Theorem 5.3 is rather different from that of its gasket counterpart (Theorem 4.3), and so we provide the details at the end of this section.

Lemma 5.4. *Let $Z = X' - X''$ where X' and X'' are two independent copies of X, so that Z is a symmetric stable process with index α. For any $0 < \delta < \beta$,*

$$\mathbb{P}^z\{Z_t = 0 \ \ \textit{for some } t \in (\delta, \beta)\} > 0.$$

Proof. The proof follows from [Ber96, Theorem 16] which says that the single points are not essentially polar for the process Z, the fact that Z has a continuous symmetric transition density with respect to Lebesgue measure, and the Markov property of the Z. □

It is well-know that symmetric stable process Z on $\mathbb{R}$ hits points (see, for example, [Ber96, Chapter VIII, Lemma 13]). Thus there exists a $0 < \beta < \infty$ so that

$$0 < \mathbb{P}^1\{Z_t = 0 \text{ for some } t \in (0, \beta)\} =: \underline{p} \text{ (say)}.$$

By scaling,

$$\mathbb{P}^\varepsilon\{Z_t = 0 \text{ for some } t \in (0, \beta\varepsilon^\alpha)\} = \underline{p}.$$

Lemma 5.5. *Suppose that X' and X'' are two independent stable processes on $\mathbb{R}$ starting at x' and x''. For any $\varepsilon > 0$,*

$$\inf_{|x'-x''| \le \varepsilon} \mathbb{P}\{X'_t = X''_t \ \textit{for some } t \in (0, \beta\varepsilon^\alpha)\} = \underline{p}.$$

Since $X'_t = X''_t$ always implies that $\exp(2\pi i X'_t) = \exp(2\pi i X''_t)$ (but converse is not true), we have the following corollary of the above lemma.

Corollary 5.6. *If Y' and Y'' are two independent stable processes on $\mathbb{T}$ starting at y' and y'', then for any $\varepsilon > 0$*

$$\inf_{|y'-y''| \le 2\pi\varepsilon} \mathbb{P}\{Y'_t = Y''_t \ \textit{for some } t \in (0, \beta\varepsilon^\alpha)\} \ge \underline{p}.$$

Lemma 5.7 ([Ber96])**.** *Suppose that X is an α-stable process on the real line. There exists a constant $C > 0$ such that*

$$\mathbb{P}^0\left\{\sup_{0 \le s \le 1} |X_s| > u\right\} \le Cu^{-\alpha}, \quad u \in \mathbb{R}_+.$$

Corollary 5.8. *(a) Let $X^1, X^2, \ldots, X^n$ be independent stable processes of index $\alpha > 1$ on $\mathbb{R}$ starting from $x^1, x^2, \ldots, x^n$ respectively. Then for each $x \in \mathbb{R}_+$ and $t > 0$,*

$$\mathbb{P}\Big\{\sup_{0 \le s \le t} |X^i_s - x^i| > u \textit{ for some } 1 \le i \le n\Big\} \le Cntu^{-\alpha}.$$

(b) The same bound holds for n independent stable processes on $\mathbb{T}$ when $u < \pi$.

Again we set $\gamma := 1/(1 - \underline{p}/5) > 1$. Fix $(\alpha - 1)/2 < \eta < \alpha - 1$ and define $h := 1 - (1 + \eta)/\alpha > 0$. Recall the definitions of τ^A_m and $\mathcal{R}(A; I)$.

Lemma 5.9. *Fix $0 < \varepsilon \le 1/2$.*
(a) There is a constant $C_1 = C_1(\varepsilon)$ such that Ξ be a set-valued coalescing stable process in $\mathbb{R}$ with $\Xi_0 = A$, then

$$\mathbb{P}\left\{\tau^A_{\lceil n\gamma^{-1}\rceil} > \beta(2\ell/n)^\alpha \ \text{ or } \mathcal{R}(A, [0, \tau_{\lceil n\gamma^{-1}\rceil}]) \not\subseteq A^{\varepsilon\ell n^{-h}}\right\} \le C_1 n^{-\eta}, \tag{5.1}$$

where $n = \#A$ and $\ell/2$ is the diameter of A.

(b) Let Ξ be the set-valued coalescing process in $\mathbb{T}$ with $\Xi_0 = A$, where A has cardinality n. Then there exists constant $C_1 = C_1(\varepsilon)$, independent of A, such that

$$\mathbb{P}\left\{\tau^A_{\lceil n\gamma^{-1}\rceil} > \beta(2/n)^\alpha \text{ or } \mathcal{R}(A, [0, \tau_{\lceil n\gamma^{-1}\rceil}]) \not\subseteq A^{\varepsilon n^{-h}}\right\} \le C_1 n^{-\eta}.$$

Proof. (a) Note that $A^{\varepsilon\ell} \subseteq [a - \ell/2, a + \ell/2]$ for some $a \in \mathbb{R}$, and this interval can be divided into $n/2$ subintervals of length $2\ell/n$. We follow closely the proof of Lemma 4.5. By considering a suitable partial coalescing particle system consisting of at least $n/4$ pairs of particles where a pair can only coalesce if they have started from the same subinterval, we have that the number of surviving particles in the original coalescing system is at most $\lceil \gamma^{-1} n\rceil$ within time $t_n := \beta(2\ell/n)^\alpha$ with error probability bounded by $\exp^{-C_1' n}$.

By Lemma 5.8, the maximum displacement of n independent stable particles on $\mathbb{R}$ within time t_n is at most

$$\varepsilon(t_n)^{1/\alpha} n^{(1+\eta)/\alpha} = 2\beta^{1/\alpha}\varepsilon\ell n^{-1+(1+\eta)/\alpha} = 2\beta^{1/\alpha}\varepsilon\ell n^{-h}$$

with error probability at most $c_2 n^{-\eta}$.

(b) The proof for part (b) is similar. □

Using strong Markov property and Lemma 5.9 repetitively as we did in the proof of Lemma 4.6, we can obtain the following lemma. We omit the details.

Lemma 5.10. *Let $0 < \varepsilon \le 1/2, \ell > 0$ be given. Let $\nu_i := \varepsilon\gamma^{-hi}$ and $\eta_i := \beta 2^\alpha \gamma^{-\alpha i}$.*
(a) Given a finite set $A \subset \mathbb{R}$, let Ξ denote the set-valued coalescing stable process in $\mathbb{R}$ with $\Xi_0 = A$. Then, there exist constants $C_2 = C_2(\varepsilon)$ such that

$$\mathbb{P}\Big\{\tau^A_{\lceil\gamma^k\rceil} > \ell^\alpha \sum_{i=k+1}^m \eta_i \text{ or } \mathcal{R}(A; [0, \tau^A_{\lceil\gamma^k\rceil}]) \not\subseteq (A)^{\ell\sum_{i=k+1}^m \nu_i}\Big\} \le C_2\gamma^{-\eta k},$$

uniformly over all sets A such that $A \subseteq [a - \ell/4, a + \ell/4]$ for some $a \in \mathbb{R}$ and $\#A = \lceil\gamma^m\rceil$.
(b) Given a finite set $A \subset \mathbb{T}$, let Ξ denote the set-valued coalescing stable process in $\mathbb{T}$ with $\Xi_0 = A$. Then, there exist constants $C_2 = C_2(\varepsilon)$ such that

$$\mathbb{P}\Big\{\tau^A_{\lceil\gamma^k\rceil} > \sum_{i=k+1}^m \eta_i \text{ or } \Xi_{\tau^A_{\lceil\gamma^k\rceil}} \not\subseteq (A)^{\sum_{i=k+1}^m \nu_i}\Big\} \le C_2\gamma^{-\eta k},$$

uniformly over all sets $A \subseteq \mathbb{T}$ such that $\#A = \lceil\gamma^m\rceil$.

Proof of Theorem 5.3. By scaling, it is enough to show that for each $0 < t_1 < t_2 < \infty$, almost surely, the set $\Xi_t \cap [-1, 1]$ is finite for each $t \in [t_1, t_2]$. Set $d := 2/\eta$. For $r \ge 1$, define

$$J_{r,1} := \Big[-\sum_{j=1}^r j^d, -\sum_{j=1}^{r-1} j^d\Big) \quad \text{and} \quad J_{r,2} := \Big[\sum_{j=1}^{r-1} j^d, \sum_{j=1}^r j^d\Big).$$

Then the collection $\{J_{r,i}\}_{r\ge1,i=1,2}$ forms a partition of the real line into bounded sets. Note that $\inf_{x\in[-1,1],y\in J_{r,i}} |x - y| \asymp r^{d+1}$ as $r \to \infty$.

Let D be a countable dense subset of Q. Run a partial coalescing system starting from D such that two particles coalesce if and only if they collide and both belonged initially to the same $J_{r,i}$. Let $(\Xi^{r,i}_t)_{t\ge0}$ denote the set-valued coalescing process consisting of the (possibly empty) subset of the particles starting from $D \cap J_{r,i}$. By a suitable coupling, for each $t > 0$,

$$\Xi_t \subseteq \bigcup_{r,i} \Xi_t^{r,i}$$

and it thus suffices to prove that the set $[-1,1] \cap \Xi_t^{r,i}$ is empty for all $t \in [t_1, t_2]$ for all but finitely many pairs (r,i) almost surely.

Fix a pair (r,i). Find $\varepsilon > 0$ such that $\sum_{i=0}^{\infty} \nu_i \leq 1/2$ which implies that $(J_{r,i})^{\sum_{i=0}^{\infty} \nu_i} \subseteq (J_{r,i})^{r^d}$. Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing sequence of finite sets such that for $\bigcup_m A_m = D \cap J_{r,i}$. Let $\widetilde{\Xi}^m$ be a coalescing set-valued stable processes such that $\widetilde{\Xi}_0^m = A_m$ and couple these processes together so that $\widetilde{\Xi}_t^1 \subseteq \widetilde{\Xi}_t^2 \subseteq \dots \subseteq \Xi_t^{r,i}$. Set $b = b(r) := (2/\eta)\lceil \log_\gamma r \rceil$. Note that by Lemma 5.10, Corollary 5.8, and the fact that there exists $c_1 > 0$ such that for all r sufficiently large

$$\inf_{x \in [-1,1], y \in J_{r,i}} |x - y| - r^d \geq c r^{d+1},$$

we can write

$$\begin{aligned}
&\mathbb{P}\Big\{\Xi_t^{r,i} \cap [-1,1] \neq \emptyset \text{ for some } t \in [t_1,t_2]\Big\} \\
&\quad = \lim_{m\to\infty} \mathbb{P}\Big\{\widetilde{\Xi}_t^m \cap [-1,1] \neq \emptyset \text{ for some } t \in [t_1,t_2]\Big\} \\
&\quad \leq \limsup_{m\to\infty} \mathbb{P}\Big\{\tau^{A_m}_{\lceil \gamma^b \rceil} > \sum_{i=b+1}^{\infty} \eta_i \text{ or } \widetilde{\Xi}^m_{\tau^{A_m}_{\lceil \gamma^b \rceil}} \not\subseteq (J_{r,i})^{r^d} \text{ or max displacement} \\
&\qquad\quad \text{of the remaining } \lceil \gamma^b \rceil \text{ coalescing particles in } [\tau^{A_m}_{\lceil \gamma^b \rceil}, t_2] > c r^{d+1}\Big\} \\
&\quad \leq \limsup_{m\to\infty} \mathbb{P}\Big\{\tau^{A_m}_{\lceil \gamma^b \rceil} > \sum_{i=r+1}^{\infty} \eta_i \text{ or } \widetilde{\Xi}^m_{\tau^{A_m}_{\lceil \gamma^b \rceil}} \not\subseteq (J_{r,i})^{r^d}\Big\} \\
&\qquad + \mathbb{P}\Big\{\text{max displacement of } \lceil \gamma^b \rceil \text{ independent particles in } [0,t_2] > c r^{d+1}\Big\} \\
&\quad \leq C_2 \gamma^{-\eta b} + C_3 \lceil \gamma^b \rceil c r^{-\alpha(d+1)} \leq C_2' r^{-2} + C_3' r^{-\alpha}
\end{aligned}$$

for suitable constants $C_2', C_3' > 0$. The proof now follows from the Borel-Cantelli lemma.

□

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