Relative Perturbation Bounds for the Unitary Polar Factor *

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July 25, 1994

Computer Science Division Technical Report UCB//CSD-94-854, University of California, Berkeley, CA 94720, December, 1994.

Abstract

Let B be an $m \times n$ $(m \geq n)$ complex matrix. It is known that there is a unique *polar decomposition* $B = QH$, where $Q^*Q = I$, the $n \times n$ identity matrix, and H is positive definite, provided B has full column rank. This paper addresses the following question: how much may Q change if B is perturbed to $\widetilde{B} = D_1^* B D_2$? Here D_1 and D_2 are two nonsingular matrices and close to the identities of suitable dimensions.

Known perturbation bounds for complex matrices indicate that in the worst case, the change in Q is proportional to the reciprocal of the smallest singular value of B. In this paper, we will prove that for the above mentioned perturbations to B, the change in Q is bounded only by the distances from D_1 and D_2 to identities!

As an application, we will consider perturbations for one-side scaling, i.e., the case when $G = D^* B$ is perturbed to $\widetilde{G} = D^* \widetilde{B}$, where D is usually a nonsingular diagonal scaling matrix but for our purpose we do not have to assume this, and B and $\widetilde{B}$ are nonsingular.

*This material is based in part upon work supported by Argonne National Laboratory under grant No. 20552402 and the University of Tennessee through the Advanced Research Projects Agency under contract No. DAAL03-91-C-0047, by the National Science Foundation under grant No. ASC-9005933, and by the National Science Infrastructure grants No. CDA-8722788 and CDA-9401156.

Let B be an $m \times n$ ($m \geq n$) complex matrix. It is known that there are Q with orthonormal column vectors, i.e., $Q^*Q = I$, and a unique positive semidefinite H such that

$$B = QH. \tag{1}$$

Hereafter I denotes an identity matrix with appropriate dimensions which should be clear from the context or specified. The decomposition (1) is called the *polar decomposition* of B. If, in addition, B has full column rank then Q is uniquely determined also. In fact,

$$H = (B^*B)^{1/2}, \quad Q = B(B^*B)^{-1/2}, \tag{2}$$

where superscript "$*$" denotes conjugate transpose. The decomposition (1) can also be computed from the *singular value decomposition* (SVD) $B = U\Sigma V^*$ by

$$H = V\Sigma_1 V^*, \quad Q = U_1 V^*, \tag{3}$$

where $U = (U_1, U_2)$ and V are unitary, U_1 is $m \times n$, $\Sigma = \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix}$ and $\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ is nonnegative.

There are many published bounds upon how much the two factor matrices Q and H may change if entries of B are perturbed in arbitrary manner [1, 2, 3, 4, 6, 5, 7, 8, 9]. In these papers, no assumption was made on how B was perturbed unlike what we are going to do here.

In this paper, we obtain some bounds for the perturbations of Q, assuming B is complex and is perturbed to $\widetilde{B} = D_1^* B D_2$, where D_1 and D_2 are two nonsingular matrices and close to the identities of suitable dimensions. Assume also B has full column rank and so do $\widetilde{B} = D_1^* B D_2$. Let

$$B = QH, \quad \widetilde{B} = \widetilde{Q}\widetilde{H} \tag{4}$$

be the polar decompositions of B and $\widetilde{B}$ respectively, and let

$$B = U\Sigma V^*, \quad \widetilde{B} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* \tag{5}$$

be the SVDs of B and $\widetilde{B}$, respectively, where $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2)$, $\widetilde{U}_1$ is $m \times n$, and $\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_1 \\ 0 \end{pmatrix}$ and $\widetilde{\Sigma}_1 = \operatorname{diag}(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$. Assume as usual that

$$\sigma_1 \geq \cdots \geq \sigma_n > 0, \quad \text{and} \quad \widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_n > 0. \tag{6}$$

It follows from (2) and (5) that

$$Q = U_1 V^*, \quad \widetilde{Q} = \widetilde{U}_1 \widetilde{V}^*.$$

In what follows, $\|X\|_{\rm F}$ denotes the Frobenius norm which is the square root of the trace of X^*X. Then

$$\begin{aligned}
\widetilde{U}^*(\widetilde{B} - B)V &= \widetilde{\Sigma}\widetilde{V}^*V - \widetilde{U}^*U\Sigma, \\
\widetilde{U}^*(\widetilde{B} - B)V &= \widetilde{U}^*(D_1^*BD_2 - D_1^*B + D_1^*B - B)V \\
&= \widetilde{U}^*\left[\widetilde{B}(I - D_2^{-1}) + (D_1^* - I)B\right]V \\
&= \widetilde{\Sigma}\widetilde{V}^*(I - D_2^{-1})V + \widetilde{U}^*(D_1^* - I)U\Sigma,
\end{aligned}$$

and similarly

$$\begin{aligned}
U^*(\widetilde{B} - B)\widetilde{V} &= U^*\widetilde{U}\widetilde{\Sigma} - \Sigma V^*\widetilde{V}, \\
U^*(\widetilde{B} - B)\widetilde{V} &= U^*(D_1^*BD_2 - BD_2 + BD_2 - B)\widetilde{V} \\
&= U^*\left[(I - D_1^{-*})\widetilde{B} + B(D_2 - I)\right]\widetilde{V} \\
&= U^*(I - D_1^{-*})\widetilde{U}\widetilde{\Sigma} + \Sigma V^*(D_2 - I)\widetilde{V}.
\end{aligned}$$

Therefore, we obtained two perturbation equations.

$$\widetilde{\Sigma}\widetilde{V}^*V - \widetilde{U}^*U\Sigma = \widetilde{\Sigma}\widetilde{V}^*(I - D_2^{-1})V + \widetilde{U}^*(D_1^* - I)U\Sigma, \tag{7}$$

$$U^*\widetilde{U}\widetilde{\Sigma} - \Sigma V^*\widetilde{V} = U^*(I - D_1^{-*})\widetilde{U}\widetilde{\Sigma} + \Sigma V^*(D_2 - I)\widetilde{V}. \tag{8}$$

The first n rows of the equation (7) yields

$$\widetilde{\Sigma}_1\widetilde{V}^*V - \widetilde{U}_1^*U_1\Sigma_1 = \widetilde{\Sigma}_1\widetilde{V}^*(I - D_2^{-1})V + \widetilde{U}_1^*(D_1^* - I)U_1\Sigma_1. \tag{9}$$

The first n rows of the equation (8) yields

$$U_1^*\widetilde{U}_1\widetilde{\Sigma}_1 - \Sigma_1 V^*\widetilde{V} = U_1^*(I - D_1^{-*})\widetilde{U}_1\widetilde{\Sigma}_1 + \Sigma_1 V^*(D_2 - I)\widetilde{V},$$

on taking conjugate transpose of which, one has

$$\widetilde{\Sigma}_1\widetilde{U}_1^*U_1 - \widetilde{V}^*V\Sigma_1 = \widetilde{\Sigma}_1\widetilde{U}_1^*(I - D_1^{-1})U_1 + \widetilde{V}^*(D_2^* - I)V\Sigma_1. \tag{10}$$

Now subtracting (10) from (9) leads to

$$\begin{aligned}
&\widetilde{\Sigma}_1(\widetilde{U}_1^*U_1 - \widetilde{V}^*V) + (\widetilde{U}_1^*U_1 - \widetilde{V}^*V)\Sigma_1 \qquad (11)\\
&= \widetilde{\Sigma}_1\left[\widetilde{U}_1^*(I - D_1^{-1})U_1 - \widetilde{V}^*(I - D_2^{-1})V\right] \\
&\quad + \left[\widetilde{V}^*(D_2^* - I)V - \widetilde{U}_1^*(D_1^* - I)U_1\right]\Sigma_1.
\end{aligned}$$

Set

$$
\begin{aligned}
X &= \widetilde{U}_1^* U_1 - \widetilde{V}^* V = (x_{ij}), &(12)\\
E &= \widetilde{U}_1^* (I - D_1^{-1}) U_1 - \widetilde{V}^* (I - D_2^{-1}) V = (e_{ij}), &(13)\\
\widetilde{E} &= \widetilde{V}^* (D_2^* - I) V - \widetilde{U}_1^* (D_1^* - I) U_1 = (\widetilde{e}_{ij}). &(14)
\end{aligned}
$$

Then the equation (11) reads $\widetilde{\Sigma}_1 X + X\Sigma_1 = \widetilde{\Sigma}_1 E + \widetilde{E}\Sigma_1$, or componentwisely, $\widetilde{\sigma}_i x_{ij} + x_{ij}\sigma_j = \widetilde{\sigma}_i e_{ij} + \widetilde{e}_{ij}\sigma_j$. Thus

$$
\begin{aligned}
|(\widetilde{\sigma}_i + \sigma_j) x_{ij}| &\leq \sqrt{\widetilde{\sigma}_i^2 + \sigma_j^2}\sqrt{|e_{ij}|^2 + |\widetilde{e}_{ij}|^2}\\
&\Rightarrow \quad |x_{ij}|^2 \leq \frac{\widetilde{\sigma}_i^2 + \sigma_j^2}{(\widetilde{\sigma}_i + \sigma_j)^2}(|e_{ij}|^2 + |\widetilde{e}_{ij}|^2) \leq |e_{ij}|^2 + |\widetilde{e}_{ij}|^2.
\end{aligned}
$$

Summing on i and j for $i, j = 1, 2, \cdots, n$ produces

$$
\|X\|_{\mathrm{F}}^2 = \sum_{i,j=1}^{n} |x_{ij}|^2 \leq \|E\|_{\mathrm{F}}^2 + \|\widetilde{E}\|_{\mathrm{F}}^2. \tag{15}
$$

Notice that

$$
\begin{aligned}
X &= \widetilde{U}_1^* U_1 - \widetilde{V}^* V = \widetilde{V}^* (\widetilde{V}\widetilde{U}_1^* U_1 V^* - I) V = \widetilde{V}^* (\widetilde{Q}^* Q - I) V,\\
&\Rightarrow \|X\|_{\mathrm{F}} = \|\widetilde{Q}^* Q - I\|_{\mathrm{F}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|E\|_{\mathrm{F}} &\leq \|I - D_1^{-1}\|_{\mathrm{F}} + \|I - D_2^{-1}\|_{\mathrm{F}},\\
\|\widetilde{E}\|_{\mathrm{F}} &\leq \|D_2^* - I\|_{\mathrm{F}} + \|D_1^* - I\|_{\mathrm{F}}.
\end{aligned}
$$

Lemma 1

$$
\begin{aligned}
&\|\widetilde{Q}^* Q - I\|_{\mathrm{F}}\\
&\quad \leq \sqrt{\left(\|I - D_1^{-1}\|_{\mathrm{F}} + \|I - D_2^{-1}\|_{\mathrm{F}}\right)^2 + (\|D_2^* - I\|_{\mathrm{F}} + \|D_1^* - I\|_{\mathrm{F}})^2}.
\end{aligned}
$$

When $m = n$, both Q and $\widetilde{Q}$ are unitary. Thus $\|\widetilde{Q}^* Q - I\|_{\mathrm{F}} = \|Q - \widetilde{Q}\|_{\mathrm{F}}$, and Lemma 1 yields

Theorem 1 *Let B and $\widetilde{B} = D_1^* B D_2$ be two $n \times n$ nonsingular complex matrices whose polar decompositions are given by (4). Then*

$$
\begin{aligned}
\|Q - \widetilde{Q}\|_{\mathrm{F}} &\leq \sqrt{\left(\|I - D_1^{-1}\|_{\mathrm{F}} + \|I - D_2^{-1}\|_{\mathrm{F}}\right)^2 + (\|D_2 - I\|_{\mathrm{F}} + \|D_1 - I\|_{\mathrm{F}})^2} \qquad (16)\\
&\leq \sqrt{2}\sqrt{\|I - D_1^{-1}\|_{\mathrm{F}}^2 + \|I - D_2^{-1}\|_{\mathrm{F}}^2 + \|D_2 - I\|_{\mathrm{F}}^2 + \|D_1 - I\|_{\mathrm{F}}^2}.
\end{aligned}
$$

If, however, $m > n$, then it follows from the last $m-n$ rows of the equations (7) and (8) that

$$\begin{aligned} \widetilde{U}_2^* U_1 \Sigma_1 &= \widetilde{U}_2^*(D_1^* - I) U_1 \Sigma_1 \qquad \text{and} \\ U_2^* \widetilde{U}_1 \widetilde{\Sigma} &= U_2^*(I - D_1^{-*}) \widetilde{U}_1 \widetilde{\Sigma}_1. \end{aligned}$$

Since we assume that both B and $\widetilde{B}$ have full column rank, both Σ_1 and $\widetilde{\Sigma}_1$ are nonsingular diagonal matrices. So

$$\widetilde{U}_2^* U_1 = \widetilde{U}_2^*(D_1^* - I) U_1 \quad \text{and} \quad U_2^* \widetilde{U}_1 = U_2^*(I - D_1^{-*}) \widetilde{U}_1.$$

Therefore, we have

$$\|\widetilde{U}_2^* U_1\|_{\mathrm{F}} \le \|D_1^* - I\|_{\mathrm{F}} \quad \text{and} \quad \|U_2^* \widetilde{U}_1\|_{\mathrm{F}} = \|I - D_1^{-*}\|_{\mathrm{F}}. \tag{17}$$

Notice that $(U_1 V^*, U_2) = (Q, U_2)$ and $(\widetilde{U}_1 \widetilde{V}^*, \widetilde{U}_2) = (\widetilde{Q}, \widetilde{U}_2)$ are unitary. Hence $U_2^* Q = 0 = \widetilde{U}_2^* \widetilde{Q}$ and

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\mathrm{F}} &= \|(Q, U_2)^*(Q - \widetilde{Q})\|_{\mathrm{F}} = \left\| \begin{pmatrix} I - Q^* \widetilde{Q} \\ -U_2^* \widetilde{Q} \end{pmatrix} \right\|_{\mathrm{F}} \\ &\le \sqrt{\|I - Q^* \widetilde{Q}\|_{\mathrm{F}}^2 + \|-U_2^* \widetilde{U}_1 \widetilde{V}^*\|_{\mathrm{F}}^2} \\ &\le \sqrt{\|I - Q^* \widetilde{Q}\|_{\mathrm{F}}^2 + \|U_2^* \widetilde{U}_1\|_{\mathrm{F}}^2} \\ &\le \sqrt{\left(\|I - D_1^{-1}\|_{\mathrm{F}} + \|I - D_2^{-1}\|_{\mathrm{F}}\right)^2 + (\|D_2^* - I\|_{\mathrm{F}} + \|D_1^* - I\|_{\mathrm{F}})^2 + \|I - D_1^{-*}\|_{\mathrm{F}}^2}. \end{aligned} \tag{18}$$

Similarly, we have

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\mathrm{F}} &= \|(\widetilde{Q}, \widetilde{U}_2)^*(Q - \widetilde{Q})\|_{\mathrm{F}} = \left\| \begin{pmatrix} \widetilde{Q}^* Q - I \\ \widetilde{U}_2 Q \end{pmatrix} \right\|_{\mathrm{F}} \\ &\le \sqrt{\left(\|I - D_1^{-1}\|_{\mathrm{F}} + \|I - D_2^{-1}\|_{\mathrm{F}}\right)^2 + (\|D_2^* - I\|_{\mathrm{F}} + \|D_1^* - I\|_{\mathrm{F}})^2 + \|D_1^* - I\|_{\mathrm{F}}^2}. \end{aligned} \tag{19}$$

Theorem 2 below follows from (18) and (19).

Theorem 2 *Let A and $\widetilde{A}$ be two $m \times n$ ($m > n$) complex matrices having full column rank and with the polar decompositions (4). Then*

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\mathrm{F}} \le \Big[&\left(\|I - D_1^{-1}\|_{\mathrm{F}} + \|I - D_2^{-1}\|_{\mathrm{F}}\right)^2 \\ &+ (\|I - D_2\|_{\mathrm{F}} + \|I - D_1\|_{\mathrm{F}})^2 + \min\left\{\|I - D_1^{-1}\|_{\mathrm{F}}^2, \|I - D_1\|_{\mathrm{F}}^2\right\}\Big]^{\frac{1}{2}} \\ \le\ & \sqrt{3}\sqrt{\|I - D_2\|_{\mathrm{F}}^2 + \|I - D_2^{-1}\|_{\mathrm{F}}^2 + \|I - D_1\|_{\mathrm{F}}^2 + \|I - D_1^{-1}\|_{\mathrm{F}}^2}. \end{aligned}$$

Now we are in the position to apply Theorem 1 to perturbations for one-side scaling (from the left). Here we consider two $n \times n$ nonsingular matrices $G = D^* B$ and $\widetilde{G} = D^* \widetilde{B}$, where D is a scaling matrix and usually diagonal (but this is not necessary to the theorem that follows). B is nonsingular and usually better conditioned than G itself. Set

$$\Delta B \stackrel{\text{def}}{=} \widetilde{B} - B.$$

$\widetilde{B}$ is also nonsingular by the condition $\|\Delta B\|_2 \|B^{-1}\|_2 < 1$ which will be assumed henceforth. Notice that

$$\widetilde{G} = D^* \widetilde{B} = D^*(B + \Delta B) = D^* B(I + B^{-1}(\Delta B)) = G(I + B^{-1}(\Delta B)).$$

So applying Theorem 1 with $D_1 = 0$ and $D_2 = I + B^{-1}(\Delta B)$ leads to

Theorem 3 *Let $G = D^* B$ and $\widetilde{G} = D^* \widetilde{B}$ be two $n \times n$ nonsingular matrices, and let*

$$G = QH \quad \text{and} \quad \widetilde{G} = \widetilde{Q}\widetilde{H}$$

be their polar decompositions. Set $\Delta B \stackrel{\text{def}}{=} \widetilde{B} - B$. If $\|\Delta B\|_2 \|B^{-1}\|_2 < 1$ then

$$\begin{aligned} \|Q - \widetilde{Q}\|_{\mathrm{F}} &\leq \sqrt{\|B^{-1}(\Delta B)\|_{\mathrm{F}}^2 + \left\| I - (I + B^{-1}(\Delta B))^{-1} \right\|_{\mathrm{F}}^2} \\ &\leq \sqrt{1 + \frac{1}{(1 - \|B^{-1}\|_2 \|\Delta B\|_2)^2}}\, \|B^{-1}\|_2 \|\Delta B\|_{\mathrm{F}}. \end{aligned}$$

One can deal with one-side scaling from the right in the similar way.

Acknowledgement: I thank Professor W. Kahan for his supervision and Professor J. Demmel for valuable discussions.

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