Geometric Continuity of Parametric Curves

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ABSTRACT

Parametric spline curves are typically constructed so that the first n parametric derivatives agree where the curve segments abut. This type of continuity condition has become known as C^n or n^{th} order parametric continuity. We show that the use of parametric continuity disallows many parametrizations which generate geometrically smooth curves.

We define n^{th} order geometric continuity (G^n), develop constraint equations that are necessary and sufficient for geometric continuity of curves, and show that geometric continuity is a relaxed form of parametric continuity. G^n continuity provides for the introduction of n quantities known as shape parameters which can be made available to a designer in a computer aided design environment to modify the shape of curves without moving control vertices. Several applications of the theory are discussed, along with topics of future research.

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1. Introduction

In recent years computer-aided geometric design (CAGD) has relied heavily on mathematical descriptions of objects based on *parametric* functions. A parametric function, such as $\mathbf{F}(u) = (X(u), Y(u))$, defines a mapping from u, called the *domain parameter*, into Euclidean two-space. This function can be used to define a curve by letting u range over some interval I_u of the u axis. If the domain parameter is thought of as time, the parametric function is used to locate the position of the particle in space at a given instant. As time passes, the particle sweeps out a path, thereby tracing the curve (see figure 1.1). A parametric function therefore defines more than just a path; there is also information about the direction and speed of the particle as it moves along the path.

A special kind of parametric function known as a *parametric spline function* is generally used in CAGD. A parametric spline function is a piecewise function where each of the segments is a parametric function. An important aspect of these functions is the manner in which the segments are joined together. The equations that govern this joining are called *continuity constraints*. In CAGD, the continuity constraints are typically chosen to impart a given order of smoothness to the spline. The order of smoothness chosen will naturally be application dependent. For some applications, such as architectural drawing, it is sufficient for the curves to be continuous only in position. Other applications, such as the design of mechanical parts, require first or second order smoothness.

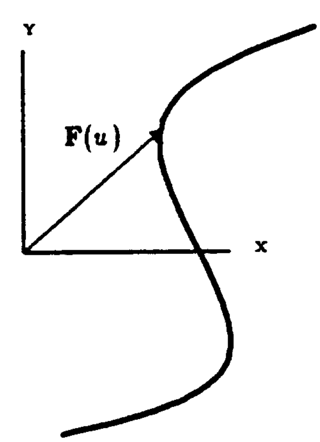


Figure 1.1. *A typical parametric curve.*

We have been intentionally vague about what is meant by "smoothness". In fact, there is more than one type of smoothness; the type that is used should be application dependent. For instance, if parametric splines are being used to define the path of an object in an animation system, it is important for the object to *move* smoothly. It is therefore not enough for the *path* of the object to be smooth, the *speed* of the object as it moves along the path must also be continuous. From elementary kinematics [Arfken70], this type of motion can be guaranteed by requiring continuity of position and the ***first parametric derivative vector***, also known as the ***velocity vector***. If higher order continuity is required, one can demand continuity of the ***second parametric*** derivative, or ***acceleration vector***. However, in CAGD only the resulting path is important; the rate the points along the curve are swept out is irrelevant. This second notion of smoothness allows discontinuities in speed as long as the resulting path is geometrically smooth. We shall refer to the first kind of smoothness as ***parametric continuity***, and to the second kind as ***geometric continuity***. N^{th} order parametric continuity (denoted by C^n) is currently being used in CAGD, largely due to the popularity of spline techniques such as parametric B-splines [Gordon74] [Riesenfeld73]. However, as we will show in section 3, parametric continuity disallows many geometrically smooth curves.

In [Barsky81] and [Barsky85], Barsky defined ***first order geometric continuity*** (G^1) as continuity of the ***unit tangent vector***. The unit tangent vector points in the same direction as the velocity vector, but has unit magnitude. Thus, the direction of motion is continuous, but the speed may change discontinuously. The magnitude of the discontinuity is controlled by the ***shape parameter*** called ***bias***, or β_1. ***Second order geometric continuity*** (G^2) was defined as continuity of the ***curvature vector***. The curvature vector is related to the second parametric derivative, but it is possible for the latter to change discontinuously with the former changing in a continuous manner. The magnitude of the discontinuity is controlled by β_1 and an additional shape parameter called ***tension***, or β_2.

In this paper, we extend the notions of geometric continuity to obtain G^n continuity, for an arbitrary integer $n \geq 0$. We show that for each level of continuity there is a new shape parameter introduced. We call these quantities shape parameters because they can be made available to a designer in a CAGD environment to change the shape of curves and surfaces. Since geometric continuity provides for the introduction of shape parameters, it is desirable to generalize existing spline techniques to obtain their geometrically continuous analogs. For instance, the geometric continuous analog of the C^2 cubic uniform B-spline is the G^2 Beta-spline [Barsky81] [Barsky85] which possesses the two shape parameters β_1 and β_2 mentioned above.

2. Mathematical Preliminaries

A scalar function such as $g(x)$, defined for all values of x on some interval $[a,b]$ of the real line, describes a ***mapping*** from $[a,b]$ to the reals ($\Re$), often written as $g : [a,b] \mapsto \Re$. That is, each point x on $[a,b]$ has an ***image point***, denoted by $g(x)$. The totality of image points is known as the ***image of g*** and is denoted by $g([a,b])$ or $\mathrm{Im}(g)$. That is,

$$g([a,b]) = \mathrm{Im}(g) = \{g(x) \mid x \in [a,b]\}\,.$$

It is also possible to define mappings from the reals into d-space ($\Re^d$). For instance, the function $\mathbf{q}(u) = (q_1(u), q_2(u))$, defined on an interval $[u_0, u_1]$, describes a mapping

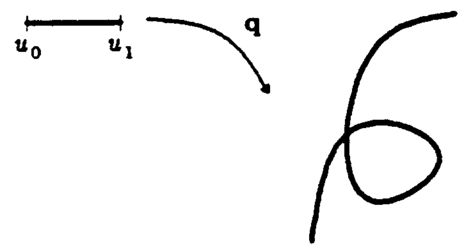


Figure 2.1. *The closed interval* $[u_0, u_1]$ *is mapped into the curve* $\mathcal{C}_{\mathbf{q}}$ *by the continuous parametrization* $\mathbf{q}$.

from $[u_0, u_1]$ into $\Re^2$, denoted by $\mathbf{q} : [u_0, u_1] \mapsto \Re^2$. The image of such a mapping, denoted by $\mathbf{q}([u_0, u_1])$ or $\mathrm{Im}(\mathbf{q})$, represents the point set

$$\mathbf{q}([u_0, u_1]) = \mathrm{Im}(\mathbf{q}) = \{\mathbf{q}(u) \mid u \in [u_0, u_1]\}.$$

Derivatives of such a mapping are defined by component-wise differentiation:

$$\frac{d^i\mathbf{q}}{du^i} = \left(\frac{d^i q_1}{du^i}, \frac{d^i q_2}{du^i}\right).$$

Loosely speaking, a map is said to be *continuous* if all neighboring points are mapped into neighboring points.

Throughout this paper, we will adhere to the following notational conventions: scalars and scalar-valued functions will be written in italics, vectors and vector-valued functions will be written in boldface, and curves (defined below) will be set in a script font. Finally, the statements of definitions, theorems, and lemmas will be set in slanted type to distinguish them more clearly from the surrounding text.

In the discussion that follows, it will be important to maintain the distinction between *functions* (mappings) and *curves*. Whereas a function is a rule for obtaining image points, we shall treat a curve as a set of points. For instance, the curve $\mathcal{G}$ traced when plotting $g(x)$ against x, $x \in [a, b]$, is the set of points

$$\mathcal{G} = \{(x, g(x)) \mid x \in [a, b]\}.$$

Each point on the curve has an X coordinate of x and a Y coordinate of $g(x)$. A more general way to define a curve is *parametrically*. Parametric definitions have the form

$$\mathcal{G} = \{(x(u), y(u)) \mid u \in I_u\}.$$

In a parametric representation, each of the coordinates is a function of u which is allowed to range over the interval I_u. Thus, we are led to the following definition of a curve.

Definition 1: *A curve in d-space is the image of a continuous mapping* $\mathbf{q} : [u_0, u_1] \mapsto \Re^d$, $u_0 < u_1$ *(see figure 2.1). We shall use the notation* $\mathcal{C}_{\mathbf{q}}$ *to mean the curve generated by* $\mathbf{q}$*; that is,* $\mathcal{C}_{\mathbf{q}} = \mathrm{Im}(\mathbf{q})$. $\mathbf{q}$ *is called a parametrization, or parametric representation for* $\mathcal{C}_{\mathbf{q}}$.

It is sometimes convenient to denote a parametrization as the restriction of a function to an interval, as in $\mathbf{q}(u), u \in [u_0, u_1]$. We will generally use $\mathbf{q}(u_0, u_1; u)$ to denote $\mathbf{q}(u), u \in [u_0, u_1]$. The independent variable u is called the *domain parameter* of the representation.

As it will often be necessary to evaluate parametrizations and their derivatives at particular points of interest, we use the notation $\frac{d^i\mathbf{q}}{du^i}\big|_{\mathbf{J}}$ to mean $\frac{d^i\mathbf{q}}{du^i}\big|_{u=u^*}$, where u^* is chosen such that $\mathbf{q}(u^*) = \mathbf{J}$. Although u^* will exist if $\mathbf{J}$ is a point on the curve, it will not be unique if $\mathbf{J}$ is a point where the curve intersects itself. We will say that a point is *simple* if it is not a point of self intersection. To avoid ambiguity, we will only use the notation $\frac{d^i\mathbf{q}}{du^i}\big|_{\mathbf{J}}$ when $\mathbf{J}$ is a simple point. Finally, if u^* is the *left parametric endpoint* ($u^* = u_0$), then $\frac{d^i\mathbf{q}}{du^i}$ is taken to be the i^{th} *right derivative* (c.f. [Buck56]); if u^* is the *right parametric endpoint* ($u^* = u_1$), then $\frac{d^i\mathbf{q}}{du^i}$ is taken to be the i^{th} *left derivative*.

The domain parameter is not *itself* a geometric quantity like a point on a curve; rather, it is used to *indirectly* define geometric quantities. For this reason, there are many (in fact, infinitely many) parametrizations for the same curve. We will say that two parametrizations $\mathbf{q}$ and $\mathbf{r}$ are *equivalent* if $C_{\mathbf{q}} = C_{\mathbf{r}}$. In other words, two parametrizations are equivalent if they generate the same curve (see figure 2.2). As we will see in section 4, a special type of equivalent of parametrizations plays a crucial role in the development of geometric continuity.

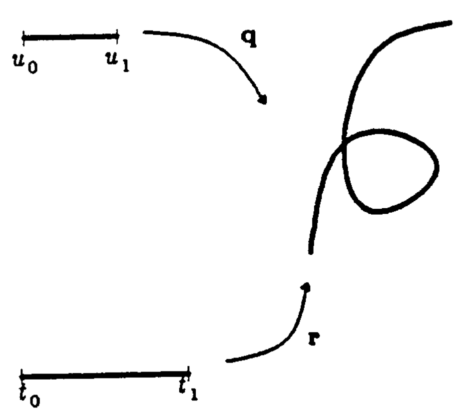


Figure 2.2. *The parametrizations* $\mathbf{q}$ *and* $\mathbf{r}$ *depicted above are equivalent because they have the same image.*

Since curves are defined as point sets, it is natural to speak of the union and intersection of two curves. However, the point set resulting from one or more of these operations may not represent a curve (see figure 2.3). In general, the union of two curves will be a curve if and only if the constituent curves have a non-null intersection. This follows from the fact that a continuous map has a *connected* (positionally continuous) image.

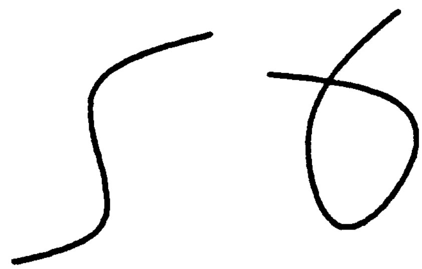


Figure 2.3. *The point set resulting from the union of the two curves above is disconnected, and therefore does not constitute a curve.*

Definition 1 is extremely general; in fact, it is too lax for many applications since it only requires that the point set comprising the curve be positionally continuous. To discuss curves of higher order smoothness, we will impose a given level of differentiability on the parametric representations. We begin with some preliminary definitions.

Definition 2: *C^n and Regularity:*

- *A scalar function $g(x)$ belongs to the class C^n on an interval I if it is n-times continuously differentiable on I. It is regular on I if*

$$\frac{dg}{dx} \neq 0 \qquad \forall x \in I.$$

- *A parametrization $\mathbf{q}(u_0, u_1; u) = (q_1(u), q_2(u), \ldots, q_d(u)), u \in [u_0, u_1]$ is C^n if each of the coordinate functions $q_i(u), i = 1, \ldots, d$ is C^n on $[u_0, u_1]$. It is regular if*

$$\frac{d\mathbf{q}}{du} \neq \mathbf{0} \qquad \forall u \in [u_0, u_1].$$

- *A curve is regular if it can be generated by a regular parametrization.*

Definition 3: *Given a parametrization $\mathbf{q}(u_0, u_1; u)$, the function $u = u(t)$, defined on an interval I_t, represents a regular C^n change of parameter if*

(i) $u(t)$ is regular and C^n on I_t,

(ii) $u(I_t) = [u_0, u_1]$.

Given a parametrization $\mathbf{q}(u_0, u_1; u)$ and a regular C^n change of parameter $u = u(t)$, the equivalent parametrization $\mathbf{r}(t_0, t_1; t)$ is simply computed by functional composition, i.e.,

$$\mathbf{r}(t) = \mathbf{q}(u(t)) \quad t \in [t_0, t_1].$$

In such an instance, $\mathbf{q}$ is said to have been *reparametrized* in terms of t. The reparametrization process is depicted in figure 2.4. A regular change of parameter $u = u(t)$ must either be monotonically increasing or monotonically decreasing since the first derivative is not allowed to vanish. If it is monotonically increasing, it is said to ***maintain sense***, or be ***sense-preserving***. Conversely, if a change of parameter is monotonically decreasing, it is said to ***reverse sense***. These terms are motivatived by the fact that if two parametrizations $\mathbf{r}$ and $\mathbf{q}$ are related by a sense-preserving change of parameter, then as their respective domain parameters are increased over their respective ranges, points on the resulting curve are generated in the same order. Conversely, if they are related by a sense reversing change of parameter, points on the curve are generated in opposite order. We will deal exclusively with sense-preserving changes.

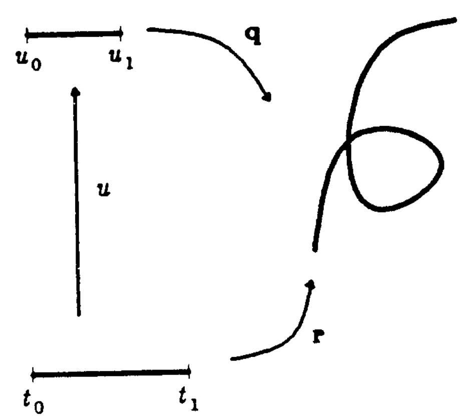


Figure 2.4. *Using the regular C^n change of parameter $u = u(t)$, the interval $[t_0, t_1]$ is mapped into $[u_0, u_1]$, and then into $C_{\mathbf{q}}$ by the mapping $\mathbf{q}$. This is identical to the composite map $\mathbf{r} = \mathbf{q} \circ u$.*

Theorem 2.1: *If $u = u(t)$ represents a regular C^n change of parameter, then its inverse $t = u^{-1}(u)$ also represents a regular C^n change of parameter.*

Proof: Corollary of the *Inverse Function Theorem*, c.f. [Buck56]. ∎

Theorem 2.2: *If $\mathbf{q}(u_0, u_1; u)$ is a regular C^n parametrization, and $u = u(t)$ represents a regular C^n change of parameter on $[t_0, t_1]$, then $\mathbf{r}(t) = \mathbf{q}(u(t)), t \in [t_0, t_1]$, is a regular C^n equivalent parametrization.*

Proof: The fact that $\mathbf{r}$ is C^n follows from the fact that composition of C^n functions results in a C^n function [Buck56]. Regularity of $\mathbf{r}$ can be shown by use of the chain rule:

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{q}}{du}\frac{du}{dt}$$

Since $\mathbf{q}$ and u are regular, $\frac{d\mathbf{r}}{dt}$ does not vanish; therefore $\mathbf{r}$ is regular. ∎

Unless otherwise stated, we will restrict the discussion to regular parametrizations, and hence to regular curves. We do so for two reasons. First, irregular parametrizations allow cusps (visual discontinuities), even though the parametrization may be as differentiable as one wishes (see figure 2.5). Second, as theorems 2.1 and 2.2 show, the transformation between equivalent regular parametrizations is smooth and invertible; this will prove to be useful in the development of a measure of continuity that is based solely on the geometric shape of the resulting curve.

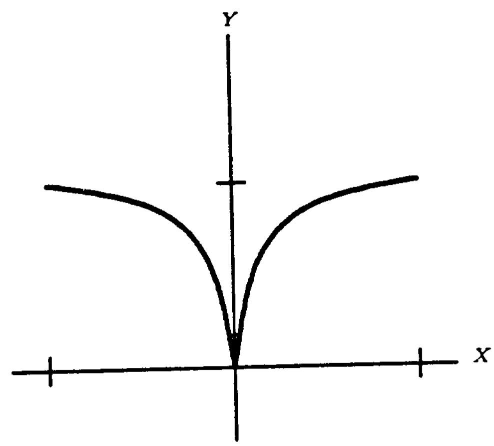


Figure 2.5. *Consider the parametrization* $\mathbf{r}(t) = (t^3, t^2), t \in [-1, 1]$. $C_{\mathbf{r}}$ *is plotted above. Note that a cusp appears at the origin even though* $\mathbf{r}$ *is* C^∞ *everywhere. This occurs because* $\mathbf{r}$ *is irregular at the origin.*

Among all the parametrizations for a curve, there are certain parametrizations that deserve special attention; they are the *arclength parametrizations*. We will only present those properties of arclength parametrizations that are needed for our discussion. The reader is referred to a standard text on differential geometry for a complete treatment [DoCarmo76] [Kreyszig59].

Definition 4: *Let* $\mathbf{r}(t_0, t_1; t)$ *be a regular* C^n, *or piecewise regular* C^n *parametrization (i.e., a finite number of derivative discontinuities are allowed), for* $n \geq 1$. *The total arclength of the curve generated by* $\mathbf{r}$ *is defined to be*

$$length(C_{\mathbf{r}}) = \int_{t_0}^{t_1} \left| \frac{d\mathbf{r}(t')}{dt'} \right| dt'. \tag{2.1}$$

By replacing the upper limit of the integral in (2.1) with t, the integral becomes an increasing function of t, denoted by $s_{\mathbf{r}}(t)$:

$$s_{\mathbf{r}}(t) = \int_{t_0}^{t} \left| \frac{d\mathbf{r}(t')}{dt'} \right| dt'. \tag{2.2}$$

Suppose that $\mathbf{r}$ is regular C^n (not piecewise), then it is easy to show that $s_{\mathbf{r}}(t)$ represents a regular C^n change of parameter, and hence by theorem 2.1, so does its inverse. This implies that arclength $s_{\mathbf{r}}$ may be introduced as a parameter using the sense-preserving regular C^n change of parameter $t = t(s_{\mathbf{r}})$. By theorem 2.2, the *arclength parametrization* $\mathbf{p}_{\mathbf{r}}(0, length(C_{\mathbf{r}}); s_{\mathbf{r}})$ thus obtained will also be regular C^n. In general, a parametrization is said to be an arclength parametrization if the first derivative vector is of unit length. That is, a particle moving along a curve in accordance with an arclength parametrization moves at constant unit speed. The particle may, however, undergo a discontinuous change in direction. By this definition, a given curve can have many arclength parametrizations (see figure 2.6).

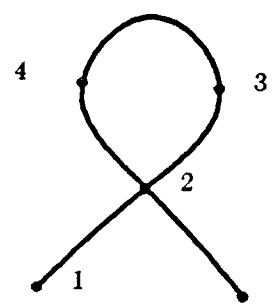


Figure 2.6. *The curve above has four arclength parametrizations. The first generates points in the order 1,2,3,4,2,5, the second in the opposite order 5,2,4,3,2,1. The last two generate points in the order 1,2,4,3,2,5, and 5,2,3,4,2,1.*

To uniquely identify one arclength parametrization, we shall call $\mathbf{p}_{\mathbf{r}}$, computed by equation (2.2), the *natural* parametrization of $\mathbf{r}$. The natural parametrization $\mathbf{p}_{\mathbf{r}}$ generates points on the curve in the same order as $\mathbf{r}$, and is as differentiable as $\mathbf{r}$, but it generates points at a different *rate*. We therefore have as a theorem:

Theorem 2.3: *If $\mathbf{r}$ is a regular C^n parametrization, then $\mathbf{p}_{\mathbf{r}}$, the natural parametrization of $\mathbf{r}$, is also regular C^n.*

3. Parametric Continuity

Let us now examine how continuity has been imposed on parametric functions in CAGD. As mentioned in section 1, it is typical to stitch pieces of parametric functions together to obtain a parametric spline. Borrowing concepts from fields such as numerical analysis and approximation theory, it seems reasonable to require that the derivatives of the pieces agree at the *joint* (the point where the segments abut). This process may be formalized as:

Definition 5: *Let $\mathbf{r}(t_0, t_1; t)$ and $\mathbf{q}(u_0, u_1; u)$ be regular C^n parametrizations such that $\mathbf{r}(t_1) = \mathbf{q}(u_0) = \mathbf{J}$. That is, the "right" endpoint of $\mathbf{r}$ agrees with the "left" endpoint of $\mathbf{q}$ (see figure 3.1). They meet with n^{th} order parametric continuity (C^n) at $\mathbf{J}$ if*

$$\frac{d^k\mathbf{r}}{dt^k}\Big|_{t_1} = \frac{d^k\mathbf{q}}{du^k}\Big|_{u_0} \qquad k = 1, ..., n. \tag{3.1}$$

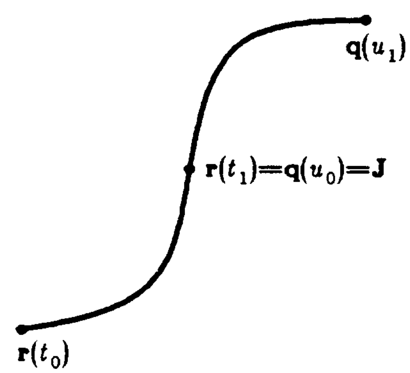


Figure 3.1. *The canonical C^n joint situation with $\mathbf{r}(t)$ and $\mathbf{q}(u)$.*

The problem with parametric continuity is that it places too much emphasis on the particulars of the parametrizations, as the following example shows.

Example 3.1: Let $\mathbf{r}(t) = (2t, t), t \in [0, 1]$ and $\mathbf{q}(u) = (2u + 2, u + 1), u \in [0, 1]$. Differentiation reveals that these parametrizations meet with C^∞ continuity at $(2, 1)$ (see figure 3.2). We can reparametrize $\mathbf{r}$ in terms of $\tilde{t}$ with the regular C^∞ change of parameter $t = 2\tilde{t}$, to obtain $\tilde{\mathbf{r}}(\tilde{t}) = (4\tilde{t}, 2\tilde{t}), \tilde{t} \in [0, 1/2]$. Since $\mathbf{r}$ and $\tilde{\mathbf{r}}$ are equivalent parametrizations, $\mathcal{C}_{\mathbf{r}} = \mathcal{C}_{\tilde{\mathbf{r}}}$, and hence the composite curve $\mathcal{C}_{\mathbf{r}} \cup \mathcal{C}_{\mathbf{q}} = \mathcal{C}_{\tilde{\mathbf{r}}} \cup \mathcal{C}_{\mathbf{q}}$. Note however that $\tilde{\mathbf{r}}$ and $\mathbf{q}$ do not meet with even C^1 continuity, even though the composite curve, and therefore the geometric appearance, has not changed. •

As example 3.1 shows, parametric continuity does not necessarily reflect the smoothness of the resulting *curve*, rather, it is a measure of the smoothness of *parametrizations*. Thus, parametric continuity is not based solely on the geometric properties of the curves generated by the parametrizations. Example 3.1 also shows that parametric continuity disallows many parametrizations that would generate visually smooth curves.

4. Geometric Continuity

Ideally, we would like a measure of continuity that is *parametrization independent* — that is, a measure of continuity that treats parametrizations as tools for describing curves, without introducing parametric artifacts. This can be accomplished by using the parametrizations to describe the point set comprising the composite curve, and then using

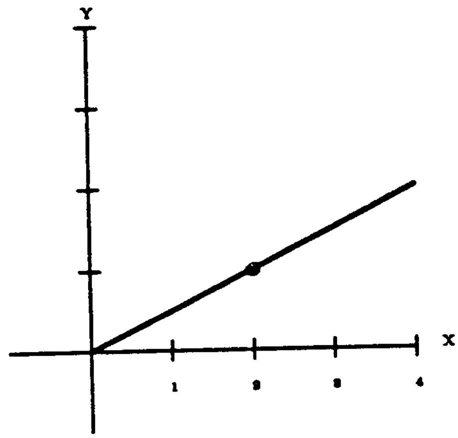


Figure 3.2. *A plot of $C_{\mathbf{r}} \cup C_{\mathbf{q}}$ as defined in example 3.1.*

an arclength parametrization for the composite curve to determine smoothness. If the composite curve has a smooth arclength parametrization, then the initial parametrizations are deemed to meet smoothly, at least in a geometric sense.

If the joint is a simple point of the union curve, then there are only two possible arclength parametrizations in the neighborhood of the joint: one whose sense is the same as $\mathbf{r}$ and $\mathbf{q}$, and one whose sense is opposite. Sense can always be reversed without changing differentiability; thus, as far as questions of continuity are concerned, it is sufficient to consider only arclength parametrizations that have the same sense as $\mathbf{r}$ and $\mathbf{q}$. Every sense-preserving arclength parametrization locally looks like the natural parametrization for $\mathbf{r}$ on one side of the joint, and like the natural parametrization for $\mathbf{q}$ on the other. Thus, if the joint is a simple point, the definition of geometric continuity may be formalized as:

Definition 6: *Let $\mathbf{r}(t_0, t_1; t)$ and $\mathbf{q}(u_0, u_1; u)$ be regular C^n parametrizations such that $\mathbf{r}(t_1) = \mathbf{q}(u_0) = \mathbf{J}$, where $\mathbf{J}$ is a simple point of $C_{\mathbf{r}} \cup C_{\mathbf{q}}$. They meet with n^{th} order geometric (G^n) continuity at $\mathbf{J}$ if the natural parametrizations of $\mathbf{r}$ and $\mathbf{q}$ meet with C^n continuity at $\mathbf{J}$.*

Complications arise if $\mathbf{J}$ is not a simple point. For instance, consider the situation depicted in figure 4.1. The union curve does have a C^1 arclength parametrization, but the sense of $\mathbf{r}$ or $\mathbf{q}$ must be reversed. Thus, there are more than two arclength parametrizations for the union curve in the neighborhood of the joint. This situation is not common in practice and adds complexity to the development that follows. We will therefore assume that joints are simple. For a complete treatment of geometric continuity for joints that are not simple points, see [DeRose85b].

The definitions above are stated in a rather abstract way, so to gain a feeling for their geometric meaning, we will consider the cases of $n = 1$ and $n = 2$ in more detail. For $n = 1$, definition 6 requires that the first derivatives with respect to arclength agree. This is equivalent to requiring that the unit tangent vectors agree at $\mathbf{J}$. Similarly, for

$n = 2$, the second derivative with respect to arclength is required to be continuous, that is, the curvature vectors must match at the joint. These are exactly the requirements of G^1 and G^2 continuity given in [Barsky81] [Barsky85]. Definition 6 therefore represents a generalization of the concepts laid down in [Barsky81] [Barsky85].

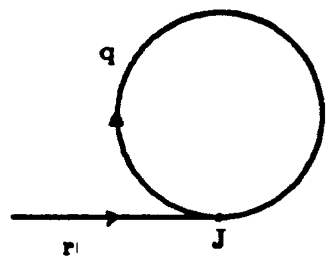


Figure 4.1. *Let* $\mathbf{q}$ *generate a circle with* $\mathbf{q}(u_0) = \mathbf{q}(u_1) = \mathbf{J}$, *and let* $\mathbf{r}$ *generate a line ending at* $\mathbf{J}$. *The sense of these parametrizations is as indicated by the arrows.*

Although definition 6 is stated in a concise manner, it does not directly provide continuity constraints necessary for the construction of geometrically continuous splines. We now undertake the task of determining constraint equations. We begin by presenting the following lemma.

Lemma 4.1: *Let* $\mathbf{r}(t_0, t_1; t)$ *and* $\mathbf{q}(u_0, u_1; u)$ *be regular* C^n *parametrizations such that* $\mathbf{r}(t_1) = \mathbf{q}(u_0) = \mathbf{J}$, *where* $\mathbf{J}$ *is a simple point of* $\mathcal{C}_\mathbf{r} \cup \mathcal{C}_\mathbf{q}$. *These parametrizations meet with* G^n *continuity at* $\mathbf{J}$ *if and only if there exist regular* C^n *equivalent parametrizations* $\widetilde{\mathbf{r}}(\widetilde{t}_0, \widetilde{t}_1; \widetilde{t})$ *and* $\widetilde{\mathbf{q}}(\widetilde{u}_0, \widetilde{u}_1; \widetilde{u})$ *that meet with* C^n *continuity at* $\mathbf{J}$.

Discussion: The lemma states that it is not necessary to check the natural parametrizations of $\mathbf{r}$ and $\mathbf{q}$ for continuity; any regular C^n equivalent parametrizations of $\mathbf{r}$ and $\mathbf{q}$ that meet with C^n continuity at $\mathbf{J}$ will do. This is fortunate since natural parametrizations are difficult to compute in general.

Proof:

Sufficiency: We assume that there exist $\widetilde{\mathbf{r}}$ and $\widetilde{\mathbf{q}}$ satisfying the above conditions. From these, we construct the piecewise parametrization $\mathbf{h}$ defined as

$$\mathbf{h}(v) = \begin{cases} \widetilde{\mathbf{r}}(v), & v \in [\widetilde{t}_0, \widetilde{t}_1]; \\ \widetilde{\mathbf{q}}(v - \widetilde{t}_1 + \widetilde{u}_0), & v \in (\widetilde{t}_1, \widetilde{t}_1 - \widetilde{u}_0 + \widetilde{u}_1]. \end{cases}$$

Since $\widetilde{\mathbf{r}}$ and $\widetilde{\mathbf{q}}$ are each regular C^n, and meet with C^n continuity at $\mathbf{J}$, $\mathbf{h}$ is also a regular C^n parametrization. By theorem 2.3, the natural parametrization for $\mathbf{h}$ is also C^n. In particular, it is C^n in the neighborhood of $\mathbf{J}$, and locally looks like the natural parametrization of $\mathbf{r}$ on one side of $\mathbf{J}$, and like the natural parametrization of $\mathbf{q}$ on the other side. We have therefore established that the natural parametrizations of $\mathbf{r}$ and $\mathbf{q}$ meet with C^n continuity at $\mathbf{J}$, which by definition 6, implies that $\mathbf{r}$ and $\mathbf{q}$ meet with G^n continuity at $\mathbf{J}$.

Necessity: We assume that $\mathbf{r}$ and $\mathbf{q}$ meet with G^n at $\mathbf{J}$, so by definition 6, the natural parametrizations $\mathbf{p_r}$ and $\mathbf{p_q}$ for $\mathbf{r}$ and $\mathbf{q}$, respectively, meet with C^n continuity at $\mathbf{J}$. The existence of $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{q}}$ is established by making the identifications $\tilde{\mathbf{r}} = \mathbf{p_r}$ and $\tilde{\mathbf{q}} = \mathbf{p_q}$. ∎

Lemma 4.1 is the first step toward our goal of a new set of continuity constraints that ensure G^n continuity. However, the lemma is an existence statement and does not directly result in concrete constraints. The following theorem completes the development.

Theorem 4.1: *Let $\mathbf{r}(t_0, t_1; t)$ and $\mathbf{q}(u_0, u_1; u)$ be regular C^n parametrizations such that $\mathbf{r}(t_1) = \mathbf{q}(u_0) = \mathbf{J}$, where $\mathbf{J}$ is a simple point of $\mathcal{C}_\mathbf{r} \cup \mathcal{C}_\mathbf{q}$. They meet with G^n continuity at $\mathbf{J}$ if and only if there exists real numbers $\beta_1 > 0$ and $\beta_2, ..., \beta_n$ such that*

$$\frac{d^k\mathbf{r}}{dt^k}\Big|_\mathbf{J} = \mathbf{g}_k \qquad k = 1, ..., n$$

where $\mathbf{g}_k$ is a vector differential expression which depends on $\mathbf{q}$, $\beta_1, ..., \beta_n$, and is computed using the following rules:

1. *Expand*

$$\frac{d^k\mathbf{q}(u(\tilde{u}))}{d\tilde{u}^k}$$

   using the chain rule, treating $u(\tilde{u})$ as a regular C^n change of parameter. The expression should only involve derivatives of $\mathbf{q}$ with respect to u, and derivatives of u with respect to $\tilde{u}$.

2. *Evaluate at $\mathbf{J}$ and make the substitutions*

$$\beta_j = \frac{d^j u}{d\tilde{u}^j}\Big|_\mathbf{J} \qquad j = 1, ..., k$$

This process generates the following set of constraints

$$\frac{d\mathbf{r}}{dt}\Big|_\mathbf{J} = \beta_1 \frac{d\mathbf{q}}{du}\Big|_\mathbf{J} \tag{4.1-1}$$

$$\frac{d^2\mathbf{r}}{dt^2}\Big|_\mathbf{J} = \beta_1^2 \frac{d^2\mathbf{q}}{du^2}\Big|_\mathbf{J} + \beta_2 \frac{d\mathbf{q}}{du}\Big|_\mathbf{J} \tag{4.1-2}$$

$$\frac{d^3\mathbf{r}}{dt^3}\Big|_\mathbf{J} = \beta_1^3 \frac{d^3\mathbf{q}}{du^3}\Big|_\mathbf{J} + 2\beta_1\beta_2 \frac{d^2\mathbf{q}}{du^2}\Big|_\mathbf{J} + \beta_3 \frac{d\mathbf{q}}{du}\Big|_\mathbf{J} \tag{4.1-3}$$

$$\frac{d^4\mathbf{r}}{dt^4}\Big|_\mathbf{J} = \beta_1^4 \frac{d^4\mathbf{q}}{du^4}\Big|_\mathbf{J} + 6\beta_1^2\beta_2 \frac{d^3\mathbf{q}}{du^3}\Big|_\mathbf{J} + (4\beta_1\beta_3 + 3\beta_2^2) \frac{d^2\mathbf{q}}{du^2}\Big|_\mathbf{J} + \beta_4 \frac{d\mathbf{q}}{du}\Big|_\mathbf{J} \tag{4.1-4}$$

$$\cdots \qquad\qquad \cdots$$

$$\frac{d^n\mathbf{r}}{dt^n}\Big|_\mathbf{J} = \beta_1^n \frac{d^n\mathbf{q}}{du^n}\Big|_\mathbf{J} + \cdots + \beta_n \frac{d\mathbf{q}}{du}\Big|_\mathbf{J} \tag{4.1-n}$$

Discussion: Although the details of the proof are fairly cumbersome, the basic idea is quite simple: ***Don't base continuity on the parametrizations at hand; reparametrize if necessary in order to find ones that meet with C^n continuity.*** The real trick is to express derivatives of the new parametrization ($\tilde{\mathbf{q}}$) in terms of derivatives of the original parametrization ($\mathbf{q}$). The chain rule is used to accomplish this, with the β's determining exactly how $\mathbf{q}$ and $\tilde{\mathbf{q}}$ are related in the neighborhood of the joint. In practice $\tilde{\mathbf{q}}$ is not computed, but the form of the constraints guarantees that such a parametrization exists.

Proof:

Sufficiency: Starting with the assumption that the G^n constraints (constraints (4.1 − 1) through (4.1 − n)) are satisfied, we will show that there exist regular C^n equivalent parametrizations for $\mathbf{r}$ and $\mathbf{q}$ that meet with C^n continuity at $\mathbf{J}$. By lemma 4.1, this will in turn show that $\mathbf{r}$ and $\mathbf{q}$ meet with G^n continuity at $\mathbf{J}$.

We will not reparametrize $\mathbf{r}$, but we will reparametrize $\mathbf{q}$ as follows: Let $u(\tilde{u})$ be any regular C^n change of parameter such that

$$\begin{aligned} u(0) &= u_0 \\ \frac{d^i u}{d\tilde{u}^i}\Big|_{\mathbf{J}} &= \beta_i \qquad i = 1, ..., n. \end{aligned} \tag{4.1}$$

Evaluation at $\mathbf{J}$ is taken to mean evaluation at $\tilde{u} = 0$ since $\mathbf{q}(u(0)) = \mathbf{J}$. A change of parameter satisfying (4.1) always exists for $\beta_1 > 0$. In fact, a polynomial of order n can be used. By theorem 2.2, $\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(u(\tilde{u}))$ is a regular C^n equivalent parametrization for $\mathbf{q}$. Now, consider the first derivative of $\tilde{\mathbf{q}}$ evaluated at the joint:

$$\begin{aligned} \frac{d\tilde{\mathbf{q}}}{d\tilde{u}}\Big|_{\mathbf{J}} &= \frac{du}{d\tilde{u}}\Big|_{\mathbf{J}} \frac{d\mathbf{q}}{du}\Big|_{\mathbf{J}} && \text{(by the chain rule)} \\ &= \beta_1 \frac{d\mathbf{q}}{du}\Big|_{\mathbf{J}} && \text{(by construction)} \\ &= \frac{d\mathbf{r}}{dt}\Big|_{\mathbf{J}}. && \text{(by assumption)} \end{aligned}$$

Thus, $\mathbf{r}$ and $\tilde{\mathbf{q}}$ meet with C^1 continuity at $\mathbf{J}$. Rules 1 and 2 of theorem 4.1 for obtaining the G^n constraints guarantee that the higher derivatives of $\mathbf{r}$ will agree with those of $\tilde{\mathbf{q}}$ at $\mathbf{J}$, and therefore $\mathbf{r}$ and $\tilde{\mathbf{q}}$ meet with C^n continuity at $\mathbf{J}$. By lemma 4.1, $\mathbf{r}$ and $\mathbf{q}$ must meet with G^n continuity at $\mathbf{J}$.

Necessity: Starting with the assumption that $\mathbf{r}$ and $\mathbf{q}$ meet with G^n continuity at $\mathbf{J}$, we must show that the G^n constraints are satisfied for some assignment of the β's.

Let $\mathbf{p_r}(0, length(C_\mathbf{r}); s_r)$ and $\mathbf{p_q}(0, length(C_\mathbf{q}); s_q)$ be the natural parametrizations of $\mathbf{r}$ and $\mathbf{q}$, respectively. Since $\mathbf{r}$ and $\mathbf{q}$ meet with G^n continuity at $\mathbf{J}$, $\mathbf{p_r}$ and $\mathbf{p_q}$ meet with C^n continuity at $\mathbf{J}$. Thus, the arclength parametrization $\mathbf{p}(s)$ for $C_\mathbf{r} \cup C_\mathbf{q}$ defined by

$$\mathbf{p}(s) = \begin{cases} \mathbf{p_r}(s) & s \in [0, length(C_\mathbf{r})]; \\ \mathbf{p_q}(s - length(C_\mathbf{r})) & s \in (length(C_\mathbf{r}), length(C_\mathbf{r} \cup C_\mathbf{q})] \end{cases}$$

is regular C^n. The idea is to use a change of parameter to reparametrize $\mathbf{p}(s)$ so that $\mathbf{r}$ is recovered on the portion of the composite curve corresponding to $\mathcal{C}_\mathbf{r}$. This can be done by constructing a regular C^n change of parameter $s(t)$ defined by

$$s(t) = \begin{cases} s_\mathbf{r}(t) & t \in [t_0, t_1]; \\ g(t) & t \in (t_1, t_2]. \end{cases}$$

For $s(t)$ to represent a regular C^n change of parameter, it is sufficient for $g(t)$ to be any regular C^n function that agrees with $s_\mathbf{r}(t)$ in position and the first n derivatives at t_1, and t_2 must be chosen so that $g(t_2) = length(\mathcal{C}_\mathbf{r} \cup \mathcal{C}_\mathbf{q})$ (see figure 4.2).

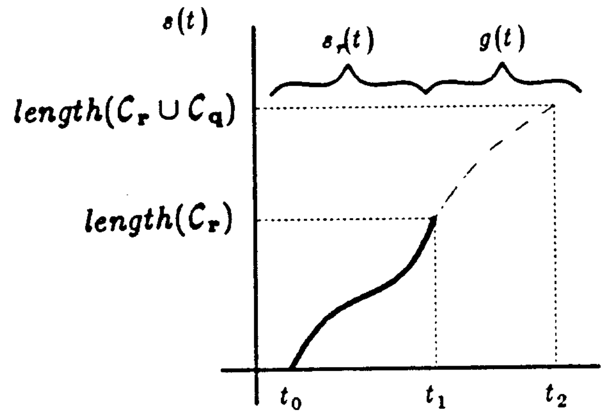


Figure 4.2. *Construction of the regular C^n change of parameter $s = s(t)$.*

Using $s(t)$, an equivalent regular C^n parametrization $\mathbf{h}(t)$ for $\mathcal{C}_\mathbf{r} \cup \mathcal{C}_\mathbf{q}$ is obtained. As mentioned above, the construction of $s(t)$ was chosen so that $\mathbf{h}(t) = \mathbf{r}(t)$ for $t \in [t_0, t_1]$. Specifically,

$$\mathbf{h}(t) = \mathbf{p}(s(t)) = \begin{cases} \mathbf{p}_\mathbf{r}(s_\mathbf{r}(t)) = \mathbf{r}(t) & t \in [t_0, t_1]; \\ \mathbf{p}_\mathbf{q}(g(t)) = \tilde{\mathbf{q}}(t) & t \in (t_1, t_2]. \end{cases}$$

The parametrization $\tilde{\mathbf{q}}$ has two important properties:

i) $\tilde{\mathbf{q}}$ and $\mathbf{r}$ meet with C^n continuity at $\mathbf{J}$. This follows from the fact that $\mathbf{h}$ is C^n at $\mathbf{J}$.

ii) $\tilde{\mathbf{q}}$ and $\mathbf{q}$ are equivalent parametrizations, and are related by a regular C^n change of parameter. This can be verified by noting that

$$\begin{aligned} \tilde{\mathbf{q}} &= \mathbf{p}_\mathbf{q} \circ g \\ &= (\mathbf{q} \circ s_q^{-1}) \circ g \\ &= \mathbf{q} \circ (s_q^{-1} \circ g) \\ &= \mathbf{q} \circ u. \end{aligned}$$

From property *i*), we know that

$$\frac{d\mathbf{r}}{dt}\Big|_{\mathbf{J}} = \frac{d\widetilde{\mathbf{q}}}{dt}\Big|_{\mathbf{J}}.$$

Using property *ii*) together with the chain rule to expand $\frac{d\widetilde{\mathbf{q}}}{dt}\big|_{\mathbf{J}}$ yields

$$\frac{d\mathbf{r}}{dt}\Big|_{\mathbf{J}} = \frac{du}{dt}\Big|_{\mathbf{J}}\frac{d\mathbf{q}}{du}\Big|_{\mathbf{J}}.$$

The quantity $\frac{du}{dt}\big|_{\mathbf{J}}$ is a positive real number (since u is an increasing function of t), so let $\beta_1 = \frac{du}{dt}\big|_{\mathbf{J}}$ to obtain the first order G^n constraint. Thus, if the composite curve $\mathcal{C}_{\mathbf{r}} \cup \mathcal{C}_{\mathbf{q}}$ has a C^1 arclength parametrization, then $\mathbf{r}$ and $\mathbf{q}$ satisfy the first order G^n constraint for some value of $\beta_1 > 0$. In general, the i^{th} derivatives of $\mathbf{r}$ and $\widetilde{\mathbf{q}}$ match at $\mathbf{J}$, so expanding derivatives of $\widetilde{\mathbf{q}}$ in terms of derivatives of $\mathbf{q}$ and u results in the i^{th} G^n constraint. Thus, if the composite curve $\mathcal{C}_{\mathbf{r}} \cup \mathcal{C}_{\mathbf{q}}$ has a C^n arclength parametrization, then $\mathbf{r}$ and $\mathbf{q}$ satisfy the G^n constraint equations for some assignment of $\beta_1 > 0$, and $\beta_2, \ldots, \beta_n$. ∎

Now that constraints have been found, the general idea is to construct splines that satisfy the G^n constraints instead of requiring that parametric derivatives match. Since these constraints are stated in terms of $\beta_1, \ldots, \beta_n$, the resulting spline will have these quantities as *parameters*; they should not, however, be confused with the *domain parameter*. Changing one of the β's will, in general, change the shape of the composite curve, but always in such a way that geometric smoothness is maintained (see section 5); we therefore call the β's *shape parameters*.

Referring back to the G^n constraints, note that the shape parameter β_i is introduced in the constraint relating the i^{th} derivatives of the parametrizations in question. For example, β_1 is introduced in (4.1-1), and therefore controls the difference between the first parametric derivatives, but always in such a way that the resulting composite curve is geometrically smooth. Suppose that $\beta_1 = 1$, implying that the first parametric derivatives agree. In this case, the shape parameter β_2 controls the difference between the second parametric derivatives. If $\beta_1 = 1$ and $\beta_2 = 0$, the first two G^n constraints reduce to the constraints for C^2 continuity. In general, if $\beta_1 = 1$ and $\beta_2 = \cdots = \beta_n = 0$, then G^n continuity reduces to C^n, showing that geometric continuity is a strict generalization of parametric continuity.

It is also important to realize that the shape parameters are *local* to a joint. If the composite curve being constructed is composed of many curve segments, each of the joints possesses its own set of shape parameters. Thus, for a composite curve of m segments ($m - 1$ joints) generated by G^n parametrizations, a total of $(m - 1)n$ shape parameters are introduced. In some applications (see section 5) it is convenient to associate the same values of the n shape parameters with each of the joints, thereby making the assignment of shape parameters *global* to the composite curve.

5. Applications

In this section, we demonstrate the use of the G^n constraints by constructing the geometric continuous analogs of some popular parametric continuous splines.

Example 5.1: Consider a piecewise quadratic spline defined as a weighted sum of *control vertices* $(\mathbf{V}_0, ..., \mathbf{V}_m)$. The sequence of control vertices is collectively called the *control polygon.* Let the i^{th} curve segment be generated by

$$\mathbf{q}_i(u) = \sum_{j=0}^{2} b_j(u)\mathbf{V}_{i+j} \qquad u \in [0,1]. \tag{5.1}$$

We will determine the weighting functions $b_j(u)$ such that $\mathbf{q}_i$ and $\mathbf{q}_{i+1}$ will meet with G^1 continuity, for $i = 0, ..., m-2$. For simplicity, we will assume a global assignment of β_1. Under these assumptions, the positional continuity constraint is

$$\mathbf{q}_{i+1}(0) = \mathbf{q}_i(1), \tag{5.2}$$

and the first derivative constraint is

$$\mathbf{q}_{i+1}^{(1)}(0) = \beta_1\, \mathbf{q}_i^{(1)}(1), \tag{5.3}$$

where superscript (p) has been used to denote the p^{th} parametric derivative. Substituting (5.1) into (5.2) and (5.3) and expanding the summation results in

$$\begin{gathered} b_0(0)\mathbf{V}_{i+1} + b_1(0)\mathbf{V}_{i+2} + b_2(0)\mathbf{V}_{i+3} = b_0(1)\mathbf{V}_i + b_1(1)\mathbf{V}_{i+1} + b_2(1)\mathbf{V}_{i+2} \\ b_0^{(1)}(0)\mathbf{V}_{i+1} + b_1^{(1)}(0)\mathbf{V}_{i+2} + b_2^{(1)}(0)\mathbf{V}_{i+3} = \beta_1\left(b_0^{(1)}(1)\mathbf{V}_i + b_1^{(1)}(1)\mathbf{V}_{i+1} + b_2^{(1)}(1)\mathbf{V}_{i+2}\right). \end{gathered} \tag{5.4}$$

If equations (5.4) are to hold for any choice of the vertices, it must be that

$$\begin{aligned} 0 &= b_0(1) & 0 &= b_0^{(1)}(1) \\ b_0(0) &= b_1(1) & b_0^{(1)}(0) &= \beta_1 b_1^{(1)}(1) \\ b_1(0) &= b_2(1) & b_1^{(1)}(0) &= \beta_1 b_2^{(1)}(1) \\ b_2(0) &= 0 & b_2^{(1)}(0) &= 0. \end{aligned} \tag{5.5}$$

Equations (5.5) represent 8 constraints on the 9 coefficients needed to uniquely determine $b_0(u), b_1(u)$, and $b_2(u)$. The additional constraint is a normalization, which is chosen to be

$$b_0(0) + b_1(0) + b_2(0) = 1.$$

Using an algebraic manipulation system such as Vaxima [Fateman82], the solution of the system of equations for the coefficients of $b_0(u), b_1(u)$ and $b_2(u)$ can be shown to be

$$\begin{aligned} b_0(u) &= \beta_1 \frac{u^2 - 2u + 1}{\beta_1 + 1}, \\ b_1(u) &= \frac{-(\beta_1 + 1)u^2 + 2\beta_1 u + 1}{\beta_1 + 1}, \\ b_2(u) &= \frac{u^2}{\beta_1 + 1}. \end{aligned} \tag{5.6}$$

We call a spline of this type a G^1*, or quadratic, Beta-spline.* When $\beta_1 = 1$, the weighting functions $b_0(u), b_1(u)$, and $b_2(u)$ reduce to the uniform quadratic B-spline weighting functions, thereby verifying that the quadratic Beta-spline is the geometrically continuous analog of the uniform quadratic B-spline. Examples of the behavior of the G^1 Beta-spline as a function of β_1 are shown in figure 5.1.

The quadratic Beta-spline is called an *approximating* technique because the curve is not guaranteed to *interpolate* (pass though) the control vertices. A G^1 spline technique that does interpolate the vertices is presented in example 5.2. •

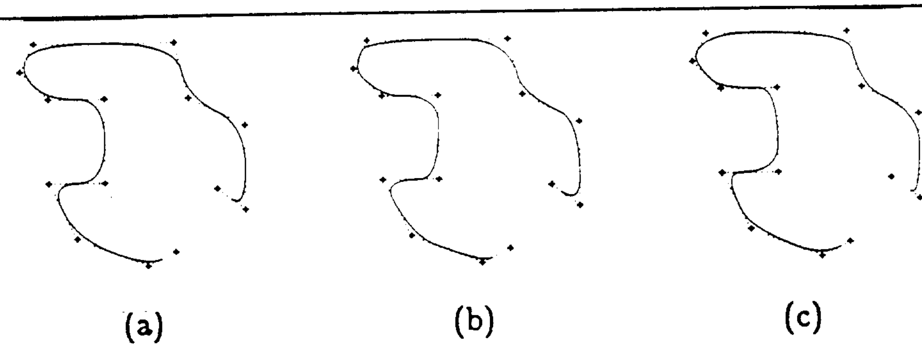


Figure 5.1. *Figure (a) is a quadratic Beta-spline with $\beta_1 = 1$, and is therefore identical to a uniform quadratic B-spline. Figures (b) and (c) are defined by the same control polygon as (a) with $\beta_1 = 1/2$ and $\beta_1 = 2$, respectively. Note that reciprocal values of β_1 bias the curve in opposite directions.*

Example 5.2: The so-called *cubic Catmull-Rom spline* [Catmull74] is a C^1 interpolating spline where the i^{th} segment of the curve is generated by

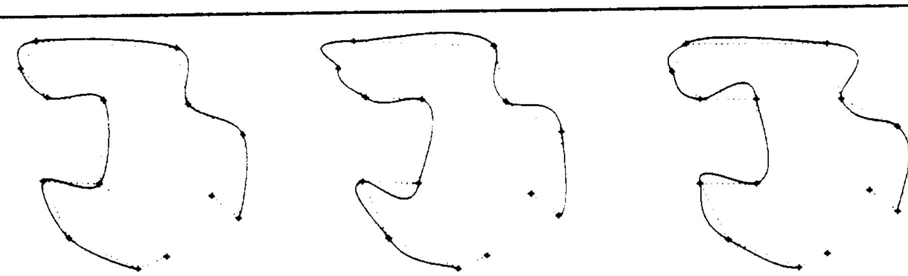
$$\mathbf{q}_i(u) = \sum_{j=-1}^{2} \phi_j(u)\mathbf{V}_{i+j} \qquad u \in [0,1]. \tag{5.7}$$

Using a process similar to example 5.1, the weighting functions $\phi_j(u)$, $j = -1, 0, 1, 2$, can be constructed so that $\mathbf{q}_i$ and $\mathbf{q}_{i+1}$ meet with G^1 continuity at their common joint

[DeRose84]. The resulting functions are

$$
\begin{aligned}
\phi_{-1}(u) &= -\beta_1^2 \frac{u^3 - 2u^2 + u}{\beta_1 + 1}, \\
\phi_0(u) &= \frac{(\beta_1^2 + \beta_1 + 1)u^3 - (2\beta_1^2 + 2\beta_1 + 1)u^2 + (\beta_1^2 - 1)u + \beta_1 + 1}{\beta_1 + 1}, \\
\phi_1(u) &= -\frac{(\beta_1^2 + \beta_1 + 1)u^3 - (2\beta_1^2 + \beta_1 + 1)u^2 - \beta_1 u}{\beta_1(\beta_1 + 1)}, \\
\phi_2(u) &= \frac{u^3 - u^2}{\beta_1(\beta_1 + 1)}.
\end{aligned}
\tag{5.8}
$$

The effect of β_1 on the shape of the spline is shown in figure 5.2. •



(a) (b) (c)

Figure 5.2. *The above curves all share the same control polygon. Curve (a) has $\beta_1 = 1$, and is therefore equivalent to the C^1 Catmull-Rom spline reported in [Catmull74]. Curves (b) and (c) have values of β_1 of $1/2$ and 2, respectively.*

Example 5.3: The Beta-spline originally introduced in [Barsky81] [Barsky85] should properly be called the G^2, or cubic, Beta-spline. The cubic Beta-spline is the geometrically continuous analog of the uniform cubic B-spline. It has been well documented elsewhere [Barsky81] [Barsky83] [Barsky85], and may be derived in a fashion similar to the method used in example 5.1. A derivation of the cubic Beta-spline with local shape parameters has been accomplished by Bartels and Beatty [Bartels84].

An aspect of the cubic Beta-spline that has not (to our knowledge) previously been pointed out follows from the proof of theorem 4.1. In that proof, the parametrization initially used on the "right" of the joint is supplanted by an equivalent parametrization designed to meet the "left" parametrization with C^n continuity. However, to do this for G^2 contintuity with a polynomial change of parameter when $\beta_2 \neq 0$, a polynomial of second degree must be employed. Since the second degree polynomial is substituted into the cubic polynomial basis functions, the resulting basis functions are of sixth degree with respect to the new domain parameter ($\tilde{u}$). Therefore, when $\beta_2 \neq 0$, the cubic Beta-spline basis functions can produce curves that cannot be realized with cubic C^2 parametrizations; to obtain a C^2 joint, a sixth degree polynomial must be used. •

Example 5.4: As our final example of the use of the G^n constraints, we will construct the geometric continuous analog of the uniform quartic B-spline called, naturally enough, the quartic Beta-spline. The i^{th} curve segment is generated by

$$\mathbf{q}_i(u) = \sum_{j=-2}^{2} b_j(u)\,\mathbf{V}_{i+j} \qquad u \in [0,1], \tag{5.9}$$

where the weighting functions are chosen to be quartic polynomials satisfying the following system of constraints resulting from the G^n constraints for G^3 continuity:

$$\begin{aligned} 0 &= b_{-2}(1) \\ b_{-2}(0) &= b_{-1}(1) \\ b_{-1}(0) &= b_0(1) \\ b_0(0) &= b_1(1) \\ b_1(0) &= b_2(1) \\ b_2(0) &= 0 \end{aligned} \qquad \begin{aligned} 0 &= \beta_1 b_{-2}^{(1)}(1) \\ b_{-2}^{(1)}(0) &= \beta_1 b_{-1}^{(1)}(1) \\ b_{-1}^{(1)}(0) &= \beta_1 b_0^{(1)}(1) \\ b_0^{(1)}(0) &= \beta_1 b_1^{(1)}(1) \\ b_1^{(1)}(0) &= \beta_1 b_2^{(1)}(1) \\ b_2^{(1)}(0) &= 0 \end{aligned}$$

$$\begin{aligned} 0 &= \beta_1^2 b_{-2}^{(2)}(1) + \beta_2 b_{-2}^{(1)}(1) \\ b_{-2}^{(2)}(0) &= \beta_1^2 b_{-1}^{(2)}(1) + \beta_2 b_{-1}^{(1)}(1) \\ b_{-1}^{(2)}(0) &= \beta_1^2 b_0^{(2)}(1) + \beta_2 b_0^{(1)}(1) \\ b_0^{(2)}(0) &= \beta_1^2 b_1^{(2)}(1) + \beta_2 b_1^{(1)}(1) \\ b_1^{(2)}(0) &= \beta_1^2 b_2^{(2)}(1) + \beta_2 b_2^{(1)}(1) \\ b_2^{(2)}(0) &= 0 \end{aligned} \qquad \begin{aligned} 0 &= \beta_1^3 b_{-2}^{(3)}(1) + 2\beta_1\beta_2 b_{-2}^{(2)}(1) + \beta_2 b_{-2}^{(1)}(1) \\ b_{-2}^{(3)}(0) &= \beta_1^3 b_{-1}^{(3)}(1) + 2\beta_1\beta_2 b_{-1}^{(2)}(1) + \beta_3 b_{-1}^{(1)}(1) \\ b_{-1}^{(3)}(0) &= \beta_1^3 b_0^{(3)}(1) + 2\beta_1\beta_2 b_0^{(2)}(1) + \beta_3 b_0^{(1)}(1) \\ b_0^{(3)}(0) &= \beta_1^3 b_1^{(3)}(1) + 2\beta_1\beta_2 b_1^{(2)}(1) + \beta_3 b_1^{(1)}(1) \\ b_1^{(3)}(0) &= \beta_1^3 b_2^{(3)}(1) + 2\beta_1\beta_2 b_2^{(2)}(1) + \beta_3 b_2^{(1)}(1) \\ b_2^{(3)}(0) &= 0 \end{aligned}$$

$$b_{-2}(0) + b_{-1}(0) + b_0(0) + b_1(0) + b_2(0) = 1.$$

The basis functions resulting from the solution of the above system are listed in the appendix. The behavior of quartic Beta-spline curves as a function of β_1, β_2, and β_3 is shown in figure 5.3. Although we might expect that a G^n Beta-spline curve would remain within the convex hull of the control vertices for positive values of the shape parameters, as figure 5.3 (i) shows, this is not the case. •

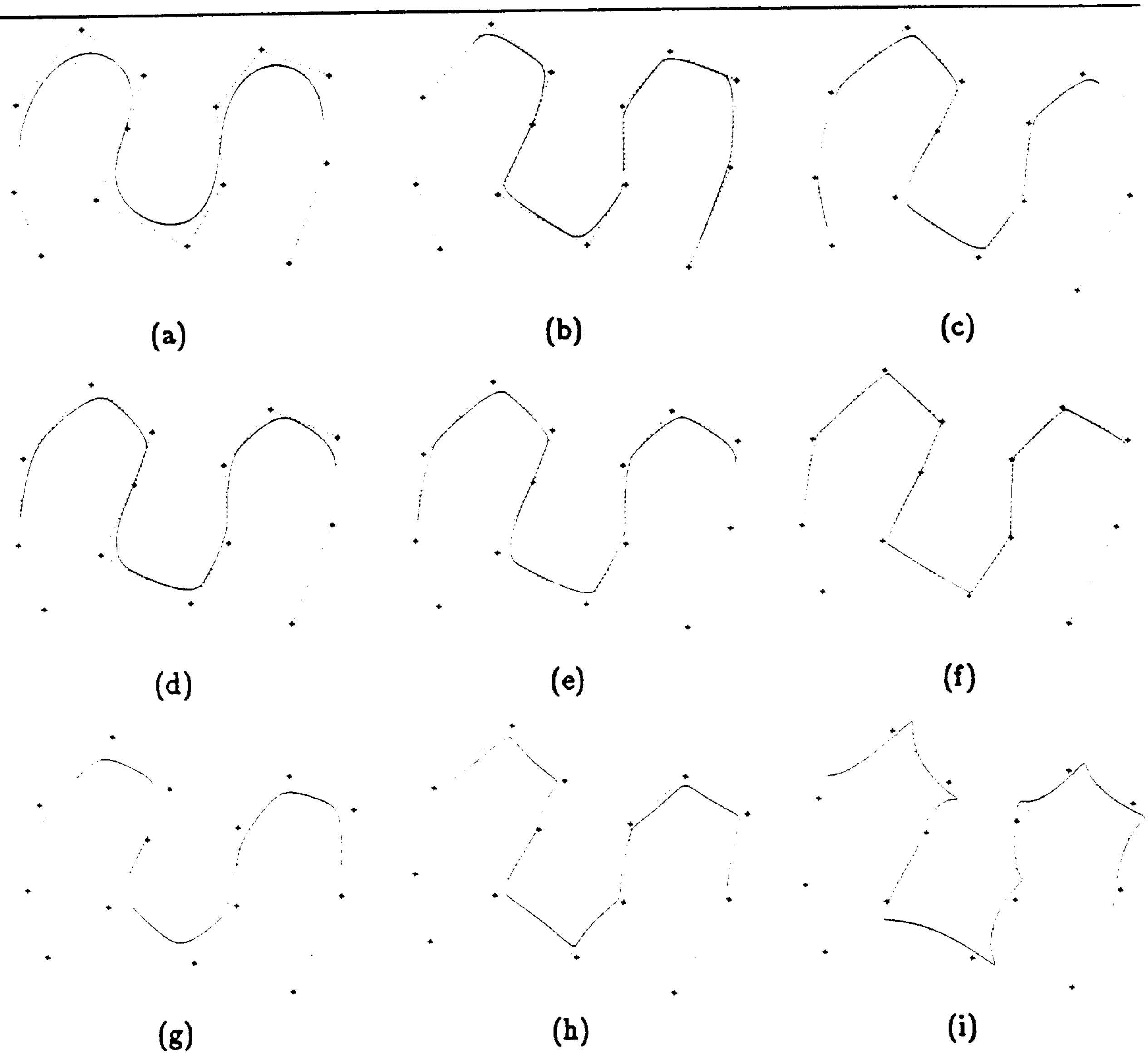


Figure 5.3. *Each of the curves above is defined by the same control polygon; only the shape parameters differ. Curve (a) has* $(\beta_1, \beta_2, \beta_3) = (1,0,0)$, *and is therefore equivalent to a uniform quartic B-spline. Curves (b) and (c) have shape parameters* $(.1,0,0)$ *and* $(10,0,0)$, *respectively, showing the effect of reciprocal values of* β_1. *Curves (d), (e), and (f) show the effect of increasing* β_2*; (d) is defined by* $(1,10,0)$, *(e) by* $(1,20,0)$, *and (f) by* $(1,100,0)$. *Curves (g), (h), and (i) show the effect of increasing* β_3*; (g) is defined by* $(1,0,20)$, *(h) by* $(1,0,50)$, *and (i) by* $(1,0,100)$. *Note that the curve leaves the convex hull, even for positive values of* β_3.

6. Future Directions

Work on geometric continuity is really just beginning. Here we list several important open questions:

- In general, how does β_i affect the shape of the curve? Early results seem to indicate that the behavior of the shape parameters intimately depends upon the order of the polynomial and whether the spline is interpolating or approximating.
- How do the concepts of geometric continuity generalize to curves with multiple points, surfaces, volumes, etc.? Research in this direction is currently underway [DeRose85a] [DeRose85b].
- The non-uniform B-spline basis functions [Gordon74] [Riesenfeld73] satisfy the C^n constraints, have ***local support***, local shape parameters, sum to one, and are non-negative. Is it always possible to construct piecewise polynomials that satisfy the G^n constraints, have local support, sum to one, and are non-negative for restricted values of the shape parameters? Some progress in this direction has been made by Goodman [Goodman84], and Bartels and Beatty [Bartels84].
- If the previous question can be answered in the affirmative, can a general theoretical formalism for the extension of non-uniform B-splines to G^n be developed? Such a theory would include:
  - A Beta-spline recurrence relation similar to the Cox/deBoor relation [Cox71] [deBoor72]. Such a relation could lead to an efficient evaluation algorithm for Beta-spline curves and surfaces. Goldman [Goldman84] has succeeded in deriving a recurrence for the special case of arbitrary β_1 with $\beta_2 = \beta_3 ... = \beta_n = 0$.
  - The Beta-spline equivalent of the Oslo algorithm [Cohen80] [Riesenfeld81] making possible ***successive refinement design*** [Knapp79] of Beta-spline curves and surfaces.

7. Conclusions

N^{th} order geometric continuity has been defined, and a set of constraints have been derived that provide necessary and sufficient conditions parametrizations to meet with G^n continuity. Using these constraints, geometric continuity has been shown shown to be a relaxed form of parametric continuity that is independent of the parametrizations of the curve segments under consideration, but is still sufficient for geometric smoothness of the resulting curve. However, geometric continuity is only appropriate for applications where the particular parametrization used is unimportant since parametric discontinuities are allowed.

By using the G^n constraints instead of requiring continuous parametric derivatives, n new degrees of freedom called shape parameters are introduced. The shape parameters may be made available to a designer in a CAGD environment as a convenient method of changing the shape of the curve without altering the control polygon.

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Appendix: Quartic Beta-spline Basis Functions

The quartic Beta-spline basis functions are of the form:

$$b_i(u) = \frac{1}{\delta}\sum_{j=0}^{4} c_{i,j}u^j \qquad i = -2, \ldots, 2$$

where

$$\delta = (\beta_1^2 - 1)\beta_3 - 3\beta_1\beta_2^2 - 6\beta_2(1+\beta_1)^3 - 6(\beta_1+1)^2(\beta_1^2+1)(\beta_1^2+\beta_1+1)$$

and

$$
\begin{aligned}
c_{-2,0} &= -6\beta_1^6\\
c_{-2,1} &= 24\beta_1^6\\
c_{-2,2} &= -36\beta_1^6\\
c_{-2,3} &= 24\beta_1^6\\
c_{-2,4} &= -6\beta_1^6\\
c_{-1,0} &= \beta_1^2\beta_3 - 3\beta_1\beta_2^2 - (6\beta_1^3 + 18\beta_1^2)\beta_2 - 6\beta_3(3\beta_1^2 + 5\beta_1 + 3)\\
c_{-1,1} &= 36\beta_1^2\beta_2 - 24\beta_1^3(\beta_1^3 - 2\beta_1 - 2)\\
c_{-1,2} &= -6\beta_1^2\beta_3 + 18\beta_1\beta_2^2 + 24\beta_1^3\beta_2 + 36\beta_1^3(\beta_1^3 + \beta_1^2 - 1)\\
c_{-1,3} &= 8\beta_1^2\beta_3 - 24\beta_1\beta_2^2 - (24\beta_1^3 + 36\beta_1^2)\beta_2 - 24\beta_1^4(\beta_1^2 + \beta_1 + 1)\\
c_{-1,4} &= -3\beta_1^2\beta_3 + 9\beta_1\beta_2^2 + (6\beta_1^3 + 18\beta_1^2)\beta_2 + 6\beta_1^3(\beta_1^3 + \beta_1^2 + \beta_1 + 1)\\
c_{0,0} &= -\beta_3 - (18\beta_1 + 6)\beta_2 - 18\beta_1^3 - 30\beta_1^2 - 18\beta_1\\
c_{0,1} &= -36\beta_1^2\beta_2 - 48\beta_1^4 - 48\beta_1^3 + 24\beta_1\\
c_{0,2} &= 6\beta_1^2\beta_3 - 18\beta_1\beta_2^2 + (12 - 24\beta_1^3)\beta_2 + 36\beta_1^2(-\beta_1^3 + \beta_1 + 1)\\
c_{0,3} &= (4 - 8\beta_1^2)\beta_3 + 24\beta_1\beta_2^2 + 6\beta_1\beta_2(4\beta_1^2 + 6\beta_1 + 36) + 24\beta_1^3(\beta_1^2 + \beta_1 + 1)\\
c_{0,4} &= (3\beta_1^2 - 3)\beta_3 - 9\beta_1\beta_2^2 - 6\beta_2(\beta_1 + 1)^3 - 6(\beta_1^5 + \beta_1^4 + 2\beta_1^3 + \beta_1^2 + \beta_1)\\
c_{1,0} &= -6\\
c_{1,1} &= -24\beta_1\\
c_{1,2} &= -12\beta_2 - 36\beta_1^2\\
c_{1,3} &= -4\beta_3 - 36\beta_1\beta_2 - 24\beta_1^3\\
c_{1,4} &= 3\beta_3 + (18\beta_1 + 6)\beta_2 + 6(\beta_1^3 + \beta_1^2 + \beta_1 + 1)\\
c_{2,0} &= c_{2,1} = c_{2,2} = c_{2,3} = 0\\
c_{2,4} &= -6
\end{aligned}
$$

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