Stationary Excursions*

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1. Introduction

This is a study of stationary excursions, built upon and including as special cases many results in the theory of stationary and Markov processes. The main result is a kind of last exit decomposition in a stationary rather than Markovian setting, formulated as part (iii) of the theorem below. This extends a result of Neveu (1977) for a discrete stationary point process. Essentially the same decomposition was obtained for Markovian excursions under duality assumptions by Getoor and Sharpe (1982), and in the Brownian case of excursions from a point by Bismut (1985), who showed how the decomposition gives a nice description of Itô's excursion law. As shown by Biane (1986), this leads to a quick derivation of the relation between Brownian excursion and Brownian bridge of Vervaat (1979). Also included as special cases of the last exit decomposition are results of Geman and Horowitz (1973), Taksar (1980) and Maisonneuve (1983) on random closed regenerative subsets of the line, all of which extend to the stationary case.

Recent work of Mitro (1984), Getoor and Steffens (1985), Fitzsimmons and Maisonneuve (1986), Dynkin (1985), shows how much of the theory of Markov processes finds its most natural expression in the setting of a stationary two sided process with random birth and death, as constructed by Kuznetsov (1974) and Mitro (1979). See also Taksar (1981). The results set out here for a homogeneous random closed set M all apply in this context. Details of this case are not given here, but readers may recognize a number of formulae in the above papers as special cases, often with M a very simple set, such as a single point at the birth time of the process, or the time it last hits a set. Another interesting M in this context is the complement of the interval on which the process is alive.

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2. The Palm measure

Let $(\Theta_t, t \in \mathbb{R})$ be a flow in a measurable space $(\Omega,\mathbf{F})$. That is to say

$$(t,\omega) \to \Theta_{t\omega}$$

is a product measurable map from $\mathbb{R}\times\Omega$ to Ω, and the maps

$$\Theta_t : \omega \to \Theta_{t\omega}$$

from Ω to Ω are such that Θ_0 is the identity and $\Theta_s \circ \Theta_t = \Theta_{s+t}$, s, $t \in \mathbb{R}$. Here $\mathbf{R}$ is given its Borel σ-field. A measure P is *invariant* under the flow if the P distribution of Θ_t is P for every t:

$$P(\Theta_t \in \cdot\,) = P(\,\cdot\,), \quad t \in \mathbf{R}.$$

Call a process $(X_{t\omega}, t \in \mathbf{R}, \omega \in \Omega) = (X_t, t \in \mathbb{R})$ *homogeneous* if

$$X_t = X_0 \circ \Theta_t, \quad t \in \mathbf{R}.$$

For example, X might be the coordinate process on any of the usual function spaces equipped with shift operators (Θ_t). Call a subset of $\mathbb{R}\times\Omega$ *homogeneous* if its indicator function is a homogeneous process. Let M be a homogeneous subset which is closed, meaning that

$$M_\omega = \{t : (t,\omega) \in M\}$$

is a closed subset of $\mathbb{R}$ for every $\omega \in \Omega$. For example, M_ω might be the closure of $\{t : X_{t\omega} \in A\}$, for a subset A of the range of a homogeneous process X.

Define

$$G_t = \sup\{s \le t : s \in M\} \quad (\sup\emptyset = -\infty)$$

$$D_t = \inf\{s > t : s \in M\} \quad (\inf\emptyset = \infty)$$

$$A = -G_0,\ R = D_0.$$

$$A_t = A \circ \Theta_t = t - G_t \quad \text{(age at t)}$$

$$R_t = R \circ \Theta_t = D_t - t \quad \text{(return time after t)}$$

$$L = \{t : R_{t-} = 0,\ R_t > 0\} \quad \text{(set of left ends of intervals comprising } M^c\text{)}.$$

It is assumed R is $\mathbf{F}$-measurable. Then so is everything else. Note that (A_t), (R_t), L are homogeneous, but (G_t) and (D_t) are not. The combination of measurability assumptions on R and (Θ_t) is too strong for some contexts. See the remark at the end of the section regarding weaker assumptions.

The measure Q introduced in the following theorem is the Palm measure on Ω associated with the homogeneous random measure on $\mathbb{R}$ which puts mass 1 at each point of L. This is a slight extension of the notion of Palm measure, in the vein of Totoki (1966), Mecke (1967), Geman and Horowitz (1973), de Sam Lazaro and Meyer (1975), Neveu (1977). See also Atkinson and Mitro (1983), Getoor and Sharpe (1984) for treatment of related measures and further references in the Markovian context.

Theorem.

Suppose P *is a σ-finite measure on* Ω *which is* (Θ_t) *invariant. For* $B \in \mathbf{F}$ *let*

$$Q(B) = P\#\{t : 0 < t < 1,\ t \in L, \Theta_t \in B\},$$

the P *integral of the number of points in L of type B per unit time. Then*

(i) Q *is a σ-finite measure on* $(\Omega,\mathbf{F})$

(ii) *For every product measurable* $f : \mathbb{R}\times\Omega \to [0,\infty)$

$$P\sum_{t\in L} f(t,\Theta_t) = \int_{\mathbb{R}}\int_{\Omega} dt Q(d\omega) f(t,\omega)$$

(iii) *The joint distribution of* Θ_{G_u} *and* $A_u = u - G_u$ *on the set* $(-\infty < G_u < u) = (0 < A_u < \infty)$ *is the same for every* $u \in \mathbb{R}$, *and given by*

$$P(A_u \in da,\ \Theta_{G_u} \in d\omega) = da Q(d\omega) 1(a < R(\omega)), \quad 0 < a < \infty.$$

(iv) $P(0 < A_u < \infty,\ \Theta_{G_u} \in d\omega) = Q(d\omega)R(\omega).$

(v) $P(F) = P(F,\ A_0 = 0 \text{ or } \infty) + Q\int_0^R 1_F(\Theta_s)ds \quad , F \in \mathbf{F}.$

(vi) $P(A_u \in da) = Q(R > a).$

(vii) *If* $Q(R > a) < \infty$, $a > 0$, *the* P *conditional distribution of* Θ_{G_u} *given* $A_u = a$ *is* $Q(\,\cdot \mid R > a)$:

$$P(\Theta_{G_u} \in d\omega | A_u = a) = Q(d\omega | R > a).$$

Proof.

That the Palm measure Q is σ-finite and formula (ii) holds can be shown by a variation of the argument of Mecke (1967). But here is a quicker argument for (i) which I learned from Maisonneuve. Take $f = \int_0^R e^{-s} g(\Theta_s)ds$ where g is chosen so $0 < g \in \mathbf{F}$ and $Pg < \infty$, using the σ-finiteness of P. Then

$$Qf = P\sum_{\substack{0<t<1 \\ t\in L}} f\circ\Theta_t \le e P \sum_{\substack{0<t<\infty \\ t\in L}} e^{-t} f\circ\Theta_t$$

$$\le e P \int_0^\infty e^{-u} g(\Theta_u)du = e Pg < \infty.$$

Since obviously $Q(R \leq 0) = 0$, and $f > 0$ on $(R > 0)$, it follows that Q is σ-finite. Now formula (ii) follows easily, just as in Mecke (1967). See also Getoor (1985) for a related argument.

Parts (iii) to (v) are generalizations of results of Neveu (1977). The proof follows the same lines as Neveu, who considered the case when $M = L$ is discrete, unbounded, and P is a probability. Here is the argument for (iii). By shift invariance, it suffices to consider the case $u = 0$. In formula (ii) take

$$f(t,\omega) = h(\omega,-t)1(R(\omega) > -t > 0), \quad \text{and let } G = G_0 = -A_0.$$

Then $f(t,\Theta_t) = 0$ for $t \geq 0$, while for $t < 0$

$$\begin{aligned} f(t,\Theta_{t\omega})1(t \in L_\omega) &= h(\Theta_{t\omega},-t)1(R \circ \Theta_{t\omega} > -t)1(t \in L_\omega) \\ &= \begin{cases} h(\Theta_G,-G) & \text{if } t = G(\omega) \in (-\infty,0) \\ 0 & \text{if } G(\omega) = -\infty \text{ or } 0 \end{cases} \end{aligned}$$

because for $t \in L_\omega$, $R \circ \Theta_{t\omega}$ is the length of the interval where left end is t, and this length exceeds $-t$ iff this interval is the one covering zero. So the formula becomes

$$Ph(\Theta_G,-G)1(\infty < G < 0) = \int_\Omega Q(d\omega) \int_0^{R(\omega)} h(\omega,s)ds,$$

which is what is meant by (iii) in this case. Appropriate substitutions in (iii) now yield (iv), (v), (vi) and (vii).

Remark. As pointed out to me by Maisonneuve, for application to Markov processes it is more convenient to assume that $(t, \omega) \rightarrow \Theta_{t\omega}$ is (Borel $\otimes$ F, G) measurable and R is G-measurable for a sub σ-field G of F. Then the same arguments show that the Theorem holds with the modifications to the various parts as follows:

(i) Q is defined on G only

(ii) f must be Borel $\otimes$ G measurable

(iii)-(vii) Θ_{G_u} has range (Ω, G).

3. Examples and applications

I. The discrete case. If P is a probability and it is assumed that M is a discrete subset of $\mathbb{R}$, unbounded above and below, then $M = L$, $P(0 < A_u < \infty) = 1$, $P(A_u = 0 \text{ or } \infty) = 0$. The first term then vanishes on the right side of (v), and the conclusions (ii), (iii) and (iv) reduce to the conclusions (20), (18) and (19) respectively of Neveu (1977) Prop II.13. In this case,

$$M = \{T_n, n \in \mathbb{Z}\},$$

where

$$\cdots T_{-1} < T_0 \leq 0 < T_1 < T_2 \cdots$$

with $T_0 = G_0$, $T_1 = R$, and T_n defined inductively by

$$T_{m+n} = T_m + T_n \circ \Theta_{T_m}.$$

Here the subset

$$(0 \in M) = (T_0 = 0) = (G_0 = 0)$$

can be any set in $\mathbf{F}$, call it B, such that the process of shifts watched on B a discrete point process. To emphasize this, write T_n^B instead of T_n, Θ_n^B instead of Θ_{T_n}. So Θ_n^B is the nth shift that hits B, and T_n^B is the time this happens. Then the family of shifts $(\Theta_n^B, n \in \mathbb{Z})$ when restricted to B defines a group of transformations on B which leave the Palm measure Q^B invariant. See Neveu Prop II.17. The shifts (Θ_t) are ergodic under P if and only if the shifts (Θ_n^B) are ergodic under Q^B. Assuming this ergodicity, and that Q^B is bounded, there is the ergodic theorem for $0 \leq Y \in \mathbf{F}$:

$$\frac{1}{n} \sum_{m=1}^{n} Y(\Theta_m^B) \to P^B(Y) \quad \text{both P and } P^B \text{ a.s.}$$

where $P^B(\cdot) = Q^B(\cdot)/Q^B(B)$ is Q^B normalized to be a probability. See for example Franken, Konig, Arndt and Schmidt (1981) or Kerstan, Matthes and Mecke (1974).

II. Excursions. The formulae of section 2 can be reformulated in terms of excursions by a change of variable. Suppose X is a (Θ_t) homogeneous process, such as the co-ordinate process in a function space with shift operators (Θ_t). For each $t \in L$ the excursion of X away from M starting at time t can be defined informally as the fragment of the path of X

$$(X_{t+s}, 0 < s < R_t)$$

where $R_t > 0$ is the lifetime of the excursion. It may also be convenient to regard some other things as part of the excursion, for example X_{t+R_t}, X_t, or X_{t-} if

X happens to have left limits. To cover all such possibilities, let $(\Omega_{ex},\mathbf{F}_{ex})$ be a measurable space, and say a measurable map $\epsilon : (0 \in L) \to \Omega_{ex}$ ***contains the excursion*** of X ***on*** (0,R) if

(i) there is an $\mathbf{F}_{ex}$ measurable map R_{ex} from Ω_{ex} to $(0,\infty]$ with

$$R_{ex} \circ \epsilon = R \quad \text{on } (0 \in L);$$

(ii) there are $\mathbf{F}_{ex}$ measurable maps X_s^{ex} defined on $(R_{ex} > s)$ such that

$$X_s^{ex} \circ \epsilon = X_s \quad \text{on } (0 \in L,\ R > s).$$

Roughly speaking, these assumptions imply that

$$(X_s,\ 0 < s < R)$$

is a function of ϵ if $(0 \in L)$. Clearly, the identity map $\epsilon = \Theta_0$ contains the excursion of X on (0,R). If Ω is any of the usual spaces of paths indexed by $\mathbb{R}$, so does the projection of Ω onto paths indexed by $\mathbb{R}^+$, and so does the operation of stopping or killing at time R after making this projection. In these cases $\Omega_{ex} \subset \Omega$, $R_{ex} = R$, $X_s^{ex} = X_s$. In general we may regard R_{ex} and X_s^{ex} as extensions of R and X_s from Ω to Ω_{ex}. In any case the 'ex' will now be dropped from the notation for these extensions of R and X_s to Ω_{ex}.

Suppose that X is a homogeneous process over a flow (Θ_t), that M is a (Θ_t) homogeneous closed set, and that $\epsilon : (0 \in L) \to \Omega_{ex}$ contains the excursion of X on (0,R), where

$$L = \text{set of left ends of M}$$

as in section 2. For $t \in L$, let $\epsilon_t = \epsilon \circ \Theta_t$, so ϵ_t is the excursion that starts at time t.

Let

$$Q_{ex}(B) = P\#\{t : 0 < t < 1,\ t \in L,\ \epsilon_t \in B\}.$$

Then Q_{ex} is a measure on $(\Omega_{ex},\mathbf{F}_{ex})$, call it the *equilibrium excursion law*. This is simply the Q distribution of ϵ, so the formulae of the theorem transfer immediately by change of variables to give corresponding formulae for excursions instead of shifts. For example, on the set $(-\infty < G_u < u)$, which is the event that $u \in M^c$ and there is some point of M to the left of u, the *excursion straddling time* u is ϵ_{G_u}. Formula (iii) gives the joint distribution of ϵ_{G_u} and $A_u = u - G_u$ as

$$P(A_u \in da,\ \epsilon_{G_u} \in de) = da Q_{ex}(de) 1(a < R(e)), \quad e \in \Omega_{ex},\ \ 0 < a < \infty.$$

Similar substitutions give excursion versions of (ii) and (iv) through (vii). These results for stationary excursions are generalizations of results that are known in various Markovian contexts. In particular, the above formula is a kind of last exit decomposition in a stationary setting, which extends results of Bismut(1985) for Brownian motion and Getoor and Sharpe (1982) for dual Markov processes. As another illustration, (v) yields

$$P(X_u \in B) = P(X_0 \in B,\ A_0 = 0 \text{ or } \infty) + Q_{ex}\int_0^R 1_B(X_s)ds, \quad u > 0.$$

These formulae are not of much interest unless Q_{ex} is σ-finite. But this is the case whenever the P distribution of X_0 is σ-finite. This can be seen using the fact that a measure μ is σ-finite if and only if there is a strictly positive measurable function f such that $\mu f < \infty$. Let f be such a function defined on the state space of X for μ the P distribution of X_0. Then formula (v) of the theorem gives

$$Pf(X_0) \geq Q\int_0^R f(X_s)ds = Q_{ex}\int_0^R f(X_s)ds \quad \text{by change of variables.}$$

But $\int_0^R f(X_s)ds > 0$ on $(R > 0)$, and $Q_{ex}(R > 0)^c = 0$, so Q_{ex} is σ-finite.

Assume now that M is *recurrent*, meaning M is P a.s. unbounded. Then $P(A_u = \infty) = 0$, and the excursion straddling time u is well defined except if $u \in M$. Parts (iii) and (iv) of the theorem then show:

The P *distribution of the excursion straddling time* u *on the event* $(u \notin M)$ *has density* R(e) *with respect to the equilibrium excursion law* $Q_{ex}(de)$*; and given that the excursion straddling* u *is* $e \in \Omega_{ex}$*, the conditional distribution of* A_u *is the uniform distribution on* $[0,R(e)]$.

Put another way:

If P_{ex} *denotes the* P *distribution of the excursion straddling an arbitrary fixed time, the equilibrium excursion law* Q_{ex} *is the measure on* $(R > 0)$ *with density* $\frac{1}{R}$ *with respect to* P_{ex}.

In the special case when P governs a reflecting Brownian motion X on $[0,\infty)$, with the P distribution of X_t equal to Lebesgue measure on $[0,\infty)$ for all t, and M the zero set of X, this amounts to a result obtained by Bismut (1985), because Q_{ex} in this case is just Itô's excursion law, as explained below.

In general, assuming (Θ_t) is ergodic, Q_{ex} describes the asymptotic rates of different types of excursions, in accordance with a ratio ergodic theorem of the type stated above in the discrete case. See Burdzy, Pitman and Yor (1986) for further details in the Markovian case.

III. Relation to Itô's excursion theory.

Suppose P governs a strong Markov process X with σ-finite invariant measure μ. So μ is the P distribution of X_t for each $t \in \mathbf{R}$.

Suppose that M is the closure of $\{t : X_t = 0\}$ where 0 is a recurrent point, meaning that the following (equivalent) conditions obtain:

$$P(R = \infty) = 0.$$

$$P(\sup M < \infty) = 0.$$

Then Q_{ex} is a multiple of the *Itô excursion law* defined by Itô (1970) as the rate measure under P^0 of the Poisson point process

$$(\epsilon_{\tau(u-)},\ u \geq 0),$$

where $(\tau(u),\ u \geq 0)$ is right continuous inverse of a local time process at zero $(U_t,\ t \geq 0)$ and $\epsilon_{\tau(u-)}$ is the excursion of X away from 0 on the interval (tau(u−), $\tau(u)$). Let $B \subset \Omega_{ex}$ be such that the process of excursions of type B is discrete, and let e_n^B be the nth excursion of type B which starts after time 0. The strong Markov property of X at the right ends of the excursion intervals implies that given $X_0 = x$ for μ almost all x, (e_n^B) is a sequence of independent and identically distributed random variables. Comparison of the law of large numbers with the ergodic interpretation of the rate measure Q_{ex} shows that

$$P(e_n^B \in \cdot\,) = Q_{ex}(\,\cdot \mid B).$$

Now let

$$N_t^B = \#\{s : s \leq t,\ s \in L,\ \epsilon_s \in B\},$$

the number of excursions of type B that have started by time t. As B passes through any increasing family of sets with finite Q_{ex} measure and union Ω_{ex}, the normalized counting process

$$(N_t^B/Q_{ex}(B),\ t \geq 0)$$

converges uniformly on compact t intervals a.s. to a continuous additive functional

$$(U_t,\ t \geq 0)$$

which serves as a local time for X as 0. The Poisson character of the time changed excursion process is then easily verified. See Greenwood and Pitman (1980a) for details.

Assuming that the local time U has been normalized as above, the Poisson character of the time changed excursion process may be expressed as follows (see e.g. Jacod (1979) (3.,34))

$$P^\lambda \sum_u H(u,\omega,\epsilon_{\tau(u-)}(\omega)) = P^\lambda \int_0^\infty du \int_{\Omega_{ex}} Q_{ex}(de) H(u,\omega,e)$$

for every positive $\mathbf{P}_{loc} \times \mathbf{F}_{ex}$-measurable function H where $\mathbf{P}_{loc}$ is the

$(\mathbf{F}_{\tau(u)}, u \geq 0)$- predictable σ-field, and it is assumed that X is $(\mathbf{F}_t)$ Markov with respect to P^λ. Applying this formula after a time change gives the Maisonneuve formula

$$P^\lambda \sum_{t\in L} F(t,\omega,\epsilon_t(\omega)) = P^\lambda \int_0^\infty dU_t \int_{\Omega_{ex}} Q_{ex}(de)F(t,\omega,e),$$

valid for every positive $[(\mathbf{F}_t) - \text{predictable}] \times \mathbf{F}_{ex}$ measurable function F. For a function $F(t,\omega,e) = F(t,e)$ depending only on t and e, this becomes

$$P^\lambda \sum_{t\in L} F(t,\epsilon_t) = P^\lambda \int_0^\infty dm_t^\lambda \int_{\Omega_{ex}} Q_{ex}(de)F(t,e)$$

where

$$m_t^\lambda = P^\lambda(U_t).$$

For $\lambda = \mu$ an invariant measure,

$$m_t^\mu = P^\mu(U_t) = c(\mu)t$$

for a constant $c(\mu)$. On the other hand, by the original definition of Q_{ex} as the rate measure of $(\epsilon_t, t \in L)$, the above formula holds for $\lambda = \mu$ with simply dt instead of dm_t^μ. Thus $c(\mu) = 1$ and the local time process defined above is normalized so that

$$P^\mu(U_t) = t.$$

In the terminology of Markov processes, U is the continuous additive functional whose characteristic measure, relative to μ, is a unit mass at 0. In particular, if X is Brownian motion on the line, and μ is Lebesgue measure, U_t is normalized as the occupation density at 0 relative to μ. For applications see Getoor (1979), Greenwood and Pitman (1980b), Pitman (1981).

In general, the constant factor between Q_{ex} defined here and Itô's excursion law depends both on the choice of invariant measure and the normalization of the local time. By formula (v) of the theorem,

$$\mu(f) = \mu(0)f(0) + Q_{ex}\int_0^R f(X_s).$$

Thus the invariant measure μ is determined on $\{0\}^c$ as a multiple of the excursion occupation measure. According to Theorem 8.1 of Getoor (1979), this formula can also be used to construct an invariant measure starting from a Markov process with a recurrent point. See also Geman and Horowitz (1973), Kaspi (1983) (1984) for related results.

IV. Relation to Maisonneuve's exit system. To focus on an important special case, suppose $X = (\Omega, \mathbf{F}, \mathbf{F}_t, X_t, \Theta_t, P^x)$ is a Hunt process which is Harris recurrent, with a single recurrent class E, and invariant reference measure μ on E. See Blumenthal and Getoor (1968), Azéma, Duflo and Revuz (1967) (1969) for background. It is well known that X can be set up as a two sided process indexed by $t \in \mathbf{R}$. Let us assume this has already been done, so that X_t and Θ_t are defined on Ω for all $t \in \mathbf{R}$.

Let M be a closed homogeneous optional subset, and let $(dA_t, \tilde{P})$ be the exit system of M as defined by Maisonneuve (1975), Definition (4.10). Thus dA_t is a homogeneous optional random measure on $(0,\infty)$, and $\tilde{P}$ the kernel from E to Ω, in the *Maisonneuve formula*:

$$P^{\bullet} \sum_{t \in L} Z_t f \circ \Theta_t = P^{\bullet} \int_0^{\infty} Z_t \tilde{P}^{X_t}(f) dA_t$$

for all optional processes $Z \geq 0$ and $\mathbf{F}$-measurable $f \geq 0$. In Maisonneuve (1975) these objects are all defined for a process indexed by $[0,\infty)$, but everything can be lifted to the two sided process, as in Mitro (1984). Also, much of this goes through even without assumptions of recurrence or quasi left continuity. See Kuznetsov (1974), Fitzsimmons and Maisonneuve (1985), Getoor and Steffens (1985). Let Q^x be the $\tilde{P}^x$ distribution of the process X killed at time R. And let $\alpha(dx)$ be the measure on E associated with dA_t via the formula

$$\alpha(h) = P^{\mu} \int_0^1 h(X_t) dA_t,$$

as in Azéma-Duflo-Revuz (1969). Let Q_{ex} be the excursion law on paths killed at time R, induced by the stationary random set M under P^{μ}, as in II above. Then a change of variables in the Maisonneuve formula shows that

$$Q_{ex} = \int_E \alpha(dx) Q^x.$$

Thus the Maisonneuve exit system provides a disintegration of the equilibrium excursion law of Q_{ex} with respect to the starting point of excursions. The definition of the exit system implies that the measure Q^x is not the zero measure, except perhaps on a α null set. Because Q_{ex} is σ-finite, the same is true of α.

The above disintegration of Q_{ex} is not unique because there is a trade off between the choice of α and the normalization of the laws Q^x. In particular problems there may be a choice more natural than the one made by Maisonneuve for the general theory. For example, if X is Brownian motion in a domain D in $\mathbf{R}^d$ with simple reflection at a smooth boundary, the invariant measure m is Lebesgue measure on the domain. The nicest formulae for the excursion laws are then obtained with α the $(d-1)$ dimensional volume measure on ∂D. See Hsu (1986) for details. Burdzy (1986) gives further results for this case.

V. Dual excursions. The equilibrium excursion law was encountered by Kaspi (1984) and Mitro (1984) who found that for a pair of recurrent Markov processes X and $\hat{X}$ in duality, the equilibrium law $\hat{Q}_{ex}$ for excursions from the dual $\hat{M}$ of a recurrent M is the Q_{ex} distribution of excursions reversed from their lifetimes. This relation may be understood in terms of Palm measures as a consequence of the fact that for each $\epsilon > 0$, the point process of left ends of intervals of M^c larger than ϵ alternates with the point process of right ends. See Neveu (1976) p. 202. The duality relation can thus be extended to more general stationary processes. In the case of dual Markovian excursions with nice transition densities, the formulae of section 2 amount to results of Getoor and Sharpe (1982).

It may also be useful to ramify excursions to keep track of the left limit of the process as it leaves M, and the right limit as it returns, for example by defining ϵ on $(0 \in L)$ by

$$X_s \circ \epsilon = \begin{cases} X_{s-}, & s < 0 \\ X_{s \wedge R}, & s \geq 0 \end{cases}.$$

The ramified excursion law Q_{ex} then admits the decomposition

$$Q_{ex}(X_{0-} \in dy,\, X_0 \in dx,\, X_{[0,\infty)} \in dw) = \beta(dy,dx)Q^x(dw),$$

where $X_{[0,\infty)} = (X_s, s \geq 0)$, where Q^x is the Maisonneuve law for excursions starting at x and stopped at time R, and β is the measure associated with the homogeneous random measure dA in the Maisonneuve exit system via the formula

$$\int\int f(y,x)\beta(dy,dx) = P^\mu \int_0^1 f(X_{t-}, X_t)dA_t.$$

Thus β is now a σ-finite measure on $E \times E$ whose projection onto the second coordinate is the α considered earlier. See Atkinson and Mitro (1983) Sharpe (1972), Getoor and Sharpe (1984) for details of these and related matters. Getoor and Sharpe (1982) and Kaspi (1983) give still finer decompositions of the excursion law according to both the endpoint and length of the excursion.

VI. The joint distribution of the age and residual life time.

Return now to the general set up of section 2 with P σ-finite and (Θ_t) invariant.

Corollary. *Suppose that* M *is closed and homogeneous, unbounded above and below a.s.. Let* $A = -G$, $V = A + R = R \circ \Theta_G$ *the overall length of the interval of* M^c *straddling 0. Let* μ *be the measure on* $[0,\infty)$ *which is the* Q *distribution of* R, *where* Q *is the Palm measure on* $(0 \in L)$*:*

$$\mu(dv) = Q(R \in dv).$$

(i) $P(V \in dv) = P(0 \in M)\delta_0(dv) + v\mu(dv), \quad v \geq 0.$

(ii) *Conditional on* Θ_G *the distribution of* A *depends only on the value of* V*, and given* $V = v$*,* A *is uniformly distributed on* $[0,v]$*, and the same holds for* $R = V - A$ *instead of* A *provided* $v < \infty$

(iii) $P(A \in da) = P(R \in da) = P(0 \in M)\delta_0(da) + \mu(a,\infty)da, \quad a \geq 0.$

Proof. These results follow from the theorem of section 2 by a change of variables, just as in Corollaries II.14 and II.15 of Neveu (1977).

If P is a probability and M forms stationary discrete point process, these are well known formulae from renewal theory for the stationary distributions of the age A and residual lifetime R, which work also in the stationary case. See for example McFadden (1962), Neveu (1977) Prop II.19. For P a probability and M a stationary regenerative set these results were established Geman and Horowitz (1973) and again by Taksar (1980) and Maisonneuve (1983). According to the corollary, these results for stationary regenerative closed sets apply just as well without the regeneration assumption, and for a σ-finite P. In the regenerative case, μ can be identified as the Lévy measure, and m as the drift parameter, of a subordinator from which M can be constructed. See Maisonneuve (1983) for details in the case P is a probability, which extend easily to the σ-finite regenerative case, corresponding to a subordinator with a null recurrent age process. In the regenerative case Taksar and Maisonneuve show that -M has the same distribution as M. This extends to the σ-finite regenerative case, see Taksar (1986) but not to the general stationary case, despite the symmetry in the joint distribution of (A,R) which is plain from the Corollary.

Example. Let Θ_t be rotation by distance t around the circumference of a circle with circumference 6,

P = uniform on circle.

$M = \{t : \Theta_t(\omega) \in A\}$ where A consists of 3 points at spacings 1,2 and 3 around the circle.

If say the spacings between points of M are

$$\cdots\ \cdots 123123123 \cdots\ \cdots$$

then going backwards they are

$$\cdots\ \cdots 321321321 \cdots\ \cdots$$

So the distributions of M and -M are different.

Warning. Even if M is discrete and recurrent, P σ-finite does *not* imply μ is σ-finite.

Example. Let $X_t = (B_t, Ue^{it})$ where B_t is a Brownian motion on $\mathbf{R}$, and U is uniformly distributed on $[0,2\pi]$, running with the stationary area measure on the surface of the infinite cylinder $\mathbb{R}\times S^1$. This is a Harris recurrent Hunt process with continuous paths. Let $M = \{t : Ue^{it} = 0\}$. Then $M = L$ is for every ω a shift of the set $2\pi\mathbb{Z}$, and the Q distribution of R is a single mass of ∞ at the point 2π. But the Q distribution of (X_0,R) is σ-finite, the product of Lebesgue measure on $\mathbb{R}$ with a point mass of $1/2\pi$ at 2π. In general, it seems a reasonable conjecture that the Q distribution of (X_0,R) will be σ-finite, provided the P distribution of X_0 is σ-finite and X has right continuous paths.

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REFERENCES.

ATKINSON, B. W. and MITRO, J. B. (1983). Applications of Revuz and Palm type measures for additive functionals in weak duality. Seminar on stochastic processes 1982. Birkhäuser, Boston.

AZEMA, J., DUFLO, M. and REVUZ, D. (1967). Mesure invariante sur les classes récurrentes des processus de Markov. *Z. Wahrscheinlichkeitstheorie* **8,** 157-181.

AZEMA, J., DUFLO, M. and REVUZ, D. (1969). Propriétés relatives des processus de Markov récurrents. *Z. Wahrscheinlichkeitstheorie* **13,** 286-314.

BIANE, P. (1986). Relations entre pont et excursion du mouvement brownien réel. *Ann. Inst. Henri. Poincaré, Prob. et Stat,* **22**, 1-7.

BISMUT, J. M. (1985). Last exit decomposition and regularity at the boundary of transition probabilities. *Z. Wahrscheinlichkeitstheorie* **69,** 65-98.

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). Markov processes and potential theory. Academic Press.

BURDZY, K. (1986). Brownian excursions from hyperplanes and smooth surfaces. *T.A.M.S.* **295,** 35-57.

BURDZY, K., PITMAN, J. W. and YOR, M. Asymptotic laws for crossings and excursions. Paper in preparation.

DE SAM LAZARO, J. and MEYER, P. A. (1975). Hélices croissantes et mesures de Palm. Séminaire de Prob. IX pp. 38-51. Lecture Notes in Math. 465.

DYNKIN, E. B. (1985). An application of flows to time shift and time reversal in stochastic processes. *T.A.M.S.* **287,** 613-619.

FITZSIMMONS, P. J. & MAISONNEUVE, B. (1986). Excessive measures and Markov processes with random birth and death. *Probability Theory and Related Fields* **72**, 319-336.

FRANKEN, P., KONIG, D., ARNDT, V., SCHMIDT, V. (1981). Queues and point processes. Wiley and sons, New York.

GEMAN, D. & HOROWITZ, J. (1973). Remarks on Palm measures. *Ann. Inst. Henri Poincaré Sec B,* **IX** 213-232.

GETOOR, R. K. (1979). Excursions of a Markov process. *Ann. Probab.* **7,** 244-266.

GETOOR, R. K. (1985). Some remarks on measures associated with homogeneous random measures. To appear. Sem. Stoch. Proc. 1985. Birkhäuser, Boston.

GETOOR, R. K. (1985). Measures that are translation invariant in one coordinate. Preprint.

GETOOR, R. K. & SHARPE, M. J. (1981). Two results on dual excursions. Seminar on stochastic processes 1981. Boston. p. 31-52. Birkhäuser.

GETOOR, R. K. & SHARPE, M. J. (1982). Excursions of Dual processes. *Adv. in Math.* **45,** 259-309

GETOOR, R. K. & SHARPE, M. J. (1984). Naturality, standardness, and weak duality for Markov processes. *Z. Wahrscheinlichkeitstheorie,* **67,** 1-62.

GETOOR, R. K. & STEFFENS, J. (1985). Capacity theory without duality. Preprint

GREENWOOD, P. and PITMAN, J. W. (1980a). Construction of local time and Poisson point processes from nested arrays. *J. London Math. Soc. (2)* **22,** 182-192.

GREENWOOD, P. and PITMAN, J. W. (1980b). Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. Appl. Prob.* **12,** 893-902.

HSU, P. (1986). On excursions of reflecting Brownian motion. To appear in T.A.M.S.

ITÔ, K. (1970). Poisson point processes attached to Markov processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* pp. 225-239. Univ. of California Press, Berkeley.

JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales.* Lecture Notes in Math. *714,* Springer-Verlag, Berlin.

KASPI, H. (1983). Excursions of Markov processes: An approach via Markov additive processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete.* **64,** 251-268.

KASPI, H. (1984). On invariant measures and dual excursions of Markov processes. *Z. Wahrscheinlichkeitstheorie* **66,** 185-204.

KERSTAN, J., MATTHES, K. and MECKE, J. (1974). Infinitely divisible point processes. Wiley and sons. New York.

KUZNETSOV, S. E. (1974). Construction of Markov processes with random times of birth and death. *Th. Prob. Appl.,* **18,** 571-574.

MAISONNEUVE, B. (1971). Ensembles régénératifs, temps locaux et subordinateurs. Sém. de Prob. V. Lecture Notes in Math 191.

MAISONNEUVE, B. (1975). Exit systems. *Ann. Prob.* **3,** 399-411.

MAISONNEUVE, B. (1983). Ensembles régénératifs de la droite. *Z. Wahrscheinlichkeitstheorie* **63,** 501-510.

MATTHES, K. (1963/4). Stationäre zufällige Punktfolgen. *Jahresbericht d. Deutsch. Math. Verein,* **66,** 66-79.

MCFADDEN, J. A. (1962). On the lengths of intervals in a stationary point process *J. Roy. Stat. Soc. Ser. B,* **24,** 364-382.

MECKE, J. (1967). Stationäre zufällige Masse auf lokal kompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie* **9,** 36-58.

MITRO, J. B. (1979). Dual Markov processes: Construction of a useful auxilliary process. *Z. Wahrscheinlichkeitstheorie* **47,** 139-156.

MITRO, J. B. (1984). Exit systems for dual Markov processes. *Z. Wahrscheinlichkeitstheorie* **66,** 259-267.

NEVEU, J. (1968). Sur la structure des processes ponctuels stationaires. *C. R. Acad. Sci, t* **267, A** p. 561.

NEVEU, J. (1976). Sur les mesures de Palm de deux processus ponctuels stationnaires. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **34,** 199-203.

NEVEU, J. (1977). Ecole d'Eté de Probabilités de Saint-Flour VI-1976. 250-446. Lecture Notes in Math. 598. Springer.

PITMAN, J. W. (1981). Lévy systems and path decompositions. Seminar on stochastic processes 1981. Birkhäuser, Boston.

SHARPE, M. J. (1972). Discontinuous additive functionals of dual Markov processes. *Z. Wahrscheinlichkeitstheorie* **21,** 81-95.

TAKSAR, M. I. (1980). Regenerative sets on the real line. Seminaire de probabilités XIV, Springer Lecture notes in math. 784.

TAKSAR, M. I. (1981). Subprocesses of a stationary Markov process. *Z. Wahrscheinlichkeitstheorie* **55,** 275-299.

TAKSAR, M. I. (1983). Enhancing of semigroups. *Z. Wahrscheinlichkeitstheorie* **63,** 445-462.

TAKSAR, M. I. (1986). Infinite excessive and invariant measures. Preprint.

TOTOKI, H. (1966). Time changes of flows.
Mem. Fac. Sci. Kyūshū Univ. Ser. A., t **20,** p. 27-55.

VERVAAT, W. (1979). A relation between Brownian bridge and Brownian excursion. *Ann. Prob.* **7,** 143-149.

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24. DOKSUM, K. and YANDELL, B. (April 1983). Tests for exponentiality. Handbook of Statistics, (P. R. Krishnaiah and P. K. Sen, eds.) 4, 1984.

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27. DOKSUM, K. A. (Dec. 1983). An extension of partial likelihood methods for proportional hazard models to general transformation models. Ann. Statist., 1987, 15, 325-345.

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