A NOTE ON WEAK STAR UNIFORMITIES

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Abstract. Consider the set $\pi(Z)$ of countably additive probabilities on Z, the set of positive integers. Endow $\pi(Z)$ with the weak star topology. The finitely additive probabilities on Z form a compactification of Z, which is not the Stone Cech compactification. Indeed, there is a bounded continuous function on $\pi(Z)$ which cannot be uniformly approximated by polynomials. Furthermore, convolution of finitely additive probabilities is non-commutative.

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1. Introduction.

In (Diaconis and Freedman, 1984, section 5), we investigated some conditions for the consistency of Bayes estimates. A key idea turned out to be the "merging" of two sequences of probabilities $\{\alpha_n\}$ and $\{\beta_n\}$, in the sense that α_n and β_n become indistinguishable from the point of view of integrating bounded continuous functions. A formal treatment involves the uniformities compatible with the weak star topology.

To review briefly, let (Z,ρ) be a metric space. Let α_n, α be probabilities in Z. Then $\alpha_n \to \alpha$ weak star iff $\int f d\alpha_n \to \int f d\alpha$ for all bounded continuous functions on Z. This defines the weak star topology T on $X = \pi(Z)$, the set of probabilities in Z. For more information, see Billingsley (1968) or Parthasarathy (1967).

Let (X,T) be any topological space. A uniformity $\mathcal{U}$ is a nonempty collection of subsets of $X \times X$, satisfying the following conditions:

a) Each member of $\mathcal{U}$ includes the diagonal $\{(x,x): x \in X\}$.

b) If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(x,y): (y,x) \in U\}$.

c) If $U \in \mathcal{U}$ then $V \cdot V \subset U$ for some $V \in \mathcal{U}$, where
$V \cdot W = \{(x,y): (x,z) \in W$ and $(z,y) \in V$ for some $z \in X\}$.

d) If U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$.

e) If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

If $A \subset X \times Y$, write $A[x] = \{y: (x,y) \in A\}$ for the x-section of A. We will say the uniformity $\mathcal{U}$ is consistent with the topology T iff for any open set W, and $x \in W$, there is a $U \in \mathcal{U}$ with $U[x] \subset W$.

The idea is that a real-valued function f on X is uniformly continuous iff for all $\varepsilon > 0$ there is a $U_\varepsilon \in \mathcal{U}$ such that $x, y \in U_\varepsilon$ implies $|f(x)-f(y)| < \varepsilon$. If $\mathcal{U}$ is consistent with T, a uniformly continuous function is continuous.

A metric ρ on X defines a natural uniformity $\mathcal{U}_\rho$ as follows: $U \in U_\rho$ iff $U \supset \{(x,y): \rho(x,y) < \delta\}$ for some δ positive. Likewise, a family of pseudo-metrics $\{\rho_\alpha: \alpha \in A\}$ on X defines a natural uniformity $\mathcal{U}_A$ as follows: $U \in \mathcal{U}_A$ iff $U \supset \{(x,y): \rho_\alpha(x,y)\} < \delta$ for all $\alpha \in F\}$, for some positive δ and finite $F \subset A$. For more information, see (Kelley, 1955, pp. 175ff).

Our main result turns out to involve the Stone Cech compactification $\tilde{X}$ of X. This is the largest possible compactification of Y; any bounded continuous function on X extends to a continuous function on $\tilde{X}$. For more information, see (Dunford and Schwartz, 1958, p. 279) or (Kelley, 1955, p. 152).

Let Z be the set of positive integers, $\pi(Z)$ the set of countably additive probabilities on Z, and $\bar{\pi}(Z)$ the set of finitely additive probabilities on Z. Endow π and $\bar{\pi}$ with the weak star topology. Thus, $\bar{\pi}$ is a compactification of π. The main result of this note is the following proposition, which will be proved in Section 3.

Proposition 1.1. $\bar{\pi}(Z)$ is not the Stone Cech compactification of $\pi(Z)$.

This issue came up in connection with work reported in Diaconis and Freedman (1984), where we considered two uniformities on $\pi(Z)$:

$\mathcal{U}_1$ induced by the pseudometrics $\rho_f(\mu,\nu) = |\int f d\mu - \int f d\nu|$ for $f \in C(Z)$

$\mathcal{U}_2$ induced by the pseudometrics $\rho_\phi(\mu,\nu) = |\phi(\mu) - \phi(\nu)|$ for $\phi \in C[\pi(Z)]$.

Here, $C(X)$ is the set of bounded, continuous functions on X; by $C(Z)$ we just mean the bounded functions on Z. Clearly, $\mathcal{U}_2$ is finer than $\mathcal{U}_1$. That the two are really different is not so obvious.

Proposition 1.2. $\mathcal{U}_1 \neq \mathcal{U}_2$.

This is fairly immediate from Proposition 1.1. Indeed, consider the algebra A_0 of functions on $\pi(Z)$ generated by the basic linear functions $\mu \to \int f d\mu$, as f varies over $C(Z)$. Thus, $A_0 \subset C[\pi(Z)]$. We will call A_0 the "polynomials." Of course, any polynomial $\phi \in A_0$ extends to $\bar{\phi} \in C[\bar{\pi}(Z)]$, and

$$(1) \qquad \sup \{|\phi(\mu)|: \mu \in \pi(Z)\} = \sup \{|\bar{\phi}(\mu)|: \mu \in \bar{\pi}(Z)\}$$

Let $A \subset C[\pi(Z)]$ be the closure of A_0 in the sup norm. As (1) implies, any $\phi \in A$ also extends to $\bar{\phi} \in C[\bar{\pi}(Z)]$.. Let $\bar{A} = \{\bar{\phi}: \phi \in A\}$.

LEMMA 1.1. $\bar{A} = C[\bar{\pi}(Z)]$.

PROOF. Use the Stone-Weierstrass theorem. □

By Proposition 1.1, A is a proper subset of $C[\pi(Z)]$. Less formally, there are bounded continuous functions ϕ on $\pi(Z)$ which cannot be uniformly approximated by the polynomials A_0.) Corollary 2.2 below completes the derivation of Proposition 1.2; the object in section 2 is to establish this corollary. (That A separates points and closed sets follows from Lemma 1.1.) Along the way, we discovered that convolution in $\bar{\pi}(Z)$ is noncommutative; we report on this in section 4. Our results can be extended to any noncompact metric space X: just identify Z with a sequence $x_j: j \in Z$ having no convergent subsequences.

2. On uniformities.

Let X be a Hausdorff space, completely regular in the sense that the bounded continuous functions separate points and closed sets, i.e., given $x \in X$ and a closed subset C with $x \notin C$, there is a continuous function f with $0 \leqslant f \leqslant 1$, $f(x) = 1$, and $f = 0$ on C. Let A be a closed subalgebra of $C(x)$, which also separates points and closed sets.

LEMMA 2.1.

a) If $f(x) = f(y)$ for all $f \in A$, then $x = y$.

b) If $\{x_\alpha\}$ is a net, and $f(x_\alpha) \to f(x)$ for all $f \in A$, then $x_\alpha \to x$.

LEMMA 2.2. X can be homeomorphically embedded as a subset of a compact Hausdorff space $\overline{X}_A$, such that A is the restriction to X of $C(\overline{X})$.

PROOF. For $f \in A$, let I_f be the closed interval $[\inf f, \sup f]$. Let $\Omega = \Pi_f I_f$, a compact Hausdorff space. Let η map X into Ω as follows:

$$[\eta(x)](f) = f(x)$$

Clearly, η is continuous. It is $1-1$ by Lemma 2.1a, and η^{-1} is continuous by Lemma 2.1b.

Let $\overline{X}$ be the closure in Ω of $\eta(X)$. Then $\overline{X}$ is compact Hausdorff; for $f \in A$ define $\overline{f}$ on $\overline{X}$ by the formula $\overline{f}(\xi) = \xi(f)$ for $\xi \in \overline{X}$. In particular, $\overline{f}$ is continuous and $\overline{f}[\eta(x)] = f(x)$ extends f.

Let $\overline{A} = \{\overline{f}: f \in A\}$. Then $\overline{A}$ is a closed subalgebra of $C(\overline{X})$ which separates points, so $\overline{A} = C(\overline{X})$ by the Stone-Weierstrass theorem. □

Notes. $\overline{X}_A$ is unique up to a homeomorphism. See (Dunford and

Schwartz, 1958, Part I, Corollary 27 on page 279). The space $\overline{X}_A$ is essentially the Stone Cech compactification of X, relative to A not $C(X)$.

Recall that A is a closed subalgebra of $C(X)$, separating points and closed sets. Let $\mathcal{U}_A$ be the uniformity generated by the seminorms $\rho_f(x,y) = |f(x) - f(y)|$ as f varies over A. Thus, any $f \in A$ is bounded and $\mathcal{U}_A$-uniformly continuous. There are no other such functions.

<u>COROLLARY 2.1.</u> If g is bounded and $\mathcal{U}_A$-uniformly continuous, then $g \in A$.

<u>PROOF.</u> We apply Lemma 2.2, and claim that g extends to $\overline{g} \in C(\overline{X}_A)$. Indeed let $\xi \in \overline{X}_A$, and $x_\alpha \in X$ with $x_\alpha \to \xi$. Then $g(x_\alpha)$ is a Cauchy net of real numbers because g is $\mathcal{U}_A$-uniformly continuous, and the net is bounded because g is. Let $\overline{g}(\xi) = \lim_\alpha g(x_\alpha)$. By standard arguments, $\overline{g}$ is well defined and continuous. So $\overline{g} \in C(\overline{X}) = \overline{A}$, and $g \in A$, as required. □

<u>COROLLARY 2.2.</u> $\mathcal{U}_A$ determines A.

3. The proof of Proposition 1.1.

Let X be a metric space. Let K be a closed subset of X. Let f be a bounded, continuous function on K. The next result is a special case of Tietze's extension theorem. See (Dunford and Schwartz, 1958, pp. 15-17).

LEMMA 3.1. f extends to a continuous function $\overline{f}$ on X, with no change of inf or sup.

PROOF OF PROPOSITION 1.1. We define a subset Q of $\pi(Z)$ as follows: $\mu \in Q$ iff $\mu = \frac{1}{2}(\delta_j + \delta_k)$ for some pair of integers j,k with $1 < j < k$, j even, k odd. As usual, δ_j is point mass at j, so $\delta_j \in \pi(Z)$.

Let ξ and ζ be remote, finitely additive, $0 - 1$ measures on Z, with ξ assigning mass 1 to the evens and ζ to the odds. So $\frac{1}{2}(\xi+\zeta) \in \overline{\pi}(Z)$. Let $\delta_{j_\alpha} \to \xi$ and $\delta_{k_\beta} \to \zeta$ weak star: α and β run through directed sets. So $\mu_{\alpha\beta} = \frac{1}{2}(\delta_{j_\alpha} + \delta_{k_\beta}) \to \frac{1}{2}(\xi + \zeta)$ weak star.

We will now construct $\phi \in C[\pi(Z)]$ such that $\phi(\mu_{\alpha\beta})$ fails to converge. More specifically,

(3.1) $$\lim_\alpha \lim_\beta \phi(\mu_{\alpha\beta}) = 1$$

while

(3.2) $$\lim_\beta \lim_\alpha \phi(\mu_{\alpha\beta}) = 0$$

To begin with, these equations hold with ϕ replaced by the discontinuous function 1_Q. Indeed, in e.g. (3.1), the double limit is just

$$\int_Z \int_Z 1_Q[\tfrac{1}{2}(\delta_j + \delta_k)]\,\zeta(dk)\xi(dj) .$$

Without changing anything, we any confine j to the evens and k to the odds. For j even, $1_Q[\frac{1}{2}(\delta_j + \delta_k)] = 1$ except for finitely many odd k, so the double integral is 1. Finally, to get ϕ, smooth 1_Q using Lemma 3.1. More specifically, take $K = \{\frac{1}{2}(\delta_j + \delta_k): j,k = 1,2,\ldots\}$. Then 1_Q is continuous on K because the latter has no points of accumulation. □

Note. This ϕ is a bounded continuous function on $\pi(Z)$ which does not extend to $\overline{\pi}(A)$, i.e., which cannot be uniformly approximated by polynomials.

4. Convolutions

While trying to prove Proposition 1.1, we came across the following point. Let ξ and ζ be finitely additive probabilities on Z. The convolution $\xi * \zeta$ can be defined as usual

$$(\xi * \zeta)(A) = \int_Z \zeta(A - j)\xi(dj)$$

where $A - j = \{a - j: a \in A\}$. This set function is finitely additive; if ξ and ζ are $0 - 1$, so is $\xi * \zeta$. However, $*$ is not commutative. Here is a preliminary.

LEMMA 4.1. There is an infinite subset S of the positive even integers and T of the odds, such that $(s,t) \to (s+t)$ is $1 - 1$ on $S \times T$.

PROOF. Inductively, we define increasing functions f and g from Z to the evens and odds respectively, such that $f(j) + g(k)$ determines (j,k); then $S = f(Z)$ and $T = g(Z)$. Let $f(1) = 2$ and $g(1) = 3$. Suppose $f(j)$ and $g(k)$ defined for $j,k \leq n$. Then

$$f(n+1) = f(n) + g(n) - 1$$

$$g(n+1) = f(n+1) + g(n)$$

As is easily seen,

$$\min_{k=1,\ldots,n} [f(n+1) + g(k)] > \max_{\substack{j=1,\ldots,n \\ k=1,\ldots,n}} [f(j) + g(k)]$$

$$\min_{j=1,\ldots,n+1} [f(j) + g(n+1)] > \max_{\substack{j=1,\ldots,n+1 \\ k=1,\ldots,n}} [f(j) + g(k)] \qquad \square$$

Proposition 4.1. There are finitely additive $0 - 1$ measures ξ

and ζ on Z such that $\xi * \zeta \neq \zeta * \xi$.

PROOF. Construct S and T as in Lemma 4.1. Let $\xi(S) = 1$ and $\zeta(T) = 1$. Let $Q = \{s+t: s \in S$ and $t \in T$ and $s < t\}$. Then

$$(\xi * \zeta)(Q) = 1 \tag{4.1}$$

$$(\zeta * \xi)(Q) = 0 \tag{4.2}$$

Only (4.2) will be argued. By definition,

$$(\zeta * \xi)(Q) = \int_Z \xi(Q - k)\zeta(dk) = \int_T \xi(Q - t)\zeta(dt)$$

because $\zeta(T) = 1$. If $t \in T$, however, $Q - t$ includes only $s \in S$ with $s < t$; this is where we need the fact that $s + t$ determines s and t. So $\xi(Q - t) = 0$. □

Remarks.

i) With bar for Stone Cech, $\overline{Z \times Z}$ seems really bigger than $\overline{Z} \times \overline{Z}$, by present construction. So a bounded function on $Z \times Z$ is not in the uniform closure of the algebra generated by $(x,y) \to u(x)$ or $v(y)$, u and v bounded.

ii) Likewise, there seems to be a bounded continuous function on Z^∞ not uniformly approximable by finitary functions, i.e. bounded and of the form $u(x_1,\ldots,x_n)$ as u and n vary, but maybe a new idea is needed, along the following lines.

Let μ be any remote $0 - 1$ finitely additive measure on Z, and let $\delta_{n_\alpha} \to \mu$ with $n_\alpha \in Z$. Let μ^k, a finitely additive $0 - 1$ measure on Z^k, be the law of the finite sequence

$$n+1, n+2, \ldots, n+k$$

with n chosen at random from μ. Likewise for μ^∞ on Z^∞. Define

$a_n, b_n \in Z^\infty$ as follows:

$$a_n = (n+1, n+2, \ldots, 2n, 2n+1, 2n+2, \ldots)$$

$$b_n = (n+1, n+2, \ldots, 2n, 0, 0, \ldots) .$$

Then $A = \{a_n\}$ and $B = \{b_n\}$ are disjoint closed sets in Z^∞. Let $f \in C(Z^\infty)$ with $0 \leqslant f \leqslant 1$, $f = 1$ on A, $f = 0$ on B. Then f is not a uniform limit of finitary functions. Indeed, for any k, for all large α, the infinite sequences $a(\alpha) = a_{n_\alpha}$ and $b(\alpha) = b_{n_\alpha}$ agree to k places. Projected on Z^k, then,

$$\lim_\alpha \delta_{a(\alpha)} = \lim_\alpha \delta_{b(\alpha)} .$$

On Z^∞, however, $f(a(\alpha)) \equiv 1$ and $f(b(\alpha)) \equiv 0$.

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