

AN ELEMENTARY PROOF OF STIRLING'S FORMULA

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AN ELEMENTARY PROOF OF STIRLING'S FORMULA

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Abstract. The object of this note is to give an elementary proof of Stirling's formula, using a direct form of Laplace's method.

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Running head. Stirling's formula.

1. Introduction. Stirling's formula is

$$(1) \quad \Gamma(\alpha) \approx \left(\frac{\alpha-1}{e}\right)^{\alpha-1} \sqrt{2\pi(\alpha-1)}$$

as $\alpha \rightarrow \infty$, in the sense that the ratio of the two sides tends to 1. By definition,

$$(2) \quad \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \quad \text{for } \alpha > 0.$$

Thus, for positive integers n , $\Gamma(n) = (n-1)!$.

A formal argument for (1) is presented in the next section, but here is the idea. Write $\exp(x) = e^x$ and $\beta = \alpha - 1$. Then

$$(3) \quad \Gamma(\alpha) = \int_0^{\infty} \exp\{\Psi_{\beta}(x)\} dx$$

where

$$(4) \quad \Psi_{\beta}(x) = \beta \log x - x.$$

Now Ψ_{β} has its maximum at $x = \beta$, and

$$(5) \quad \Psi_{\beta}(\beta+y) \doteq \beta \log \beta - \beta - \frac{1}{2\beta} y^2.$$

Here, \doteq means "nearly equal," and is used informally. So

$$(6) \quad \begin{aligned} \Gamma(\alpha) &\doteq \exp(\beta \log \beta - \beta) * \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\beta} y^2\right) dy \\ &= \left(\frac{\beta}{e}\right)^{\beta} \sqrt{2\pi\beta} \end{aligned}$$

because

$$(7) \quad \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} z^2\right) dz = \sqrt{2\pi}.$$

2. The argument. Recall Ψ_β from (4). The rigorous version of (5) is the following identity:

$$(8) \quad \Psi_\beta(\beta+y) = \beta \log \beta - \beta - \beta g(y/\beta)$$

where

$$(9) \quad g(v) = v - \log(1+v) .$$

Substitute (8) into (3) and change variables:

$$(10) \quad \Gamma(\alpha) = \left(\frac{\beta}{e}\right)^\beta \int_{-\beta}^{\infty} \exp\{-\beta g(y/\beta)\} dy .$$

Change variables again, putting $y = \sqrt{\beta} z$,

$$(11) \quad \Gamma(\alpha) = \left(\frac{\beta}{e}\right)^\beta \sqrt{2\pi\beta} \Gamma_1(\beta)$$

where

$$(12) \quad \Gamma_1(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\beta}}^{\infty} \exp\{-\beta g(z/\sqrt{\beta})\} dz .$$

Our proof of Stirling's formula is reduced to the following.

LEMMA.
$$\lim_{\beta \rightarrow \infty} \Gamma_1(\beta) = 1$$

PROOF. Fix L large but finite. Then $\Gamma_1(\beta) = \Gamma_L(\beta) + \tau_L(\beta)$

where

$$(13) \quad \begin{aligned} \Gamma_L(\beta) &= \frac{1}{\sqrt{2\pi}} \int_{-L}^L \exp\{-\beta g(z/\sqrt{\beta})\} dz \\ &\rightarrow \frac{1}{\sqrt{2\pi}} \int_{-L}^L \exp\{-\tfrac{1}{2}z^2\} dz \end{aligned}$$

because

$$(14) \quad \beta g(z/\sqrt{\beta}) \rightarrow \frac{1}{2} z^2 \quad \text{as } \beta \rightarrow \infty, \text{ uniformly for } z \in [-L, L] .$$

Relationship (14) holds because for v small enough,

$$(15) \quad (1-\epsilon)^{\frac{1}{2}} v^2 < g(v) < (1+\epsilon)^{\frac{1}{2}} v^2$$

It remains to estimate $\tau_L(\beta)$. We take the upper tail first.

Abbreviate $h(z) = \beta g(z/\sqrt{\beta})$; the dependence of h on β is implicit.

Then

$$\begin{aligned} (16) \quad \int_L^\infty \exp\{-h(z)\} dz &\leq \frac{1}{h'(L)} \int_L^\infty h'(z) \exp\{-h(z)\} dz \\ &= \frac{1}{h'(L)} \exp\{-h(L)\} \\ &\rightarrow \frac{1}{L} \exp\{-\frac{1}{2} L^2\} \quad \text{as } \alpha \rightarrow \infty; \end{aligned}$$

the first inequality holds because $h'(z) = \sqrt{\beta} z / (\sqrt{\beta} + z)$ is increasing in $z > 0$. The lower tail is similar, so for β large,

$$(17) \quad \tau_L(\beta) \leq \frac{2}{\sqrt{2\pi}} \frac{1}{L} \exp\{-\frac{1}{2} L^2\} + \epsilon$$

□

The present approximation comes out in powers of $\beta = \alpha - 1$; if powers of α are preferred, a preliminary change of variables can be made ($x = e^u$), so

$$\Gamma(\alpha) = \int_{-\infty}^{\infty} u^\alpha \exp(-e^u) du .$$

A refinement of the argument leads to the usual asymptotic development,

$$\Gamma_1(\beta) = 1 + \frac{1}{12\beta} - \dots$$

3. History. The first proofs of Stirling's formula were given by de Moivre (1730) and Stirling (1730). Both used what is now called the Euler-MacLaurin formula to approximate $\log 2 + \log 3 + \dots + \log n$. De Moivre proved the result on the way to the normal approximation for the binomial distribution. His first derivation did not explicitly determine the constant $\sqrt{2\pi}$. In a 1731 addendum, he acknowledged that Stirling was able to determine the constant, using Wallis' formula. To statisticians, the most familiar version of this argument is Robbins (1955); also see Feller (1968). The approach yields upper and lower bounds for $n!$ but does not extend to $\Gamma(\alpha)$.

In essence, we are using a direct form of La Place's (1774) method to estimate the integral in (12), with quite explicit bounds. For a more general treatment, see de Bruijn (1981, sec. 4.5). For one very similar to ours, see Woodroffe (1975, p. 127). We know of three other approaches to proving Stirling's formula. Modern analysts extend Γ into the complex plane, and have a proof of (1) using the saddlepoint method: see de Bruijn (1981, sec. 6.9). Artin (1964) presents a fascinating discussion of the Γ -function and its properties, as well as a proof of Stirling's formula based on the following theorem: $\Gamma(\alpha)$ is the only log convex function on $(0, \infty)$ satisfying $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, with $\Gamma(1) = 1$. A third approach using the residue calculus is due to Lindelof; for a modern exposition, see Ahlfors (1979).

We stumbled on our proof while working on finite forms of de Finetti's theorem for exponential families (Diaconis and Freedman, 1980, 1984). As an example, we were thinking of the gamma shape parameter $\alpha > 0$ in the family

$$(20) \quad \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad \text{for } 0 \leq x < \infty.$$

As $\alpha \rightarrow \infty$, the density (20) tends to normal, with mean β and variance β . The argument for Stirling's formula was a by-product.

REFERENCES

- AHLFORS, L. (1979). Complex Analysis. 3rd ed. McGraw-Hill, New York.
- ARTIN, E. (1964). The Gamma Function. Holt, Rinehart, and Winston, New York.
- de BRUIJN, N. G. (1981). Asymptotic Methods in Analysis. Dover, New York.
- de MOIVRE, A. (1730). Miscellanea Analytica de Seriebus et Quadraturis, London.
- DIACONIS, P. and FREEDMAN, D. (1980). Finite exchangeable sequences. Ann. Prob. 8 745-764.
- DIACONIS P. and FREEDMAN, D. (1984). Finite versions of de Finetti's theorem for certain exponential families. Technical report, University of California, Berkeley.
- FELLER, W. (1968). An Introduction to Probability Theory and its Application, Vol. 1, 3rd ed. Wiley, New York.
- LA PLACE, P.S. (1774). Mémoire sur la probabilité des causes par les événements. Mémoires de Mathématique et de Physique Présentés à L'académie Royale des Sciences, par Divers Savants, & Lûs dans ses Assemblées. 6. (Reprinted in Laplace's Oeuvres Completes. 8, 27-65, English translation by S. Stigler, Technical Report #164, Department of Statistics, University of Chicago, September, 1984).
- ROBBINS, H. (1955). A remark on Stirling's formula. Amer. Math. Monthly 62 26-29.
- STIRLING, J. (1730). Methodus Differentialis. London.
- WOODROOFE, M. (1975). Probability with applications. McGraw-Hill, New York.