SOME CONTRASTING EXAMPLES OF THE TIME AND FREQUENCY DOMAIN APPROACHES TO TIME SERIES ANALYSIS

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ABSTRACT

Two distinct approaches to the analysis of time series data are in common use - the time side and the frequency side. The frequency approach involves essential use of sinusoids and bands of (angular) frequency, with Fourier transforms playing an important role. The time approach makes little use of these. Certain useful techniques are hybrids of these two approaches. This work proceeding via examples, compares and contrasts the two approaches with respect to modelling, statistical inference and researchers' aims.

1 INTRODUCTION

Many, many time series analyses have been carried out at this point in time. Some of these analyses have been carried out totally in the time domain, some have proceeded essentially in the frequency domain, and some have made substantial use of both domains. There are numerous examples in hydrology of each type of analysis. It seems useful to examine some time series analyses to attempt to recognize the strengths and weaknesses of each approach and to try to discern just what lead the researchers involved to adopt the particular approach that they did.

This work presents descriptions of a number of time series analyses that the author has been involved with. Some of these have been frequency side, some have been time side and some have been hybrids. Some have been parametric, some have been nonparamet-

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ric. Some have involved linear systems, some have been concerned with nonlinear systems. None of the studies are hydrological, however it is clear that analagous situations do arise in hydrology. It seemed best to present examples that the author knew all details concerning.

2 TIME SERIES ANALYSIS

Tukey (1978) defines our field of study as follows:

> "Time series analysis consists of all the techniques that, when applied to time series data, yield, at least sometimes, either insight or knowledge, AND everything that helps us choose or understand these procedures."

In that paper he further lists some of the aims of time series analysis. These are: 1. discovery of phenomena, 2. modelling, 3. preparation for further inquiry, 4. reaching conclusions, 5. assessment of predictability and 6. description of variability. As one attempts to understand the relative merits of the various approaches and techniques of time series analysis, it is worthwhile to keep the above definition and aims in mind.

Most researchers would seem agreed on what is a time side analysis. There is uncertainty over just what constitutes the frequency side. The following variant of a statement in Bloomfield et al (1981) is helpful: frequency side analysis is thinking of systems, their inputs, outputs and behavior in sinusoidal terms.

It is easier to list techniques that are time side, frequency side or hybrids. On the time side one may list: state space, autoregressive moving average (ARMA) and econometric modelling, trend analysis, regression, pulse probing of systems and empirical orthogonal functions among other things. On the frequency side one may list: spectral and cepstral analysis, seasonal adjustment, harmonic decomposition and sinusoidal probing of systems. Hybrid techniques include: complex demodulation, moving spectrum analysis and the probing of systems by chirps. In practice it seems that there is usually a frequency version of a time side procedure, and vice versa. It further seems that these techniques are generally allies, rather than competitors.

A number of practical time series analyses will now be described and their type of analysis commented on.

3 THE CHANDLER WOBBLE

The point of intersection of the Earth's axis of rotation with the polar cap does not remain fixed, rather it wanders about within a region of the approximate size of a tennis court. Let $(X(t), Y(t))$ denote the coordinates of the point at time t, relative to its long run average position. Set $Z(t) = X(t) + iY(t)$, then (from Munk and MacDonald (1960)) the equations of motion are

$$\frac{dZ(t)}{dt} = \alpha Z(t) + \frac{d\Phi(t)}{dt}$$

with $\Phi(t)$ the excitation function whose increments $d\Phi(t)$ describe the change in the Earth's inertia tensor in the time interval $(t, t+dt)$. Supposing the process Φ to have stationary increments, the power spectrum of the series Z is given by

$$f_{ZZ}(\lambda) = |i\lambda - \alpha|^{-2} f_{\Phi\Phi}(\lambda) \quad .$$

What is of interest here is to derive an estimate of α and to derive characteristics of the excitation process Φ. It is known that the excitation process contains an annual component, due to the alternation of seasons in the southern and northern hemispheres. To build a specific model, suppose that the increments of seasonally adjusted Φ are white noise with variance σ^2. The spectrum of the seasonally adjusted Z is then $|i\lambda - \alpha|^{-2}\sigma^2/2\pi$. The data available for analysis is $Z(t)$ perturbed by measurement error for $t = 0,\dots,T-1$. (In Brillinger (1973) it is monthly data from 1902 to 1969.) Supposing the variance of the measurement error series to be ρ^2, the power spectrum of the series of first differences of the seasonally adjusted discrete data is given by

$$\frac{\sigma^2}{2\pi}\frac{1-e^{-2\beta}}{2\beta}\frac{1}{1-2e^{-\beta}\cos(\lambda-\gamma)+e^{-2\beta}} + \frac{\rho^2|1-e^{-i\lambda}|^2}{2\pi} = f(\lambda)$$

where $\alpha = -\beta + i\gamma$. Given the data one would like estimates of the parameters $\alpha,\beta,\gamma,\rho,\sigma$ and to examine the validity of the model. These things are possible on the frequency side.

Let $d^T(\lambda)$ denote the finite Fourier transform of the series of first differences of the seasonaly corrected data. The periodogram of this data is then $I^T(\lambda)$. The periodogram ordinates $I^T(2\pi s/T)$, $s = 1,2,\ldots$ being approximately independent exponential variates with means $f(2\pi s/T)$, $s = 1,2,\ldots$ respectively, estimating the parameters by maximizing the "Gaussian" likelihood

$$\prod_s f(\tfrac{2\pi s}{T})^{-1} \exp(-I^T(\tfrac{2\pi s}{T})/f(\tfrac{2\pi s}{T}))$$

is one way to proceed. (In essence this procedure is suggested in Whittle (1954).) Estimates derived in this fashion, and estimates of their standard errors are presented in Brillinger (1973). Figure 4 of that paper is an estimate of the power spectrum derived by smoothing the periodogram together with the estimated above functional form. The fit is quite good.

However the nonparametric estimate does show a minor peak at frequency .154 cycles/month that is suspiciously large. This frequency was further investigated by the method of complex demodulation. Complex demodulation is a hybrid frequential-temporal technique. If $X(t)$ denotes the series of concern, then the steps involved are: i. form $U(t) = X(t)\exp(-i\lambda t)$, for λ the frequency of interest, ii. smooth the series $U(t)$ to obtain the series $V(t)$, this is the complex demodulate at frequency λ, iii. graph $|V(t)|^2$ and $\arg V(t)$. One of the important uses of complex demodulation is the detection of changes with time in a frequency band of interest. For the frequency .154 special activity seems to be present only for the period 1905 to 1914.

The above analysis took place principally in the frequency domain, but partially in the hybrid domain as well. The advantages of the frequency domain included: a. operations on the series (sampling, seasonal adjustment, differencing) could be handled directly, b. measurement noise was easily dealt with, c. estimation became a problem of maximizing an elementary function, d. standard errors were a byproduct of the estimation procedure. It is further evident that a frequency component present for only a restricted time period could only be discovered by a hybrid procedure. This was why complex demodulation was so useful.

4 FREE OSCILLATIONS OF THE EARTH

For a time interval after a major earthquake the Earth rings at certain fundamental frequencies. This motion is called its free oscillations. The frequencies are called its eigenfrequencies. The estimation of the values of the eigenfrequencies and their associated decay rates is a problem of fundamental importance to geophysicists building models of the Earth. The problem is that of how to estimate these parameters given the seismogram of a major earthquake. The frequency domain provides an effective means of doing this. Complex demodulation provides an effective means of checking the mechanical model.

Dynamical considerations suggest the following model for the seismogram,

$$X(t) = \sum_{k=1}^{K} \alpha_k \exp(-\beta_k t) \cos(\gamma_k t + \delta_k) + \varepsilon(t)$$

for $t > 0$, with the γ_k the eigenfrequencies of interest, the β_k their decay rates, α_k and δ_k constants and ε a noise series. Crude estimates of the γ_k may be derived by graphing the periodogram of a data stretch. The model may be examined by complex demodulating at estimated γ_k. If the smoothing filter has a bandwidth small enough to exclude other eigenfrequencies, and if the above model holds with the noise not too substantial, then a graph of log $|V(t)|$ will fall off in a linear fashion (slope approximately $-\beta_k$) and arg $V(t)$ will be approximately constant (if the estimated frequency is close enough to the true one.) Bolt and Brillinger (1979) present such graphs for the record made at Trieste of the great Chilean earthquake of 1960. The model seems confirmed. What is needed now are precise estimates of the unknown parameters and estimates of their standard errors. These may be constructed as follows.

Let $a = \gamma + i\beta$, $b = \alpha\exp(i\delta)$,

$$d_X^T(\lambda) = \sum_{t=0}^{T-1} X(t)\exp(-i\lambda t) \quad \text{and} \quad \Delta^T(\lambda) = \sum_{t=0}^{T-1} \exp(-i\lambda t) \quad .$$

For λ in an interval I_k near γ_k, one has $d_X^T(\lambda) \doteq b_k\Delta^T(\lambda - a_k) + d_\varepsilon^T(\lambda)$. Now if the noise series, ε, is stationary and such that well-separated values are only weakly dependent, then the finite Fourier transform values $d_\varepsilon^T(2\pi s/T)$, for s an integer with $2\pi s/T$ near λ, will be approximately independent complex normal variates with mean 0 and variance $2\pi T f_{\varepsilon\varepsilon}(\lambda)$. (See Brillinger (1981) for example.) The maximum likelihood estimates of the unknown parameters are thus the least squares estimates found by minimizing

$$\sum \left|d_X^T\left(\frac{2\pi s}{T}\right) - b_k\,\Delta^T\left(\frac{2\pi s}{T} - a_k\right)\right|^2$$

where the summation is over frequencies $2\pi s/T$ in I_k. Further the asymptotic distribution of these estimates may be found directly and so standard errors estimated and confidence intervals constructed. Details are given in Bolt and Brillinger (1979).

Once again, by going over to the frequency domain a direct estimation procedure has been found. Because estimates of standard errors are part of the procedure, estimated eigenfrequencies from different seismograms may now be combined efficiently. Further the approximate sampling properties of the estimates are clear, being based on normal variates. A hybrid procedure allowed confirmation of the model.

5 THE HUMAN PUPILLARY SYSTEM

The pupil of the eye exhibits a number of nonlinear characteristics. When it is probed with narrow bandwidth sinusoidal light, the motions of its diameter display second and possibly third order harmonics of the fundamental frequency. Further the shape of the transfer function estimated by such sinusoidal probing changes as the amplitude of the stimulus is varied and finally a dynamic asymmetry is exhibited between responses to on and off stimuli. It is apparent that a nonlinear model needs to be developed in order to describe the pupillary system.

A useful model for nonlinear systems is the following one discussed by Tick (1961),

$$Y(t) = \alpha + \int a(t-u)X(u)du + \iint b(t-u,t-v)X(u)X(v)dudv + \varepsilon(t)$$

with X, the system input stationary and Gaussian, with Y the system output and with ε a stationary noise series. Let A and B denote the linear and quadratic transfer functions of this system,

$$A(\lambda) = \int a(u)\exp(-i\lambda u)du$$

$$B(\lambda,\mu) = \int b(u,v)\exp(-i\lambda u - i\mu v)dudv \quad ,$$

then, in this case of Gaussian stimulation, one has the relationships

$$f_{YX}(\lambda) = A(\lambda)f_{XX}(\lambda)$$

$$f_{XXY}(-\lambda,-\mu) = 2B(\lambda,\mu)f_{XX}(\lambda)f_{XX}(\mu) \quad .$$

Here f_{XX} is the power spectrum of the input, f_{YX} the cross-spectrum of the input and the output and f_{XXY} the cross-bispectrum of the input and the output. (This last is the Fourier transform of the third order cross-moment function.)

These last relationships allow the computation of estimates of

A and B once estimates of the spectra involved have been computed. The spectral estimates may be based directly on the Fourier transforms of the data stretches available. As a final step a and b may be estimated by back Fourier transforming the estimates of A and B, taking care to insert convergence factors in the process. Hung et al (1979) present the specific computational formulas involved and present an example of this system identification procedure for the human pupillary system. The estimated a and b are found to make sense physiologically and to be consistent with characteristics noted in other types of experiment with the system.

The extent of linearity of the system may be measured by the (linear) coherence

$$|R_{YX}(\lambda)|^2 = |f_{YX}(\lambda)|^2 / f_{XX}(\lambda) f_{YY}(\lambda) \quad ,$$

with $|R|^2 \leq 1$ and the nearer it is to 1, the more strongly linear the system. Setting $W(t) = \iint b(t-u, t-v) X(u) X(v) du dv$, the quadratic coherence is defined as

$$|R_{YW}(\lambda)|^2 = \frac{1}{2 f_{YY}(\lambda)} \int \frac{|f_{XXY}(\lambda-\mu, \mu)|^2}{f_{XX}(\lambda-\mu) f_{XX}(\mu)} d\mu \quad .$$

This too is bounded by 1, with its nearness to 1 indicating how purely quadratic the system is. The strength of linear plus pure quadratic relationship is measured by $|R_{YX}|^2 + |R_{YW}|^2$. Estimates of the linear and quadratic coherence for the human pupil are presented in Hung et al (1979). The linear coherence is larger, but the quadratic is important as well.

The above analysis is a frequency domain one. Had the input series been Gaussian white noise, a and b could have been estimated directly by cross-correlation, however in the experiments of Hung et al X could not be taken to be white noise. (A side remark is that even in the white noise case, the cross-correlations might be better computed via a (fast) Fourier transform.) In the non-white case a form of deconvolution needs to be carried out and this is done effectively via frequency domain procedures.

Proceeding via the frequency domain lead to the definition of the linear and quadratic coherences. These are frequency side parameters that prove exceedingly useful in practice. There seem to be no useful time side analogs.

6 A LINEAR DESCRIPTION OF NEURON FIRING

In an important class of neurophysiological experiments a sequence of constant amplitude electrical impulses is taken as input to a neuron. The neuron in turn emits a train of near constant amplitude electrical impulses. The neurophysiologist is interested in describing and understanding the process by which an input train is converted to an output train.

To develop a formal description of such a process it is convenient to assimilate the input and output pulse trains to point processes M and N with M(t) the number of input pulses in the time interval (0,t] and N(t) the corresponding number of output pulses. A linear model relating two point processes is described by

$$\text{Prob}(N \text{ point in } (t,t+h) \mid M) \sim [\mu + \sum_j a(t-\sigma_j)]h$$

for small h, where the σ_j are the times of input pulses. It is of interest to estimate the function a and to construct a measure of how appropriate this model is in practical situations. These things may be done by means of a frequency side approach.

The basic statistic is once again a finite Fourier transform,

$$d_M^T(\lambda) = \sum_{0<\sigma_j<T} \exp(-i\lambda\sigma_j) = \int_0^T \exp(-i\lambda t)dM(t) \quad .$$

The periodogram of the data is defined as $I_{MM}^T(\lambda) = (2\pi T)^{-1}|d_M^T(\lambda)|^2$. The power spectrum, $f_{MM}(\lambda)$, may be defined for $\lambda \neq 0$ as the limit, T going to ∞, of $E\, I_{MM}^T(\lambda)$. At $\lambda = 0$ it may be defined by continuity. As in the case of ordinary time series, the power spectrum may be estimated by smoothing the periodogram. The cross-periodogram and cross-spectrum may be defined and estimated in a similar fashion.

The model leads directly to the relationship $f_{NM}(\lambda) = A(\lambda)f_{MM}(\lambda)$, with A the Fourier transform of a . This relationship provides estimates of A and a in turn. Quite a number of such estimates are given in Brillinger et al (1976) for neurons of the sea hare. One factor causing the forms of A and a to vary substantially is whether the synapse is excitatory (input tends to increase the output rate) or inhibitory (input decreases the output rate). The time lag from input to output shows up in the estimates as well, as does the refractory period (output pulses may not be spaced arbitrarily closely together).

The degree to which the output train may be determined from the input via the model presented is conveniently measured by the coherence function, $|R_{NM}(\lambda)|^2 = |f_{NM}(\lambda)|^2/f_{NN}(\lambda)f_{MM}(\lambda)$, once again. In the examples of Brillinger et al (1976) this function is found to vary substantially with frequency. Generally it is much larger at the lower frequencies. It is surprisingly large in many cases given the essential nonlinearity of the system under study.

The frequency side approach is naturally effective in detecting periodicities that are present and in one of the Brillinger et al (1976) examples the estimated power spectrum displays a minor peak corresponding to a periodicity that really could not be seen on the time side. However, as the above development makes clear, the frequency approach further allows the deconvolution of input from system characteristics and leads to the definition of a useful measure of linear time invariant association.

The cited reference presents a frequency side solution to an important problem for which no other solution is presently known. It concerned the physiological connections of three neurons, L2, L3 and L10, of the sea hare. The three neurons were clearly related (there was substantial coherence between all pairs of covarying pulse trains). It was known that L10 was the driving neuron; however it was not known if the neurons were in series L10 $\rightarrow$ L2 $\rightarrow$ L3 or L10 $\rightarrow$ L2 $\rightarrow$ L3 or if L3 and L2 had no direct connection, but L10 $\rightarrow$ L2 and

$L10 \longrightarrow L3$ only.

Partial coherence analysis is a useful tool for examining such questions. Denote the spike trains by A, B, C respectively. The partial coherence between trains A and B is defined to be the coherence between the trains A and B with the linear time invariant effects of C removed. It is given by the modulus-squared of

$$(f_{CC}f_{AB} - f_{AC}f_{CB})/\sqrt{(f_{CC}f_{BB} - f_{BC}f_{CB})(f_{CC}f_{AA} - f_{CA}f_{AC})} \quad .$$

In this expression dependence on λ has been suppressed for convenience. In the case referred to, the partial coherence of L3 and L2 with the effects of L10 removed was not significant and the presence of a direct L2 to L3 connector could be ruled out essentially.

7 THE THRESHOLD MODEL OF NEURON FIRING

Suppose that a neuron receives as input the fluctuating electrical signal $X(t)$. Physiological considerations suggest the following description of its firing. A membrane potential

$$U(t) = \int_0^{B(t)} a(u)X(t-u)du$$

is formed internally, where $a(\cdot)$ describes a summation process and $B(t)$ denotes the time, at t, since the neuron last fired. The neuron then fires when $U(t)$ crosses a threshold $\theta + \varepsilon(t)$, ε being a noise process. Given experimental data it is of interest to verify and fit this model.

Frequency analyses may be carried out in the manner of the previous section given stretches of input and corresponding output data. However given the essential nonlinearity of the system and the feedback from output to input (due to the presence of $B(t)$) these may not be expected to be effective. (As will be mentioned later, in the case that X can be taken to be Gaussian stationary they are of some use.) In Brillinger and Segundo (1979) a time side solution is provided.

Let X_t, U_t, Y_t, $t = 0, \pm 1, \cdots$ denote the sampled versions of the series involved. One has $Y_t = 1$ if the neuron fired at time t and $Y_t = 0$ otherwise. Suppose that the noise is Gaussian white, then

$$\mathrm{Prob}(Y_t = 1 \mid U_t) = \Phi(U_t - \theta)$$

with Φ the normal cumulative. Further, conditional on the given input the likelihood function of the data is

$$\prod_{t=0}^{T-1} \Phi(U_t - \theta)^{Y_t} \left(1 - \Phi(U_t - \theta)\right)^{1-Y_t} \quad .$$

The parameters a_u and θ may now be estimated by maximizing this likelihood. Brillinger and Segundo (1979) present a number of estimates found in this fashion for the neurons R2 and L5 of the sea hare. Once these estimates have been obtained, the function $\mathrm{Prob}(Y = 1 \mid u)$ may be estimated. This was done. It was found to have the sigmoidal shape of Φ .

In the case that the input X is Gaussian and the feedback effect is not large, it may be shown that the estimated a_u derived via cross-spectral analysis are, up to sampling fluctuations, proportional to the desired a_u. (See Brillinger (1977).) Such estimates are given in Brillinger and Segundo (1979) and good agreement found.

For this problem, a frequency analysis could not suffice. The system had a nonlinearity and a feedback was present. By choice of special input, (Gaussian), and if the feedback was not strong, the frequency analysis gave approximate answers; however it is better to address the system directly.

8 . NICHOLSON'S DATA ON SHEEP BLOWFLIES

During the 1950's the Australian entomologist A.J. Nicholson carried out an extensive series of experiments concerning the population variation of Lucilia cuprina (the sheep blowfly) under various conditions. Nicholson maintained populations of the flies on various diets (some constant, some fluctuating), experiencing different forms of competition (between larvae and adults, for egg laying space, etc.), and other many other conditions. The paper Brillinger <u>et al</u> (1980) reports the analysis of population data for a cage maintained under constant conditions. The basic data were the numbers of flies emerging and flies dieing in successive two day intervals. From these series, and the initial conditions, the number of adults alive at time t could be computed. The amount of food provided the flies was constant and limited. This caused the population size to oscillate dramatically, for when many flies were present the females did not receive enough protein to realize their maximum fecundity. In consequence many fewer eggs were laid and the next generation smaller. Nicholson ran the experiment for approximately 700 days.

The life cycle of a blowfly lasts 35 to 40 days. The aggregate numbers displayed an oscillation with this period throughout much of the experiment. Ostor (1977) presents the Fourier spectrum of the data and a peak does stand out. However, while the data does have substantial stationary features, it also has a chaotic appearance in one stretch. Spectrum analysis does not take notice of alternate behavior in separate stretches. Complex demodulation was not especially informative either. A cross-spectrum analysis of the number of deaths, D_t, on the number of emergences, E_t, led to a plausible shape for the impulse response, however the coherence was not high. It seemed that a much better description must be obtainable for such fine experimental data.

Considerations of the biology involved suggested that the probability of a blowfly dieing, in a two day period, would depend on its age, i, on the number, N, it was competing with, and the number, N-, it had competed with last time period. An expression that worked well was

$$q_{i,N,N-}(\theta) = (1 - \alpha_i)(1 - \beta N)(1 - \gamma N-)$$

with θ denoting the unknown parameter values α_i, β, γ. A state space approach, (Gupta and Mehra (1974), Lipster and Shiryayev (1978)), was then taken for the description of the data. A state vector $\underset{\sim}{N}_t$ was defined whose entries gave the (unobservable except for age 0) members of each age group. The Kalman-Bucy filter was set up for

$$\underset{\sim}{m}_t = E(\underset{\sim}{N}_t \mid N_u, u \leq t)$$

and maximum likelihood estimation came down to choosing θ to minimize

$$\sum_{t=2}^{T} (D_t - \sum_{i=1}^{I} q_{i-1,N_{t-1},N_{t-2}}(\theta)\, m_{i-1,t-1})^2 / N_{t-1}^2 \quad .$$

Specific details may be found in Brillinger et al (1980). The model was found to provide an effective description.

For this data nonlinearities were present. Further different subgroups of the population were behaving differently. Despite the presence of understood oscillations, a frequency approach was not very revealing.

9 DISCUSSION

This paper has described a number of time series analyses proceeding from a frequency side analysis to a time side analysis with some hybrid analyses in between. In each case no initial commitment was made to one side or the other, rather at some stage one approach became much more revealing than the other.

Because of space limitations some of the bases for deciding on the final approach will simply be listed. These are: goals and circumstances, ease of (physical) interpretation, simplicity and parsimony, sampling fluctuations, computational difficulty, sensitivity, physical theory (versus black box), data quality, data quantity, ease in dealing with complications, expertness available, real time versus dead time, efficiency, dangers (eg. overtight parameterization), bandwidth of phenomenon, presence and type of nonlinearities, type of nonstationarity.

10 REFERENCES

Bloomfield, P., Brillinger, D.R., Cleveland, W.S. and Tukey, J.W., 1981. The Practice of Spectrum Analysis. In preparation, 233 pp.

Bolt, B.A. and Brillinger, D.R., 1979. Estimation of uncertainties in eigenspectral estimates. Geophys. J. R. astr. Soc.,59:593-603.

Brillinger, D.R., 1973. An empirical investigation of the Chandler wobble. Bull. Internat. Statist. Inst., 45: 413-433.

Brillinger, D.R., 1977. The identification of a particular nonlinear time series system. Biometrika, 64: 509-515.

Brillinger, D.R., 1981. Time Series: Data Analysis and Theory. Holden-Day, San Francisco,540 pp.

Brillinger, D.R., Bryant, H.L. and Segundo, J.P., 1976. Identification of synaptic interactions. Biol. Cybernetics, 22: 213-228.

Brillinger, D.R., Guckenheimer, J., Guttorp, P. and Oster, G., 1980. Lectures on Math. in Life Sci. 13. Amer. Math. Soc., Providence.

Brillinger, D.R. and Segundo, J.P., 1979 . Empirical examination of the threshold model of neuron. Biol. Cybernetics, 35: 213-220.

Gupta, N.K. and Mehra, R.K., 1974 . Computational aspects of maximum likelihood estimation. IEEE Trans. Aut. Control, AC-19: 771-783.

Hung, G., Brillinger, D.R. and Stark, L., 1979 . Interpretation of kernels. II. Math. Biosciences, 46: 159-187.

Lipster, R.S. and Shiryayev, A.N., 1978. Statistics of Random Processes. Springer, New York.

Munk, W.H. and MacDonald, G.J.F., 1960 . The Rotation of the Earth. Cambridge Press, Cambridge.

Oster, G., 1977. Modern Modelling of Continuum Phenomena, R. DiPrima (Editor). Amer. Math. Soc., Providence.

Tick, L.J., 1961. The estimation of transfer functions of quadratic systems. Technometrics, 3: 563-567.

Tukey, J.W., 1978. Can we predict where "time series". In: D.R. Brillinger and G.C. Tiao (Editors) Directions in Time Series. IMS.

Whittle, P., 1954. Some recent contributions to the theory of stationary processes. In: H. Wold, A Study in the Analysis of Stationary Time-Series. Almqvist and Wiksell, Uppsala.