Some Extensions of the Arc Sine Law as (Partial) Consequences of the Scaling Property of Brownian Motion

By

Ph. Carmona, F. Petit and M. Yor

Technical Report No. 398
August 1993

Department of Statistics
University of California
Berkeley, California 94720

July 1993

Preliminary version

Some extensions of the arc sine law as (partial) consequences of the scaling property of Brownian motion.

Ph. Carmona, F. Petit, M. Yor

Université Paris VI - Laboratoire de Probabilités associé au C.N.R.S. N° 224 - 4, Place Jussieu - Tour 56 - 3ème Etage - 75252 PARIS 05

1. Introduction.

(1.1) Let $(B_t \; ; \; t \geq 0)$ be a 1-dimensional motion, starting from 0.

Define $$A_t^+ = \int_0^t ds\, 1_{(B_s \geq 0)} \qquad \text{and} \qquad A_t^- = \int_0^t ds\, 1_{(B_s < 0)} .$$

Lévy ([10], 1939) showed that, for each $t > 0$, $\frac{1}{t} A^+(t)$ is arc sine distributed, i.e. :

(1.a) $$P\left(\frac{A^+(t)}{t} \in du\right) = \frac{du}{\pi\sqrt{u(1-u)}} \qquad (0 < u < 1).$$

On his way to his result, Lévy proved that : for any $t > 0, s > 0$,

(1.b) $$\frac{1}{t} A^+(t) \overset{(\text{law})}{=} \frac{A^+(\tau(s))}{\tau(s)} \left(\equiv \frac{A^+(\tau(s))}{A^+(\tau(s))+A^-(\tau(s))}\right)$$

where $(\tau(s), s \geq 0)$ denotes the right-continuous inverse of the local time $(\ell_t, t \geq 0)$ of Brownian motion at 0.

The identity *(1.a)* is an easy consequence of *(1.b)* since, by excursion theory, $(A^+(\tau(s)), s \geq 0)$ and $(A^-(\tau(s)), s \geq 0)$ are two independent stable $(\frac{1}{2})$ subordinators, which satisfy :

$$A^+(\tau(s)) \overset{(\text{law})}{=} A^-(\tau(s)) \overset{(\text{law})}{=} \frac{s^2}{4N^2} ,$$

where N is a standard gaussian, centered, reduced variable, so that from *(1.b)*, we obtain :

$$\text{(1.c)} \qquad \frac{1}{t} A^{+}(t) \overset{\text{(law)}}{=} \frac{N_{-}^{2}}{N_{+}^{2}+N_{-}^{2}} ,$$

where N_{+} and N_{-} are two independent copies of N ; since it is well known that the right-hand side of *(1.c)* is arc sine distributed, the identity *(1.c)* implies *(1.a)*.

(1.2) Barlow-Pitman-Yor [2] obtained the following reinforcement of *(1.b)* : for every fixed $t > 0$, and $s > 0$,

$$\text{(1.d)} \qquad \frac{1}{\ell_{t}^{2}} (A^{-}(t),A^{-}(t)) \overset{\text{(law)}}{=} \frac{1}{s^{2}} (A^{+}(\tau(s)),A^{-}(\tau(s))).$$

To see that this is indeed a strenghtening of *(1.b)*, remark that *(1.d)* is equivalent (by elementary algebraic manipulations) to :

$$\text{(1.d')} \qquad \frac{1}{t} (A^{+}(t),\ell_{t}^{2}) \overset{\text{(law)}}{=} \left(\frac{A^{+}(\tau(s))}{\tau(s)} \; ; \; \frac{s^{2}}{\tau(s)}\right).$$

The proof of *(1.d)* presented in [2] is done by replacing t on the left-hand side of *(1.d)* by T, an exponential time independent of B, and using excursion theory. A short summary of this approach is presented in Revuz-Yor ([19], Exercise 2.17, p. 449-450).

A remarkable feature of *(1.d)* is that the laws of the 2-dimensional functional :

$$F(u) \equiv \frac{1}{\ell_{u}^{2}} (A^{+}(u),A^{-}(u))$$

taken at a fixed time $u = t$, where $B_{t} \neq 0$, a.s., and at time $u = \tau(s)$, where $B_{\tau(s)} = 0$, a.s., are the same. In order to understand better what lies behind this coïncidence, Pitman-Yor [16] and Perman-Pitman-Yor [13] present some infinite dimensional identities (see, e.g., Theorem (1.1) of [16]) which, again, strenghten *(1.d)* ; in particular, there exists a rearrangement of the trajectory of the pseudo-Brownian bridge (using the terminology in [16]) :

$$\left(\frac{1}{\sqrt{\tau_1}} B_{u\tau_1} \ ; \ u \le 1\right)$$

from which the law of $(B_t \ ; \ t \le g)$, where $g \equiv \sup\{t < 1 : B_t = 0\}$, is recovered (see [16], Theorem 1.3, and [13], Theorem 3.8).

(1.3) Brownian excursion theory plays an essential part in the proofs given in [16] and [13], and, as a consequence, it seemed a quite difficult task to modify the arguments of [16] and [13] to prove the following variant of *(1.d)*, which is due to the second author ([14], [15]) : let $\mu > 0$, and $t > 0$, $s > 0$; then, the identity in law

$$\text{(1.e)} \qquad \frac{1}{(\ell_t^{(\mu)})^2} \left(A^{\mu,+}(t), A^{\mu,-}(t)\right) \overset{(\text{law})}{=} \frac{1}{s^2} \left(A^{\mu,+}(\tau^{\mu}(s)), A^{\mu,-}(\tau^{\mu}(s))\right)$$

where $A^{\mu,\pm}(t) = \int_0^t ds \, 1_{(|B_s| - \mu\ell_s \in \mathbb{R}_\pm)}$,

$(\ell_t^{(\mu)}, t \ge 0)$ denotes the local time at 0 of $(|B_t| - \mu\ell_t \ ; \ t \ge 0)$, and $(\tau^{\mu}(s), s \ge 0)$ is the right-continuous inverse of $(\ell_t^{(\mu)} \ ; \ t \ge 0)$.

As explained in [15] and [23], but only partly proven, both sides of *(1.e)* are distributed as :

$$\text{(1.f)} \qquad \frac{1}{8} \left(\frac{1}{Z_{1/2}}, \frac{1}{Z_{1/2\mu}}\right)$$

where, here, and in the sequel, Z_a will denote a gamma variable with parameter a, i.e :

$$P(Z_a \in dt) = dt \, t^{a-1} e^{-t} \qquad (t > 0)$$

and the two gamma variables featured in *(1.f)* are independent.

The following extension of Lévy's arc sine law *(1.a)* is a consequence of the identity in law between the variables in *(1.e)* and *(1.f)* :

(1.g) $$A_1^{\mu,-} \overset{\text{(law)}}{=} Z_{1/2,1/2\mu} \,,$$

where $Z_{a,b}$ denotes a beta variable with parameters a and b, i.e.

$$P(Z_{a,b} \in dt) = \frac{dt}{B(a,b)}\, t^{a-1}(1-t)^{b-1}dt \qquad (0 < t < 1)$$

(1.4) A few words of explanation may be in order concerning our interest in the variables $A^{\mu,\pm}(t)$: it was found in [8] that the random variables

$$A^{\mu,\pm}(\tau(1)) \equiv \int_0^{\tau(1)} ds\, 1_{(|B_s|-\mu\ell_s\in\mathbb{R}_\pm)}$$ play an important role in the

expressions of the limits in law of the winding numbers of 3-dimensional Brownian motion around curves going to infinity in $\mathbb{R}^3$; henceforth, it seemed natural to study the distributions of $A^{\mu,\pm}(t)$, for fixed time t. We now remark that these random variables occur similarly as the limits in law for two families of natural quantities related to 1-dimensional Brownian motion $(B_t\ ;\ t \geq 0)$:

(a) let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an integrable function, and define :

$$F(t) = \int_0^t du\, f(B_u), \text{ and } A_t^f = \int_0^t ds\, 1_{(|B_s| \geq F(s))}.$$

Then, denoting : $\bar{f} = \int_{-\infty}^{+\infty} dx\, f(x)$, it is not difficult to prove :

(1.h) $$\frac{1}{t} A_t^f \xrightarrow[t\to\infty]{\text{(law)}} A_1^{\bar{f},+} \equiv \int_0^1 du\, 1_{(|B_u| \geq \bar{f}\ell_u)}.$$

Indeed, using the scaling property of B, and the occupation time density formula, we have :

$$\frac{1}{t}\,A_t^f \overset{(law)}{=} \int_0^1 du\, 1_{\left(|B_u| \geq \sqrt{t}\int_0^u dh\, f(\sqrt{t}\,B_h)\right)}$$

$$\overset{(law)}{=} \int_0^1 du\, 1_{\left(|B_u| \geq \int dx\, f(x)\ell_u^{x/\sqrt{t}}\right)}$$

and we obtain *(1.h)* by letting $t \longrightarrow \infty$.

We remark that, in the case $\overline{f} = 1$, which occurs in particular when f is a probability density, the right-hand side of *(1.h)* is arc-sine distributed, since $(|B_u| - \ell_u \;;\; u \geq 0)$ is a Brownian motion.

(b) The random variables $A^{\mu,\pm}(1)$ also occur as limits in law of the following random variables :

$$\frac{1}{t}\,E_t^{(\alpha)} \overset{def}{=} \frac{1}{t}\int_0^t ds\, 1_{\left\{\exp(B_s) \geq \left(\frac{1}{s}\int_0^s du\, \exp B_u\right)^{\alpha}\right\}}$$

which represents the fraction of time spent by the geometric Brownian motion $\{\exp(B_s),\ s \leq t\}$ above the α^{th}-power of its average ; we now prove :

(1.i) $$\frac{1}{t}\,E_t^{(\alpha)} \xrightarrow[t\to\infty]{(law)} A_1^{\bar{\alpha},-} \equiv \int_0^1 du\, 1_{(|B_u| \leq \bar{\alpha}\ell_u)}\ , \text{ where } \bar{\alpha} = 1-\alpha.$$

(Obviously, in the case $\alpha \geq 1$, the right-hand side of *(1.i)* is equal to 0).
To prove *(1.i)*, we remark that :

$$\frac{1}{t}\,E_t^{(\alpha)} \overset{(law)}{=} \int_0^1 du\, 1_{\left(B_u \geq \frac{\alpha}{\sqrt{t}}\log\left(\frac{1}{u}\int_0^u dh\, \exp(\sqrt{t}\,B_h)\right)\right)}$$

and the right-hand side converges in law, as $t \longrightarrow \infty$, towards :

$$\int_0^1 du\, 1_{(B_u \geq \alpha S_u)} \quad , \text{ where } \quad S_u = \sup_{s \leq u} B_s .$$

Now, using Lévy's equivalence : $(|B_u|, \ell_u \, ; \, u \geq 0) \overset{(law)}{=} (S_u - B_u, S_u \, ; \, u \geq 0)$, we obtain :

$$\int_0^1 du\, 1_{(B_u \geq \alpha S_u)} \overset{(law)}{=} \int_0^1 du\, 1_{(|B_u| \leq \bar{\alpha} \ell_u)} ,$$

which finishes the proof of *(1.i)*.

(1.5) The main objective of this paper is to give a simple proof of the identity in law *(1.e)*, relying essentially on Brownian scaling arguments, and on the independence of the processes

$$(A^{\mu,+}(\tau^{\mu}(s)), s \geq 0) \quad \text{and} \quad (A^{\mu,-}(\tau^{\mu}(s)), s \geq 0).$$

This will be done in the third section of this paper, by modifying and developing some of the arguments of D. Williams [22], involving the process $\alpha_t^+ \equiv \inf\{u : A_u^+ > t\}$; for the reader's convenience, such modifications will be first presented in the second section of the paper, in order to derive *(1.d)* independently of the arguments of Barlow-Pitman-Yor [2] and Pitman-Yor [16].

To keep this introduction reasonably short, we briefly recall here that D. Williams' proof of the arc sine law *(1.a)* relies upon the identity :

(1.j) $$\alpha_t^+ = t + A^-(\alpha^+(t)) \equiv t + A_\tau^-(\ell_{\alpha^+(t)}) , \qquad t \geq 0, \quad ^{(*)}$$

and on the essential fact that the processes :

(*) For notational convenience, we shall write sometimes $(A_\tau^-(u), u \geq 0)$ or $(A^-(\tau(u)), u \geq 0)$ for the process $(A_{\tau(u)}^-, u \geq 0)$, and similarly for A^+, and $A^{\mu,\pm}$.

(1.k) $(A^-(\tau(u)), u \geq 0)$ *and* $(\ell_{\alpha^+(t)}, t \geq 0) \equiv ((A^+_\tau)^{-1}(t),\ t \geq 0)$ *are independent.*

This approach is detailed in Karatzas-Shreve [7], but, strangely enough, perhaps due to its apparent asymmetry, it is not discussed in either [2] or [16], in relation with *(1.d)*.

In section 4, we develop some studies related to the process $(X_t = |B_t| - \mu\ell_t\ ;\ t \geq 0)$; in particular, we compare the law of $(X_t, t \leq 1)$, conditionned by $X_1 = 0$, to those of $\left(\frac{1}{\sqrt{\tau_1^\mu}} X_{t\tau_1^\mu}\ ;\ t \leq 1\right)$ and of $\left(\frac{1}{\sqrt{g_1^\mu}} X_{tg_1^\mu}\ ;\ t \leq 1\right)$, where $g_1^\mu = \sup\{s < 1 : X_s = 0\}$.

The first result is obtained just as in the Brownian case $(\mu=1)$, but the second is quite different, and seems to necessitate some involved computations.

In section 5, we show how the proof of *(1.d)* can be modified to obtain, in a similar way as above, some multidimensional extension of the arc sine law for Walsh's Brownian motions and Bessel processes taking values in n rays in the plane ; the original result, which is the identiy *(5.a)* below, was also obtained in [2].

(1.6) Our incentive to develop thoroughly these various extensions of *(1.d)* has two origins :

- the first origin is that, as explained in **(1.3)** above, we wanted to give a simple explanation of the identity in law between the left-hand side of *(1.e)*, and *(1.f)* ;

- the second origin is the result recently obtained by S. Watanabe [21] that the distributions featured in [2], for the time spent in $\mathbb{R}_+$ by a skew Bessel process, are essentially the only possible limits in law, as $t \longrightarrow \infty$, of the quantities :

$$\frac{1}{t} A_t \equiv \frac{1}{t} \int_0^t ds\ 1_{(X_s > 0)}\ ,$$

where X is a generalized diffusion. To be precise, these distributions are the laws of the following ratios :

$$\frac{p^{1/\mu}\, T}{p^{1/\mu}\, T + q^{1/\mu}\, T'}\,, \tag{1.ℓ}$$

where $0 < \mu < 1$, $p + q = 1$, and T and T' are two independent, one-sided stable variables, with index μ. (J. Lamperti showed that the variables in *(1.ℓ)* have a simple enough density ; see, e.g., [16] p. 343).

2. D. Williams' proof of the arc sine law and the identity (1.d).

(2.1) To begin with, we show how, using *(1.j)* and scaling arguments, one deduces *(1.b)* ; this is also presented succinctly in [23], p. 104-105.

We remark that, from *(1.j)* and *(1.k)*, we have, by scaling :

$$\alpha_1^+ \overset{(\text{law})}{=} 1 + (\ell^2_{\alpha^+(1)})\, (A^-(\tau(1))) \overset{(\text{law})}{=} 1 + \frac{A^-(\tau(1))}{A^+(\tau(1))} \equiv \frac{\tau(1)}{A^+(\tau(1))}\,,$$

and, finally, again by scaling :

$$A_1^+ \overset{(\text{law})}{=} \frac{1}{\alpha_1^+} \overset{(\text{law})}{=} \frac{A^+(\tau(1))}{\tau(1)}$$

which proves *(1.b)*.

(2.2) Bootstrapping on the previous arguments, we shall prove the identity *(1.d)*, as a consequence of the following

Proposition 2.1 : *Let* $F : C[0,1] \longrightarrow \mathbb{R}_+$ *be a measurable functional. Then, we have :*

$$E\left[F(B_u\ ;\ u \le 1)\ 1_{(B_1>0)}\right] = E\left[F\left(\frac{1}{\sqrt{\alpha_1^+}}\, B_{s\alpha_1^+}\ ;\ s \le 1\right)\frac{1}{\alpha_1^+}\right]. \tag{2.a}$$

Proof : Let T be an $\mathbb{R}_+$-valued random time, which is independent of B, and whose law is given by : $P(T \in dt) = h(t)dt$,

for some probability density h (e.g. : $h(t) = \exp(-t)$, but any probability density will do). Then, we have :

$$E\left[F(B_u \; ; \; u \le 1) \; 1_{(B_1>0)}\right] = E\left[F\left(\frac{1}{\sqrt{T}} B_{uT} \; ; \; u \le 1\right) \; 1_{(B_T>0)}\right]$$

$$= \int_0^{+\infty} dt \; h(t) \; E\left[1_{(B_t>0)} \; F\left(\frac{1}{\sqrt{t}} B_{st} \; ; \; s \le 1\right)\right]$$

$$= E\left[\int_0^{+\infty} dA_t^+ \; h(t) \; F\left(\frac{1}{\sqrt{t}} B_{ut} \; ; \; u \le 1\right)\right]$$

$$= E\left[\int_0^{+\infty} du \; h(\alpha_u^+) \; F\left(\frac{1}{\sqrt{\alpha_u^+}} B_{s\alpha_u^+} \; ; \; s \le 1\right)\right]$$

$$= \int_0^{+\infty} du \; E\left[h(u\alpha_1^+) F\left(\frac{1}{\sqrt{\alpha_1^+}} B_{s\alpha_1^+} \; ; \; s \le 1\right)\right] \qquad \text{(by scaling)}$$

$$= E\left[\frac{1}{\alpha_1^+} \left(\int_0^{+\infty} dv \; h(v)\right) F\left(\frac{1}{\sqrt{\alpha_1^+}} B_{s\alpha_1^+} \; ; \; s \le 1\right)\right] \qquad \text{(taking : } v = u\alpha_1^+\text{)}$$

$$= E\left[\frac{1}{\alpha_1^+} F\left(\frac{1}{\sqrt{\alpha_1^+}} B_{s\alpha_1^+} \; ; \; s \le 1\right)\right] \quad \text{(since } h \text{ is a probability density).} \qquad \square$$

<u>**Corollary 2.1.1**</u> : *(i) Let* $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a Borel function ; then :*

$$E\left[f\left(\frac{A_1^+, A_1^-}{\ell_1^2}\right) 1_{(B_1>0)}\right] = E\left[\frac{A^+(\tau(1))}{\tau(1)} \left(A^+_{\tau(1)}, A^-_{\tau(1)}\right)\right]. \tag{2.b$_+$}$$

(ii) The identity in law

(1.d) $$\frac{1}{\ell_1^2}(A_1^+, A_1^-) \overset{(law)}{=} (A^+(\tau(1)), A^-(\tau(1)))$$

holds ;

(iii) $$P(B_1 > 0 | A_1^+ = a, \ell_1) = a.$$

Proof : (i) From *(2.a)*, the left-hand side of *(2.b)*$_+$ is equal to :

$$E\left[\frac{1}{\alpha_1^+} f\left(\frac{1}{\ell^2_{\alpha_1^+}} ; \frac{A^-_{\alpha^+(1)}}{\ell^2_{\alpha_1^+}}\right)\right]$$

$$= E\left[\frac{1}{1+A_\tau^-(\ell_{\alpha^+(1)})} f\left(\frac{1}{\ell^2_{\alpha^+(1)}} ; \frac{A_\tau^-(\ell_{\alpha^+(1)})}{\ell^2_{\alpha^+(1)}}\right)\right] \qquad \text{(from } (1.j)).$$

Using the same scaling arguments as in subsection **(2.1)**, we find that the last written quantity is equal to the right-hand side of *(2.b)*$_+$.

(ii) Replacing B by -B in *(2.b)*$_+$, we also obtain :

(2.b)$_-$ $$E\left[f\left(\frac{A_1^+, A_1^-}{\ell_1^2}\right) 1_{(B_1<0)}\right] = E\left[\frac{A^-(\tau(1))}{\tau(1)} f(A^+_{\tau(1)}, A^-_{\tau(1)})\right],$$

so that, adding *(2.b)*$_+$ and *(2.b)*$_-$, we obtain :

$$E\left[f\left(\frac{A_1^+, A_1^-}{\ell_1^2}\right)\right] = E[f(A^+_{\tau(1)}, A^-_{\tau(1)})],$$

which is equivalent to *(1.d)*.

(iii) Making use jointly of *(2.b)*$_+$ and *(1.d)*, we obtain :

$$E\left[f\left(\frac{A_1^+, A_1^-}{\ell_1^2}\right) \frac{1_{(B_1>0)}}{A_1^+}\right] = E[f(A^+_{\tau(1)}, A^-_{\tau(1)})] = E\left[f\left(\frac{A_1^+, A_1^-}{\ell_1^2}\right)\right],$$

so that : $$P(B_1 > 0 \mid A_1^+, \ell_1) = A_1^+. \qquad \square$$

If we use, together with the identity *(2.a)*, the well-known result :

(2.c) $\qquad (B_{\alpha^+(t)}, t \geq 0)$ *is a reflecting Brownian motion,*

(see, e.g. : Mc Kean [11], Karatzas-Shreve [7],...),

we obtain the following description of the joint law of (A_1^+, ℓ_1, B_1), which, as the reader may easily check, agrees with the formula given by Karatzas-Shreve ([7], p. 423).

Corollary 2.1.2 : *We use the notation :* $A_1^{\varepsilon} = A_1^+$, *if* $B_1 > 0$; $A_1^{\varepsilon} = A_1^-$, *if* $B_1 < 0$.

Then, we have for every Borel $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, *and* $a_+, a_- \geq 0$ *:*

$$E\left[g\left(\frac{|B_1|}{(A_1^{\varepsilon})^{1/2}}\right) \,\middle|\, \frac{A_1^+}{\ell_1^2} = a_+ \,,\, \frac{A_1^-}{\ell_1^2} = a_-\right]$$

(2.d)

$$= \left(\frac{a_+}{a_+ + a_-}\right) E\left[g(|B_1|) \,\middle|\, \ell_1 = \frac{1}{2\sqrt{a_+}}\right] + \frac{a_-}{a_+ + a_-} E\left[g(|B_1|) \,\middle|\, \ell_1 = \frac{1}{2\sqrt{a_-}}\right].$$

Proof : a) Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, and $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two Borel functions. Then, we have, from formula *(2.a)* :

$$E\left[1_{(B_1 > 0)} \, f\left(\frac{(A_1^+, A_1^-)}{\ell_1^2}\right) g\left(\frac{B_1}{(A_1^+)^{1/2}}\right)\right] = E\left[\frac{1}{\alpha_1^+} \, f\left(\frac{(1, A^-_{\alpha^+(1)})}{\ell^2_{\alpha^+(1)}}\right) g(B_{\alpha^+(1)})\right]$$

$$= E\left[\frac{1}{(1 + A^-_{\alpha^+(1)})} \, f\left(\frac{(1, A^-_{\alpha^+(1)})}{\ell^2_{\alpha^+(1)}}\right) g(B_{\alpha^+(1)})\right]$$

$$= E\left[\frac{1}{1+A^-(\tau(\ell_{\alpha_1^+}))}\, f\left(\frac{(1, A^-_{\alpha^+(1)})}{\ell^2_{\alpha^+(1)}}\right)\, g(B_{\alpha^+(1)})\right]$$

$$(2.e) \qquad = E\left[\frac{1}{1+(\ell^2_{\alpha^+(1)})(A^-(\tau(1)))}\, f\left(\frac{1}{\ell^2_{\alpha^+(1)}}\, ;\, A^-(\tau(1))\right)\, g(B_{\alpha^+(1)})\right] \text{ (by scaling)}$$

$$(2.f) \qquad = E\left[\frac{1}{(1+\ell_1^2 T^-)}\, f\left(\frac{1}{4\ell_1^2}\, ;\, \frac{1}{4}\, T^-\right)\, g(|B_1|)\right]$$

where, for the last two equalities, $4A^-(\tau(1)) \overset{(law)}{=} T^-$ denotes a standard one-sided stable $(\frac{1}{2})$ variable, which is independent of the reflecting Brownian motion $(B_{\alpha^+(t)}, t \geq 0)$ in *(2.e)*, and of the pair $(|B_1|, \ell_1)$ in *(2.f)*.
To justify the last equality, we have used *(2.c)*.

b) By symmetry, we may now write :

$$E\left[f\left(\frac{(A_1^+, A_1^-)}{\ell_1^2}\right)\, g\left(\frac{|B_1|}{(A_1^{\varepsilon})^{1/2}}\right)\right]$$

$$(2.g) \qquad = E\left[\frac{\tilde{\ell}_1^2}{(\ell_1^2+\tilde{\ell}_1^2)}\, f\left(\frac{1}{4\ell_1^2}, \frac{1}{4\tilde{\ell}_1^2}\right)\, g(|B_1|)\right] + E\left[\frac{\ell_1^2}{(\ell_1^2+\tilde{\ell}_1^2)}\, f\left(\frac{1}{4\ell_1^2}, \frac{1}{4\tilde{\ell}_1^2}\right)\, g(|\tilde{B}_1|)\right],$$

where B and $\tilde{B}$ denote two independent 1-dimensional Brownian motions, and ℓ and $\tilde{\ell}$ their respective local times at 0.
The identity *(2.d)* now follows easily from *(2.g)*. □

3. <u>Some extensions of the arc sine law to perturbed reflecting Brownian motion.</u>

(3.1) <u>Some notation</u>. Throughout this section, μ will denote a fixed positive real, and $(X_t = |B_t| - \mu\ell_t\, ;\, t \geq 0)$ is the reflecting Brownian motion

$(|B_t|, t \geq 0)$ perturbed by the subtraction of μ times the local time of B at 0.

As announced in the Introduction, we are interested in the computation of the distribution of :

$$A_t^{\mu,+} \overset{\mathrm{def}}{=} \int_0^t ds\, 1_{(X_s > 0)} ,$$

and, as above, the local time $(\ell_t^{(\mu)}, t \geq 0)$ of X at 0 will play an important role, together with its right continuous inverse $(\tau^{\mu}(s), s \geq 0)$.

(3.2) The methodology of the proof of *(1.e)* which is adopted here is the same as that of *(1.d)*, developed in Section 2 above. However, in order to make this methodology effective, we first need to describe some essential properties of the 2-dimensional process $\{A^{\mu,+}(\tau^{\mu}(s)), A^{\mu,-}(\tau^{\mu}(s))\ ;\ s \geq 0\}$.

Proposition 3.1 : *(i) The processes* $(A^{\mu,+}(\tau^{\mu}(s)), s \geq 0)$ *and* $(A^{\mu,-}(\tau^{\mu}(s)), s \geq 0)$ *are independent ;*

(ii) For every $\lambda > 0$, *one has :*

$$(A^{\mu,\pm}(\tau^{\mu}(\lambda s)), s \geq 0) \overset{(law)}{=} (\lambda^2 A^{\mu,\pm}(\tau^{\mu}(s)), s \geq 0)$$

(iii) For every $s > 0$, *one has :*

$$\frac{1}{s^2} A^{\mu,+}(\tau^{\mu}(s)) \overset{(law)}{=} \frac{1}{8 Z_{1/2}} \quad \text{and} \quad \frac{1}{s^2} A^{\mu,-}(\tau^{\mu}(s)) \overset{(law)}{=} \frac{1}{8 Z_{1/2\mu}}.$$

Proof : (i) This independence result is a particular consequence of the more general statement made in Theorem 3.2 below.

(ii) This point follows immediately from the scaling property of B.

(iii) This is proven in Chapter 9 of [23], Theorem 9.1 and Corollary 9.1.1. ; this Theorem 9.1 is a Ray-Knight theorem for the local times of X considered up to time τ_s^μ ; a generalized version of it is presented in Theorem 3.3. below. □

It should now be clear to the reader that the main identities of Section 2 extend when B is replaced by X, α^+ by $\alpha^{\mu,+}$, τ by τ^μ, and so on ; in particular, we have :

- the μ-variant of *(2.a)* :

$$(3.a) \qquad E\left[F(X_u \; ; \; u \le 1)\, 1_{(X_1>0)}\right] = E\left[F\left(\frac{1}{\sqrt{\alpha_1^{\mu,+}}}\, X_{s\alpha_1^{\mu,+}} \; ; \; s \le 1\right) \frac{1}{\alpha_1^{\mu,+}}\right]$$

- the μ-variant of *(2.b)$_+$* :

$$(3.b)_+ \qquad E\left[f\left(\frac{A_1^{\mu,+}, A_1^{\mu,-}}{(\ell_1^{(\mu)})^2}\right) 1_{(X_1>0)}\right] = E\left[\frac{A_1^{\mu,+}(\tau^\mu(1))}{\tau^\mu(1)}\, f\left(A^{\mu,+}_{\tau^\mu(1)}, A^{\mu,-}_{\tau^\mu(1)}\right)\right]$$

- the μ-variant of *(1.d)* : for $t > 0$, and $s > 0$,

$$(1.e) \qquad \frac{1}{(\ell_t^{(\mu)})^2}\,(A_t^{\mu,+}, A_t^{\mu,-}) \overset{(\text{law})}{=} \frac{1}{s^2}\,(A^{\mu,+}(\tau^\mu(s)), A^{\mu,-}(\tau^\mu(s))).$$

from which we deduce *(1.f)* and *(1.g)*, thanks to Proposition 3.1.

- the μ-variant of point (iii) in Corollary 2.1.1 :

$$(3.c) \qquad P(X_1 > 0 \,|\, A^{\mu,+} = a, \; \ell_1^{(\mu)}) = a$$

(3.3) We now complete the proof of Proposition 3.1 by showing the more general

Theorem 3.2 : *For* $t \ge 0$, *define* $\left\{L_t^+ = (\ell_{\tau_t^\mu}^{(\mu),x} \ ; \ x \ge 0) \ ; \ t \ge 0\right\}$ and $\left\{L_t^- = (\ell_{\tau_t^\mu}^{(\mu),-x} \ ; \ x \ge 0) \ ; \ t \ge 0\right\}$ *two continuous processes* [*as functions of* $t \ge 0$] *taking their values in the space* $\Sigma = C_c(\mathbb{R}_+,\mathbb{R}_+)$ *of continuous functions* $f : \begin{array}{c} x \longrightarrow f(x) \\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \end{array}$ *with compact support. Then*

(i) *the processes* $(L_t^+ \ ; \ t \ge 0)$ *and* $(L_t^- \ ; \ t \ge 0)$ *are independent ;*

(ii) *each of them is an homogeneous Markov process ;*

(iii) *the process* $(L_t^+ \ ; \ t \ge 0)$ *has independent increments, and for each* $t > 0$, *the distribution of the variable* L_t^+ *is* Q_t^0 , *the law of the square of a 0-dimensional Bessel process starting from* t.

Proof : 1) We first remark that $(\ell_t^{(\mu)} \ ; \ t \ge 0)$ is an additive functional of the 2-dimensional Markov process $\{\tilde{B}_t \equiv (|B_t|,\ell_t) \ ; \ t \ge 0\}$; as a consequence, the process $(\hat{B}_t \overset{\text{def}}{=} \tilde{B}_{\tau_t^\mu} \ ; \ t \ge 0)$ is also an homogeneous Markov process ; we then remark that the two components of $\hat{B}_t$, namely : $|B_{\tau_t^\mu}|$ and $\ell_{\tau_t^\mu}$ are related by : $|B_{\tau_t^\mu}| = \mu \, \ell_{\tau_t^\mu}$; hence, the process $(|B_{\tau_t^\mu}| \ ; \ t \ge 0)$ is itself an homogeneous Markov process ; since $-\mu\ell_{\tau_t^\mu} = \inf_{s \le \tau_t^\mu} X_s$, the r.v. $|B_{\tau_t^\mu}|$ is measurable with respect to the σ-field generated by L_t^-.

The same arguments prove that $(L_t \equiv (L_t^+ \, , \, L_t^-) \ ; \ t \ge 0)$ is an homogeneous Markov process. Moreover, since, for every t, $|B_{\tau_t^\mu}|$ is measurable with respect to $\sigma(L_t^-)$, it is obvious that $L^- \equiv (L_t^- ; t \geq 0)$ is, by itself, an homogeneous Markov process.

2) We now proceed to the proof of the independence of the processes $(L_t^+ ; t \geq 0)$ and $(L_t^- ; t \geq 0)$; this will be obtained from a recurrence argument bearing upon the dimension k of the marginals $(L_{t_1}^+,\dots,L_{t_k}^+)$ and $(L_{t_1}^-,\dots,L_{t_k}^-)$ for $t_1 < t_2 <\dots< t_k$, of the processes $(L_t^+ ; t \geq 0)$ and $(L_t^- ; t \geq 0)$.

- first, we already know, for $k = 1$, by Theorem 9.1 in [23], that for a given $t_1 \equiv t > 0$, L_t^+ and L_t^- are independent ;

- next, we assume that, for $t_1 < t_2 <\dots< t_{k-1} < t_k$, the $(k-1)$ dimensional marginals $(L_{t_1}^+,\dots,L_{t_{k-1}}^+)$ and $(L_{t_1}^-,\dots,L_{t_{k-1}}^-)$ are independent.

Then, we know, from the Markov property of the process $((L_t^+,L_t^-) ; t \geq 0)$, that for any measurable $F : \Sigma \times \Sigma \longrightarrow \mathbb{R}_+$:

$$E\left[F(L_{t_k}^+,L_{t_k}^-)\,|\,\sigma\{L_s ; s \leq t_{k-1}\}\right] = E\left[F(L_{t_k}^+,L_{t_k}^-)\,|\,L_{t_{k-1}}^+,L_{t_{k-1}}^-\right],$$

so that, to finish the recurrence argument, it remains to prove that for two positive reals $s < t$,

the pairs (L_s^+,L_t^+) *and* (L_s^-,L_t^-) *are independent,*

or, equivalently, for $F_i(\ell) \equiv \exp(-<\ell,\varphi_i>)$ and $G_i(\ell) \equiv \exp(-<\ell,\psi_i>)$, $i = 1,2$, where $\{\varphi_i , \psi_i ; i = 1,2\}$ are four continuous functions with compact support on $\mathbb{R}_+$, and : $<\ell,f> = \int_0^{+\infty} dx\, \ell^x f(x)$, we have :

$$E[F_1(L_s^+)G_1(L_s^-)F_2(L_t^+)G_2(L_t^-)] = E[F_1(L_s^+)F_2(L_t^+)]E[G_1(L_s^-)G_2(L_t^-)]. \tag{3.d}$$

The left-hand side of *(3.d)* is equal to :

$$E[\exp\{-<L_s^+,\varphi_1> - <L_s^-,\psi_1> - <L_t^+,\varphi_1> - <L_t^-,\psi_2>\}]$$

$$= E[\exp\{-<L_s^+,\varphi_1+\varphi_2> - <L_s^-,\psi_1+\psi_2>\}E_{\hat{B}_s}(\exp\{-<L_{t-s}^+,\varphi_2> - <L_{t-s}^-,\psi_2>\})]$$

(from the Markov property for $(L_t,t \geq 0)$)

$$= E[\exp(-<L_s^+ ; \varphi_1+\varphi_2>)]\ E[\exp(-<L_s^- ; \psi_1+\psi_2>)\ E_{\hat{B}_s}(\exp\{-<L_{t-s}^+,\varphi_2> - <L_{t-s}^-,\psi_2>\})]$$

from the independence of L_s^+ and L_s^-, and the fact that $\hat{B}_s$ is measurable with respect to $\sigma(L_s^-)$.

It is now clear that the identity *(3.d)* will be proven, together with the independence and the homogeneity of the increments of the process $(L_t^+ ; t \geq 0)$ if we show :

$$E_{\hat{B}_s}(\exp\{-<L_{t-s}^+,\varphi_2> - <L_{t-s}^-,\psi_2>\})$$

(3.e)

$$= E[\exp(-<L_{t-s}^+,\varphi_2>)]\ E_{\hat{B}_s}[\exp(-<L_{t-s}^-,\psi_2>)].$$

In *(3.e)*, the notation $E_{\hat{B}_s}$ refers to the family of distributions of the Markov process $(|B_t|,\ell_t ; t \geq 0)$ starting from (a,ξ) with, furthermore :

$$a = |B_{\tau_s^\mu}| \quad , \text{ and } \quad \xi = \ell_{\tau_s^\mu} = \frac{a}{\mu}.$$

Since $(\ell_t,t \geq 0)$ is an additive functional of $(|B_t|,t \geq 0)$, we have, in general :

$$E_{a,\xi}[F(|B_t|,\ell_t ; t \geq 0)] = E_a[F(|B_t|,\ell_t + \xi ; t \geq 0)],$$

where P_a is now simply the distribution of $(|B_t|,t \geq 0)$, starting from a

(and, in *(3.e)*, E refers to P_o).

Once this notation has been made precise, we remark that :

$$(3.f)\qquad E_{a,\frac{a}{\mu}}[\exp\{-\langle L^+_{t-s},\varphi_2\rangle - \langle L^-_{t-s},\psi_2\rangle\}] = E_a[\exp\{-\langle L^{a,+}_{t-s},\varphi_2\rangle - \langle L^{a,-}_{t-s},\psi_2\rangle\}]$$

where :

$$(3.g)\qquad L^{a,+}_t \equiv (\ell^{\mu,a+x}_{\tau^{\mu,a}_t}\ ;\ x \geq 0)\ ;\ L^{a,-}_t \equiv (\ell^{\mu,a-x}_{\tau^{\mu,a}_t}\ ;\ x \geq 0).$$

Here, $(\ell^{\mu,y}_u\ ;\ u \geq 0)$ denotes the local time at level y of the process $(X_u \equiv |B_u| - \mu\ell_u\ ;\ u \geq 0)$, while $(\tau^{\mu,a}_t\ ;\ t \geq 0)$ is the right continuous inverse of $(\ell^{\mu,a}_u\ ;\ u \geq 0)$.

It now follows from the Ray-Knight theorem stated as Theorem 3.3. below that the right-hand side of *(3.f)* is equal to :

$$E_a\left[\exp\left\{-\langle L^{a,+}_{t-s},\varphi_2\rangle\right\}\right]\ E_a\left[\exp\left\{-\langle L^{a,-}_{t-s},\psi_2\rangle\right\}\right]$$

$$= E\left[\exp\left\{-\langle L^+_{t-s},\varphi_2\rangle\right\}\right]\ E_a\left[\exp\left\{-\langle L^{a,-}_{t-s},\psi_2\rangle\right\}\right],$$

which proves *(3.e)*. □

In order to complete the above proof, we state a Ray-Knight theorem which describes the law of the local times processes in *(3.g)* ; this theorem generalizes Theorem 9.1 in [23], with an analogous proof ; hence, details will not be reproduced.

Theorem 3.3 : *Let* $a \geq 0$, *and* $t > 0$ *be fixed.*

Consider $(|B_t|, t \geq 0)$ *a reflecting Brownian motion starting from* a, *and*

$(\ell^{\mu,x}_{\tau^{\mu,a}_t}\ ;\ x \in \mathbb{R})$ *the family of local times of* $(X_u \equiv |B_u| - \mu\ell_u\ ;\ u \geq 0)$*,*
considered at time $\tau^{\mu,a}_t \equiv \inf\{u : \ell^{\mu,a}_u > t\}$. *Then :*

(i) the two processes $L^{a,+}_t \equiv (\ell^{\mu,x+a}_{\tau^{\mu,a}_t}\ ;\ x \geq 0)$ *and* $L^{a,-}_t \equiv (\ell^{\mu,a-x}_{\tau^{\mu,a}_t}\ ;\ x \geq 0)$ *are independent ;*

(ii) $L^{a,+}_t$ *is, as a process in* $x \geq 0$*, a* $BESQ^0_t$ *, that is : the square starting at* t*, of a* 0*-dimensional Bessel process ;*

(iii) $L^{a,-}_t$ *is, as a process in* $x \geq 0$*, an inhomogeneous Markov process, which is a* $BESQ^0_t$ *on the* x*-interval* $[0,a]$*, and a* $BESQ^{2-\frac{2}{\mu}}$ *process on* $[a,\infty[$*; both processes are absorbed at* 0*.*

<u>**Important remark**</u> : Theorem 3.3 extends, for all $\mu > 0$, the two main Ray-Knight theorems known for Brownian local times $(\mu = 1)$ and, moreover, it allows to unify their statements, with the introduction of the stopping times $\tau^{\mu,a}_t$.
To see this, we recall these two theorems (see, e.g., [19], Chapter 11, paragraph 2), by refering ourselves to particular cases considered in Theorem 3.3 :

α) if we take $\mu = 1$, and $a = 0$, then $L^{0,+}_t$ and $L^{0,-}_t$ are two independent $BESQ^0_t$ processes indexed by $x \in \mathbb{R}_+$;

β) if we take $\mu = 1$, $t = 0$, and $a > 0$, then : $\tau^{1,a}_0 \equiv \inf\{t : |B_t|-\ell_t = a\}$ is the first hitting time of a by the 1-dimensional Brownian motion $\{|B_t|-\ell_t\ ;\ t \geq 0\}$ and, from (iii) above, $L^{a,-}_0$ is, as a process in $x \geq 0$, an inhomogeneous Markov process which is a $BESQ^2_0$ on the x-interval $[0,a]$, and a $BESQ^0$ on $[a,\infty[$. □

Independently of its interest for the proof of Theorem 3.2, we will use Theomem 3.3 in section 4 for the proof of Theorem 4.7.

We now give a last Ray-Knight theorem from which we will deduce the distribution of $T^{\mu,a} \equiv \inf\{u \; ; \; |B_u| - \mu\ell_u = a\}$, at least for a>0.

Theorem 3.4 : *Let* $a \ge 0$, *and* $t > 0$ *be fixed. Consider* $(B_t ; t \ge 0)$ *a standard Brownian motion, and* $(\ell^{\mu,x}_{\tau^{\mu,a}_t} ; x \in \mathbb{R})$ *the family of local times of* $(X_u \equiv |B_u| - \mu\ell_u \; ; \; u \ge 0)$, *considered at time* $\tau^{\mu,a}_t \equiv \inf\{u : \ell^{\mu,a}_u > t\}$. *Then :*

(i) the two processes $L^{a,+}_t \equiv (\ell^{\mu,x+a}_{\tau^{\mu,a}_t} ; x \ge 0)$ *and* $L^{a,-}_t \equiv (\ell^{\mu,a-x}_{\tau^{\mu,a}_t} ; x \ge 0)$ *are independent ;*

(ii) $L^{a,+}_t$ *is, as a process in* $x \ge 0$, *a* $BESQ^0_t$ *;*

(iii) $L^{a,-}_t$ *is, as a process in* $x \ge 0$, *an inhomogeneous Markov process, which is a* $BESQ^2_t$ *on the* x*-interval* $[0,a]$, *and a* $BESQ^{2-\frac{2}{\mu}}$ *process absorbed at* 0 *on* $[a,\infty[$.

From this, we deduce the:

Corollary 3.4.1 : *Let* $T^{\mu,a} \equiv \inf\{u : |B_u| - \mu\ell_u = a\}$.

(i) if $a>0$, *then,*

$$E\left[\exp\left(-\frac{\lambda^2}{2} T^{\mu,a}\right)\right] = \int_0^{+\infty} \frac{(\mathrm{sh}(\lambda a))^{1/\mu}\, dx}{(\mathrm{sh}(\mu x + \lambda a))^{1+1/\mu}}$$

$$= \int_0^{+\infty} dt \, \exp\left(-\frac{\lambda^2}{2}t\right) \sqrt{2/\pi}\; t^{-3/2} \frac{a}{\mu+1} \sum_{n\ge 0} (2n+1) \frac{\left(\frac{\mu-1}{2\mu}\right)_n}{\left(\frac{3\mu+1}{2\mu}\right)_n} \exp(-a^2(2n+1)^2/2t)$$

where $(\alpha)_n \equiv \alpha(\alpha+1)\ldots(\alpha+n-1)$, *and* $(\alpha)_0 \equiv 1$, *and,*

$$a + \mu\, \ell_{T^{\mu,a}} \overset{(law)}{=} \frac{a}{Z_{1/\mu,1}}$$

(ii) if $a \le 0$, $T^{\mu,a}$ *has the same law as the first hitting time of* $(-a/\mu)$ *by a standard Brownian motion.*

Proof : (i) We remark that : $\tau_0^{\mu,a} \overset{\text{def}}{=} \inf\{u \; ; \; \ell_u^{\mu,a} > 0\}$ is precisely equal to $T^{\mu,a}$. Then, according to Theorem 3.4 and usual computations about squares of Bessel processes, we have:

$$E[\exp-\frac{\lambda^2}{2} T^{\mu,a}] = \lim_{t\to 0} E[\exp-\frac{\lambda^2}{2} \tau_t^{\mu,a}]$$

$$= \lim_{t\to 0} Q_t^0[\exp(-\frac{\lambda^2}{2}\int_0^{+\infty} Y_x dx)]\, Q_t^2[\exp(-\frac{\lambda^2}{2}\int_0^a Y_x dx)\, Q_{Y_a}^{2-2/\mu}[\exp(-\frac{\lambda^2}{2}\int_0^{T_0} Z_x dx)]]$$

$$= \lim_{t\to 0} \exp-\frac{\lambda}{2}t \; \frac{\Gamma(\frac{\mu+1}{2\mu})}{\sqrt{\pi}\;\Gamma(\frac{1}{\mu})}\, Q_t^2[(\lambda Y_a)^{1/2\mu} K_{1/2\mu}(\frac{\lambda}{2}Y_a)\exp(-\frac{\lambda^2}{2}\int_0^a Y_x dx)]$$

$$= \lim_{t\to 0} \exp-\frac{\lambda}{2}t \; \frac{1}{\Gamma(\frac{1}{2\mu})}\int_0^{+\infty} Q_t^2[\exp(i\frac{N\lambda}{2}Y_a/\sqrt{2s} - \frac{\lambda^2}{2}\int_0^a Y_x dx)]\, e^{-s} s^{1/2\mu-1}\, ds$$

where N is an independent standard gaussian, centered, reduced variable. The result follows, after computations.

The law of $T^{\mu,a}$ may also be obtained by the resolution of a Skorohod problem (Jeulin-Yor [6], Proposition 4.4 with $k(x)=h(x)=\frac{1}{\mu x+a}$), which gives the law of $\ell_{T^{\mu,a}}$.

(ii) It follows from the equality $T^{\mu,a} = \tau_{-a/\mu}(B)$.

In fact from the inequality

$a=|B_{T^{\mu,a}}| - \mu\ell_{T^{\mu,a}} \geq -\mu\ell_{T^{\mu,a}}$,

we deduce: $T^{\mu,a} \geq \tau_{-a/\mu}(B)$.

But, as $X_{\tau_{-a/\mu}} = |B_{\tau_{-a/\mu}}| - \mu\ell_{\tau_{-a/\mu}} = 0 - \mu(-a/\mu) = a$,

we have $T^{\mu,a} = \tau_{-a/\mu}(B)$. □

4. Several results about the process $(X_t \equiv |B_t|-\mu\ell_t \; ; \; t \geq 0)$.

(4.1) Towards a general principle ?

After reading sections 2 and 3 above, the reader may come very naturally to the "conclusion" that, at least as far as the "arc-scenery" is concerned, identities in law valid for Brownian motion (such as *(1.d)*, for instance) "always" extend to the process X, either literally, or with "little" change. The aim of this section is to show that there is no such "principle", and to present precisely how some of the well-known representations of the Brownian bridge have to be modified in the context of the "μ-process" X, conditionned by $X_1 = 0$.

(4.2) Some notation.

For short, we call $(X_t^{\mu} \equiv |B_t| - \mu \ell_t \, , \, t \geq 0)$ the μ-process ;

- we shall write $(p_{\mu}(t), t \leq 1)$ for the μ-bridge, i.e. :

the μ-process $(X_t^{\mu} \, ; \, t \leq 1)$ conditionned by : $X_1^{\mu} = 0$;

- we shall also consider the pseudo-μ-bridge :

$$\left(p_{\mu}^{\neq}(t) \overset{\text{def}}{=} \frac{1}{\sqrt{\tau_1^{\mu}}} X^{\mu}(t\tau_1^{\mu}) \; ; \; t \leq 1\right).$$

Now we remark that, in the case $\mu = 1$, $(X_t, t \geq 0)$ is a 1-dimensional Brownian motion, and the $(\mu \equiv)1$-bridge is simply the Brownian bridge, which we shall denote by $(p(t) \; ; \; t \leq 1)$; $(\lambda_t \; ; \; t \leq 1)$ denotes the local time at 0 of $(p(t) \; ; \; t \leq 1)$.

- finally, it is also natural to introduce the μ-process of the Brownian bridge ; precisely : $(q_{\mu}(t) \overset{\text{def}}{=} (|p(t)| - \mu\lambda_t \; ; \; t \leq 1)$.

(4.3) An absolute continuity relationship.

Another fairly straightforward extension of the results valid in the Brownian case $(\mu = 1)$ is the following

Proposition 4.1 : *For every measurable functional* $F : C([0,1,\mathbb{R}]) \longrightarrow \mathbb{R}_+$ *, we have :*

(4.a) $$E[F(p_{\mu}(t) \; ; \; t \leq 1)] = \sqrt{\frac{\pi}{2}} \left(\frac{1+\mu}{2}\right) E\left[\frac{1}{\sqrt{\tau_1^{\mu}}} F(p_{\mu}^{\neq}(t) \; ; \; t \leq 1)\right].$$

Proof : It suffices to follow the steps of the proof in [3] ; here again, as for Proposition 2.1, the scaling property is essential. A unification of these various consequences of the scaling property will be presented in [24]. □

It is easy to show that the local time at 0 of $(p^{\neq}_{\mu}(t), t \le 1)$ is $\dfrac{1}{\sqrt{\tau^{\mu}_{1}}}$.

Hence, we deduce from *(4.a)*, with the help of the identity *(1.e)*, the following

Corollary 4.1.1 : *Let* $f : [0,1] \times \mathbb{R}_{+} \longrightarrow \mathbb{R}_{+}$ *be a Borel function ; then, we have :*

$$(4.b) \qquad E\left[f\left(\int_0^1 dt\, 1_{(p_{\mu}(t)\le 0)}\,,\ \lambda^{\mu}_{1}\right)\right] = \sqrt{\frac{\pi}{2}}\,\left(\frac{1+\mu}{2}\right)\, E\left[f(A^{\mu,-}_{1}, \ell^{\mu}_{1})\right],$$

where $(\lambda^{\mu}_{t}\ ;\ t \le 1)$ *denotes the local time at* 0 *of* p_{μ}.

The absolute continuity relationship *(4.a)*, considered for $\mu = 1$, may also be used to obtain the following results concerning the processes q_{ν}.

Proposition 4.2 : *Let* $\nu > 0$. *Define* $A^{-}_{t}(q_{\nu}) \equiv \displaystyle\int_0^t ds\, 1_{(q_{\nu}(s)\le 0)}$, *and let* $(\ell_t(q_{\nu}), t \le 1)$ *be the local time of* q_{ν} *at* 0.

Then, if ν *and* μ *are related by :* $\dfrac{1}{\nu} = 1 + \dfrac{1}{\mu}$, *we have :*

$$(4.c) \qquad E\left[f(A^{-}_{1}(q_{\nu})\ ;\ \ell_{1}(q_{\nu}))\right] = \sqrt{\frac{\pi}{2}}\,\left(\frac{1+\mu}{2}\right) E\left[f(A^{\mu,-}_{1}, \ell^{\mu}_{1})\ell^{\mu}_{1}\right]$$

for every Borel function $f : [0,1] \times \mathbb{R}_{+} \longrightarrow \mathbb{R}_{+}$.

Comparing relations *(4.b)* and *(4.c)*, we obtain the following

Corollary 4.2.1 : *If* μ *and* ν *are related by :* $\frac{1}{\nu} = 1 + \frac{1}{\mu}$ *, then :*

$$(4.d) \qquad (A_1(q_\nu) \; ; \; \ell_1(q_\nu)) \overset{(law)}{=} \left(\int_0^1 dt \; 1_{(p_\mu(t) \le 0)} \; ; \; \lambda_1^\mu \right).$$

In the particular case $\mu = 1, \; \nu = \frac{1}{2}$, the identity in law *(4.d)* follows from a more general result obtained by Pitman-Yor [17] :

(4.e) *the processes of local times, in the space variable* $x \in \mathbb{R}$*, taken at time* 1*, of the Brownian bridge* $(p(t) \; ; \; t \le 1)$ *and of the process* $(q_{1/2}(t) \equiv |p(t)| - \frac{1}{2} \lambda_t \; ; \; t \le 1)$ *are identically distributed.*

The identities in law *(4.d)* and *(4.e)* have led us naturally to the following

Theorem 4.3 : *Let* $\nu > 0, \; \mu > 0$ *be such that :* $\frac{1}{\nu} = 1 + \frac{1}{\mu}$.

The processes $(\ell_1^x(q_\nu) \; ; \; x \in \mathbb{R})$ *and* $(\ell_1^x(p_\mu) \; ; \; x \in \mathbb{R})$ *of local times are identically distributed.*

Before we prove this theorem, we present another interesting identity in law which follows from Theorem 4.3, and we identify the common distribution.

Proposition 4.4 : *If* μ *and* ν *are related by :* $\frac{1}{\nu} = 1 + \frac{1}{\mu}$ *, then :*

$$\sup_{0 \le t \le 1} p_\mu(t) \overset{(law)}{=} \sup_{0 \le t \le 1} q_\nu(t) \equiv S_\nu \; ;$$

furthermore, if N *is a centered reduced gaussian variable, which is independent of* S_ν *, one has :*

$$\exp(2|N|S_\nu) - 1 \overset{(law)}{=} \left(Z_{1,1/2\nu}\right)^{\frac{1-Z_{1,1/\nu}}{Z_{1,1/\nu}}}$$

where, on the right-hand side, the two beta variables are independent.

Here is now a

Proof of Theorem 4.3 : We will show that for every Borel $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ we have :

$$(4.f) \qquad E\left[\exp(-\int f(x)\ell_1^x(p_\mu)dx)\right] = E\left[\exp(-\int f(x)\ell_1^x(q_\nu)dx)\right].$$

Using the absolute continuity relationship *(4.a)* considered for a general μ and for $\mu = 1$, it is equivalent to show :

$$(4.g) \qquad \frac{1+\mu}{2}\left[\frac{1}{\sqrt{\tau_1^\mu}} \exp(-\frac{1}{\tau_1^\mu}\int f\left(\frac{x}{\sqrt{\tau_1^\mu}}\right) \ell^x_{\tau_1^\mu}(X^\mu)\ dx)\right]$$

$$= E\left[\frac{1}{\sqrt{\tau_1}} \exp(-\frac{1}{\tau_1}\int f\left(\frac{x}{\sqrt{\tau_1}}\right) \ell^x_{\tau_1}(X^\nu)\ dx)\right]$$

Let $f_\pm : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two Borel functions such that $f(x) = f_+(x)$ if $x \geq 0$, $f(x) = f_-(-x)$ if $x < 0$. We note :

$$(\tau_1^\mu)^\pm \equiv \int_0^{+\infty} \ell^{\pm x}_{\tau_1^\mu}(X^\mu)dx \ ; \qquad \tau_1^\pm \equiv \int_0^{+\infty} \ell^{\pm x}_{\tau_1}(X^\mu)dx.$$

(Beware $\tau_1^\pm$ depends on ν !)

The main tools we use are :

i) the scaling property of the square of a Bessel process ;

ii) the Ray-Knight theorem which describes the process of the local times of the ν-process considered up to time τ_1 , as an inhomogeneous Markov process (Le Gall-Yor [9]) ;

iii) Theorem 9.1 in Chapter 9 of [23] for the local times of the μ-process considered up to time τ_1^μ ; this is, in fact, another Ray-Knight theorem.

Then, we are able to prove the following :

1) the variables $\dfrac{\tau_1^+}{[\ell^0_{\tau_1}(X^\nu)]^2}$ and $\dfrac{\tau_1^-}{[\ell^0_{\tau_1}(X^\nu)]^2}$ are independent.

More precisely,

$$\frac{\mu+1}{2}\,\mathbb{P}[(\tau_1^\mu)^+ \in du]\,\mathbb{P}[(\tau_1^\mu)^- \in dv]$$

$$= E\left[(\ell^0_{\tau_1}(X^\nu))^{-1}\,\Big|\,\frac{\tau_1^-}{[\ell^0_{\tau_1}(X^\nu)]^2} = v\right]\,\mathbb{P}\left[\frac{\tau_1^+}{[\ell^0_{\tau_1}(X^\nu)]^2} \in du\right]\,\mathbb{P}\left[\frac{\tau_1^-}{[\ell^0_{\tau_1}(X^\nu)]^2} \in dv\right]$$

2) $$E\left[(\ell^0_{\tau_1}(X^\nu)^{-1}\exp\left(-\int_0^{+\infty} f_+\left(\frac{x}{\ell^0_{\tau_1}(X^\nu)}\right)\frac{\ell^x_{\tau_1}(X^\nu)}{(\ell^0_{\tau_1}(X^\nu))^2}\,dx\right)\,\Big|\,\frac{\tau_1^+}{(\ell^0_{\tau_1}(X^\nu))^2}=u,\ \frac{\tau_1^-}{(\ell^0_{\tau_1}(X^\nu))^2}=v\right]$$

$$= \mathbb{Q}_0^{2/\nu}\left[\frac{1}{Y_\nu}\,\Big|\,\frac{\int_0^\nu Y_x dx}{Y_\nu^2} = v\right]\ \mathbb{Q}_1^0\left[\exp\left(-\int_0^{+\infty} f_+(x)Y_x dx\right)\,\Big|\,\int_0^{+\infty} Y_x dx = u\right]$$

$$= E\left[(\ell^0_{\tau_1}(X^\nu))^{-1}\,\Big|\,\frac{\tau_1^-}{(\ell^0_{\tau_1}(X^\nu))^2} = v\right]\ E\left[\exp\left(-\int_0^{+\infty} f_+(x)\,\ell^0_{\tau_1^\mu}(X^\mu)\,dx\right)\,\Big|\,(\tau_1^\mu)^+ = u\right].$$

3) $$E\left[(\ell^0_{\tau_1}(X^\nu))^{-1}\exp\left(-\int_0^{+\infty} f_-\left(\frac{x}{\ell^0_{\tau_1}(X^\nu)}\right)\frac{\ell^{-x}_{\tau_1}(X^\nu)}{(\ell^0_{\tau_1}(X^\nu))^2}dx\right)\,\Big|\,\frac{\tau_1^+}{(\ell^0_{\tau_1}(X^\nu))^2}=u,\ \frac{\tau_1^-}{(\ell^0_{\tau_1}(X^\nu))^2}=v\right]$$

$$= c_\nu\,\frac{\mathbb{Q}_0^{2/\nu}\left[\int_0^{L_1} Y_x dx \in dv\right]}{\mathbb{Q}_0^{2/\nu}\left[\frac{\int_0^\nu Y_x dx}{Y_\nu^2} \in dv\right]} = v\Big] \cdot \mathbb{Q}_0^{2/\nu}\left[\exp\left(-\int_0^{L_1} f_-(L_1-x)Y_x dx\right)\,\Big|\,\int_0^{L_1} Y_x dx = v\right]$$

$$= E\left[(\ell^0_{\tau_1}(X^\nu)^{-1}\,\Big|\,\frac{\tau_1^-}{(\ell^0_{\tau_1}(X^\nu))^2} = v\right] \cdot E\left[\exp\left(-\int_0^{+\infty} f_-(x)\,\ell^{-x}_{\tau_1^\mu}(X^\mu)\,dx\right)\,\Big|\,(\tau_1^\mu)^- = v\right].$$

It now follows that for every Borel function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ we have :

$$(4.h)\qquad \begin{aligned} &\frac{1+\mu}{2}\, E\left[\psi((\tau_1^\mu)^+ \,;\, (\tau_1^\mu)^-)\, \exp(-\frac{1}{\tau_1^\mu}\int f\left(\frac{x}{\sqrt{\tau_1^\mu}}\right) \ell^x_{\tau_1^\mu}(X^\mu)dx)\right] \\ &= E\left[\frac{1}{\ell^0_{\tau_1}(X^\nu)}\, \psi\left(\frac{\tau_1^+}{(\ell^0_{\tau_1}(X^\nu))^2} \,;\, \frac{\tau_1^-}{(\ell^0_{\tau_1}(X^\nu))^2}\right) \exp(-\frac{1}{\tau_1}\int f\left(\frac{x}{\sqrt{\tau_1}}\right) \ell^x_{\tau_1}(X^\nu)\, dx)\right]\end{aligned}$$

from which we deduce *(4.g)* by taking $\psi(s,t) = \dfrac{1}{\sqrt{s+t}}$. □

(4.4) About another proof of the arc sine law.

4.4.1. In the case $\mu = 1$, one may prove that $A_1^- \equiv \int_0^1 ds\, 1_{(B_s<0)}$ is arc-sine distributed by first proving that : $a^- \equiv \int_0^1 du\, 1_{(p(u)\le 0)}$ is uniformly distributed on $[0,1]$, and then using the identity :

$$(4.i)\qquad A_1^- \overset{(law)}{=} a^-\cdot g + \varepsilon(1-g),$$

where $g = \sup\{s < 1 : B_s = 0\}$ is also arc-sine distributed, $\varepsilon = 1_{(B_1<0)}$, and (a^-,g,ε) are independent.

(4.i) follows immediately from the fact that : $\left(\pi(t) \equiv \frac{1}{\sqrt{g}} B_{tg} \,;\, t \le 1\right)$ is a Brownian bridge, which is independent of $\sigma\{g \,;\, B_{g+u}, u \ge 0\}$.

Furthermore, the fact that a^- is uniformly distributed on $[0,1]$ follows easily from the absolute continuity relationship *(4.a)*, from which we deduce :

$$E[f(a^-)] = \sqrt{\frac{\pi}{2}}\, E\left[\frac{1}{\sqrt{\tau(1)}}\, f\left(\frac{A^-(\tau(1))}{\tau(1)}\right)\right].$$

4.4.2. From the previous subsection, the question arises naturally whether the process :

$$\pi_\mu(t) = \frac{1}{\sqrt{g_1^\mu}} X^\mu(tg_1^\mu) \ , \ t \le 1, \text{ where } \ g_1^\mu = \sup\{t < 1 : X^\mu(t) = 0\},$$

is independent from $\sigma\{g_1^\mu \ ; \ X^\mu(g_1^\mu+u), u \ge 0\}$, and also whether π_μ and p_μ have the same distribution.

To discuss these questions which, as we shall see, have an affirmative answer only in the case $\mu = 1$, we shall use again, in an essential way, the scaling property of Brownian motion, which will allow us to express the following expression I_μ in several different, but equivalent, forms :

$$I_\mu \overset{\text{def}}{=} \int_0^{+\infty} ds\ h(s)\ E\left[\ k(g_s^\mu)\ F\left(\frac{1}{\sqrt{g_s^\mu}} X^\mu(vg_s^\mu) \ ; \ v \le 1\right) \right],$$

where $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $k : [0,1] \longrightarrow \mathbb{R}_+$ are two Borel functions, $F : C([0,1],\mathbb{R}) \longrightarrow \mathbb{R}_+$ is a measurable functional, and g_s^μ is the last zero of X^μ before time s.

Decomposing the above time integral with respect to the excursions of X^μ away from 0, we obtain :

$$\begin{aligned} I_\mu &= E\left[\sum_{u>0} \int_{\tau_{u-}^\mu}^{\tau_u^\mu} ds\ h(s)\ k(\tau_{u-}^\mu)\ F\left(\frac{1}{\sqrt{\tau_{u-}^\mu}} X^\mu(v\tau_{u-}^\mu) \ ; \ v \le 1\right)\right] \\ (4.j) \qquad &= E\left[\sum_{u>0} k(\tau_{u-}^\mu)\ F\left(\frac{1}{\sqrt{\tau_{u-}^\mu}} X^\mu(v\tau_{u-}^\mu) \ ; \ v \le 1\right) \int_0^{\tau_u^\mu-\tau_{u-}^\mu} ds\ h(s+\tau_{u-}^\mu)\right]. \end{aligned}$$

To simplify notation, we now introduce

$$\varphi_u = k(\tau_{u-}^\mu)\ F\left(\frac{1}{\sqrt{\tau_{u-}^\mu}} X^\mu(v\tau_{u-}^\mu) \ ; \ v \le 1\right) \qquad (u > 0)$$

which is a previsible process with respect to the filtration $(\mathcal{F}_{\tau_u^\mu}, u \ge 0)$.

The key to the next developments is the following

Lemma 4.5 : *For every* $\mathbb{R}_+$*-valued process* $(\psi_u\ ;\ u > 0)$ *, which is previsible with respect to the filtration* $(\mathcal{F}_{\tau^\mu_u}\ ;\ u \geq 0)$ *, and every Borel function* $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *, one has :*

$$E\left[\sum_{u>0} \psi_u\left(\int_0^{\tau^\mu_u - \tau^\mu_{u-}} ds\ h(s+\tau^\mu_{u-})\right)\right] = E\left[\int_0^\infty du\ \psi_u \int_0^\infty \frac{ds}{\sqrt{s}}\ h(s+\tau^\mu_u)\theta_\mu\left(\frac{1}{s}\ (B(\tau^\mu_u))^2\right)\right]$$

where $(\theta_\mu(x)\ ,\ x > 0)$ *is given by :*

$$\theta_\mu(x) = \frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{x}\ B(\frac{1}{2}\ ,\frac{1}{2\mu})}\int_0^{+\infty} |\sin\ t|^{\frac{1}{\mu}-1}\ \exp\left(-\ \frac{t^2}{2x}\right)dt.$$

Remark 4.6 : 1. In the particular case $\mu = 1$, θ_μ is a constant ; precisely, $\theta_1(x) = \sqrt{\frac{2}{\pi}}$. A posteriori, we may say that the independence of g_1 and π_1 appears as a consequence of the constancy of the function θ_1 ; of course, there are more direct and well-known proofs of this result, and of the identity in law between π_1 and p. (see, for example, [19], Exercise , p.).

2. In the language of the general theory of random processes, the identity obtained in Lemma 4.5 is equivalent to the following property : if $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Borel function, and if we denote $H(x) = \int_0^x ds\ h(s)$, then the $\left(\mathcal{F}_{\tau^\mu_t}\ ,t \geq 0\right)$ predictable projection of $\sum_{u\leq t} H\left(\tau^\mu_u - \tau^\mu_{u-}\right)$ is :

$$\int_0^t du \int_0^{+\infty}\frac{ds}{\sqrt{s}}\ h(s)\ \theta_\mu\left(\frac{1}{s}\ (B(\tau^\mu_u))^2\right). \qquad \square$$

We postpone the proof of the Lemma, and, for the moment, we apply it to $\psi = \varphi$

in *(4.j)* in order to relate the laws of π_μ and $p^{\neq}_\mu$, or p_μ.
Thus, we obtain :

$$I_\mu = \iint_{\mathbb{R}^2_+} \frac{du\ ds}{\sqrt{s}} E[k(u^2\tau^\mu_1) F\left(\frac{X_\mu(v\tau^\mu_1)}{\sqrt{\tau^\mu_1}} ; v \le 1\right) h(s+u^2\tau^\mu_1) \theta_\mu\left(\frac{u^2}{s} B^2(\tau^\mu_1)\right)]$$

(by scaling).
Making the change of variables $y = u^2\tau^\mu_1$ in the integral in (du), we obtain :

$$I_\mu = \iint_{\mathbb{R}^2_+} \frac{dy\ ds}{2\sqrt{ys}} k(y) E\left[\frac{h(s+y)}{\sqrt{\tau^\mu_1}} \theta_\mu\left(\frac{y}{s} \frac{\mu^2\ell^2(\tau^\mu_1)}{\tau^\mu_1}\right) F\left(\frac{X^\mu(v\tau^\mu_1)}{\sqrt{\tau^\mu_1}} ; v \le 1\right)\right]$$

$$= \iint_{\mathbb{R}^2_+} \frac{k(y)dy\ ds}{\sqrt{2ys}} h(y+s)E\left[\frac{1}{\sqrt{\tau^\mu_1}} F(p^{\neq}_\mu(v) ; v \le 1) \theta_\mu(\frac{y}{s} i^2(p^{\neq}_\mu))\right]$$

where : $i(p^{\neq}_\mu) = \inf_{s\le 1} p^{\neq}_\mu(s)$.

Thus, we obtain :

$$(4.k) \qquad I_\mu = \int_0^{+\infty} dt\ h(t) \int_0^t \frac{dy\ k(y)}{2\sqrt{y(t-y)}} E\left[\frac{1}{\sqrt{\tau^\mu_1}} F(p^{\neq}_\mu(v) ; v \le 1) \theta_\mu(\frac{y}{t-y} i^2(p^{\neq}_\mu))\right].$$

On the other hand, from the definition of I_μ , we obtain, by scaling :

$$(4.\ell) \qquad I_\mu = \int_0^{+\infty} ds\ h(s) E[k(sg^\mu_1) F(\pi_\mu(v) ; v \le 1)].$$

Now, comparing *(4.k)* and *(4.ℓ)*, we obtain :

$$E[k(g^\mu_1) F(\pi_\mu(v) ; v \le 1)]$$

$$(4.m) \qquad = \int_0^1 \frac{dy\ k(y)}{2\sqrt{y(1-y)}} E\left[\frac{1}{\sqrt{\tau^\mu_1}} \theta_\mu(\frac{y}{1-y} i^2(p^{\neq}_\mu)) F(p^{\neq}_\mu(v) ; v \le 1)\right]$$

$$= \int_0^1 \frac{dy\ k(y)}{\sqrt{y(1-y)}}\ c_\mu\ E\left[\theta_\mu\left(\frac{y}{1-y}\ i^2(p_\mu^{\neq})\right)\ F(p_\mu^{\neq}(v)\ ;\ v \le 1)\right]$$

where $c_\mu = \frac{1}{(1+\mu)}\sqrt{\frac{2}{\pi}}$, and the equality *(4.m)* follows from *(4.a)*.

Below, we shall exploit formula *(4.m)* to describe the law of g_1^μ and to relate the laws of π_μ and p_μ.

But, first, we give a proof of Lemma 4.5 which, from well-known arguments relating discontinuous martingales of a "nice" Markov process to its Lévy system (see, e.g., Meyer [12]) may be seen as a consequence of the following partial determination of the infinitesimal generator A of the two-dimensional Markov process $(|B_{\tau_t^\mu}|, \tau_t^\mu\ ;\ t \ge 0)$.

Theorem 4.7 : *Let* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *be a* C^1 *function, with suitable integrability conditions. Then,* f, *considered as a function of two variables* (a,z), *belongs to the domain of* A, *and :*

$$Af(a,z) = \int_0^{+\infty} f'(z+s)\ \theta_\mu\left(\frac{a^2}{s}\right) \frac{ds}{\sqrt{s}}\ .$$

Proof of Theorem 4.7 : We proceed as for the generator of the generalized Watanabe process $(|B_{\tau_t^\mu}|)_{t\ge 0}$ (see Carmona-Petit-Yor [4] , section (4.2)).

Then, we obtain that

the semi-group $(P_t)_{t\ge 0}$ of the Markov process $(|B_{\tau_t^\mu}|\ ;\ \tau_t^\mu)_{t\ge 0}$ is given by :

$$P_t f(a;z) = E_a\left[f(|B_{\tau_t^{\mu,a}}|\ ;\ z + \tau_t^{\mu,a})\right]$$

where $\tau_t^{\mu,a}$ is the inverse of the local time at the point a of the μ-process built with a Brownian motion starting at a.

In the particular case where $f(a,z) = \exp(-\frac{\lambda^2}{2} z)$, we deduce from Theorem 3.3 that :

$$P_t f(a;z) = \exp(-\frac{\lambda^2}{2} z)\, E_a\left[\exp(-\frac{\lambda^2}{2}\tau_t^{\mu,a})\right]$$

$$= \exp(-\frac{\lambda^2}{2} z)\, \mathbb{Q}_t^o\left[\exp(-\frac{\lambda^2}{2}\int_0^{T_o} Y_x dx)\right]$$

$$\times\left\{\mathbb{Q}_t^o\left[1_{T_o \le a}\exp(-\frac{\lambda^2}{2}\int_0^{T_o} Y_x dx)\right] + \mathbb{Q}_t^o\left[1_{T_o > a}\exp(-\frac{\lambda^2}{2}\int_0^{a} Y_x dx)\, \mathbb{Q}_{Y_a}^{2-2/\mu}\left(\exp(-\frac{\lambda^2}{2}\int_0^{T_o} Z_x dx)\right)\right]\right\}$$

With the calculations made for the proof of the Corollary 3.4.1, we have :

$$P_t f(a;z) = \exp(-\frac{\lambda^2}{2} z)\,\frac{\exp(-\lambda t/2)}{\Gamma(\frac{1}{2\mu})}\int_0^{+\infty} e^{-s}\, s^{\frac{1}{2\mu}-1}\, ds\, \mathbb{Q}_t\left(\exp(-\frac{\lambda^2}{2}\int_0^a Y_u du + \frac{iN\lambda Y_a}{2\sqrt{2s}})\right)$$

then, with usual computations on Bessel processes,

$$Af(a,z) = \lim_{t\downarrow 0} \exp(-\frac{\lambda^2}{2} z)\, E_a\left[\frac{\exp(-\frac{\lambda^2}{2}\tau_t^{\mu,a}) - 1}{t}\right]$$

$$= -\lambda^2 \exp(-\frac{\lambda^2}{2} z)\, E\left[\frac{1}{\lambda}\left\{1 + \frac{\sqrt{2Z_{1/2\mu}} + iN}{\sqrt{2Z_{1/2\mu}} - iN}\exp(-2a\lambda)\right\}^{-1}\right]$$

where N is a standard gaussian variable which is independent of $Z_{1/2\mu}$.
Then, we develop in serie the term inside the expectation, and we invert the Laplace transforms $\frac{1}{\lambda}\exp(-2an\lambda)$ in $\frac{\lambda^2}{2}$. The theorem follows for each function $f(a,z)=f(z)$ with suitable integrability conditions, for example, for quickly decreasing functions. □

We now discuss shortly the identity *(4.m)*.

Proposition 4.8 : *1) Taking* $F \equiv 1$, *in (4.m), we obtain after some calcula-*

tions :

$$(4.n) \qquad \mathbb{P}[g_1^\mu \in dy] = c_\mu \frac{1_{]0,1[}(y)dy}{\sqrt{y(1-y)}} E\left[\theta_\mu(\frac{y}{1-y}\, i^2(p_\mu))\right]$$

$$= \frac{1}{\pi(1+\mu)} \frac{dy}{\sqrt{y(1-y)}} 1_{]0,1[}(y) + \frac{\Gamma\left(\frac{\mu+1}{2\mu}\right)^2}{\left|\Gamma\left(\frac{\mu+1}{2\mu}(1+i\sqrt{\frac{1-y}{y}})\right)\right|^2} \frac{1_{]0,1[}(y)dy}{2\mu y\, \mathrm{sh}\left[\pi\frac{\mu+1}{2\mu}\sqrt{\frac{1-y}{y}}\right]}$$

2) The identity (4.m) gives the law of $(\pi_\mu(v)\ ;\ v \le 1)$ *conditionally on* g_1^μ

$$(4.o) \qquad E[F(\pi_\mu(v)\ ;\ v \le 1) \mid g_1^\mu = y] = \frac{E\left[\theta_\mu(\frac{y}{1-y}\, i^2(p_\mu))\, F(p_\mu(v)\ ;\ v\le 1)\right]}{E\left[\theta_\mu(\frac{y}{1-y}\, i^2(p_\mu))\right]}$$

3) g_1^μ *and* $(\pi_\mu(v)\ ;\ v \le 1)$ *are independent conditionally on* $i(\pi_\mu) \equiv \inf_{v\le 1} \pi_\mu(v)$.

5. Application to Walsh's processes.

We now present some variants for Walsh's Brownian motions and Bessel processes of the results obtained in the previous sections ; we recall (see [1], [2], [20]) that these Markov processes $(X_t, t \ge 0)$, which take values in $E = \bigcup_{i=1}^{n} I_i$, the union of n rays in the plane, are defined as follows : let $(p_i\ ;\ 1 \le i \le n)$ be a probability on $\{1,2,\dots,n\}$. Consider n rays $(I_i)_{1\le i\le n}$ meeting at the origin. Suppose $(X_t)_{t\ge 0}$ starts at the origin, that its radial part is a Bessel process of dimension $\delta = 2(1-\mu)$, with $\delta \in]0,2[$, and that, when (X_t) reaches the origin, it chooses, at least, heuristically, the i^{th} ray I_i with probability p_i. This process $(X_t)_{t\ge 0}$ may be constructed rigorously using excursion theory : the characteristic (Itô) measure of its excursions away from the origin is given by : $\sum_{i=1}^{n} p_i n_i$, where n_i, the characteristic measure of excursion in I_i , is obtained in a canonical way from the measure of excursions of a δ-dimensional Bessel process (see [2] for more details). In particular, when $n = 2$, and $\delta = 1$, $(X_t)_{t\geq 0}$ is the so-called skew Brownian motion, with $P(X_t > 0) = p_1 \equiv p$ and $P(X_t < 0) = p_2 \equiv 1-p$. (See Walsh [20]).

Let $(\ell_t \ ; \ t \geq 0)$ be the Markovian local time at 0 of $(X_t, t \geq 0)$, or, of its radial part $(|X_t|, t \geq 0)$; $(\ell_t \ ; \ t \geq 0)$ is defined up to a multiplicative constant, which we choose such that $(\tau_u \ ; \ u \geq 0)$, the right continuous inverse of $(\ell_t \ ; \ t \geq 0)$ be a standard stable subordinator of index μ, i.e :

$$E\left[\exp(-\lambda\tau_u)\right] = \exp(-u\lambda^{\mu}), \qquad , \text{ for every } \ u \geq 0,\ \lambda \geq 0.$$

We now define the multidimensional process of times spent in the n rays :

$$\left(A_t^i = \int_0^t ds\ 1_{(X_s \in I_i)} \ ; \ 1 \leq i \leq n \ ; \ t \geq 0\right).$$

We recall the main result of [2]

<u>**Proposition 5.1**</u> : *Let* $(T_1, T_2, \ldots, T_n)$ *be* n *independent one-sided stable variables of index* μ. *We have, for any fixed* $t > 0$ *:*

(5.a) $$\left(\frac{1}{\ell_t^{1/\mu}} A_t^i \ ; \ 1 \leq i \leq n\right) \overset{(law)}{=} (p_i^{1/\mu}\, T_i \ ; \ 1 \leq i \leq n).$$

We now give a short proof of *(4.a)*, following the method developed above in section 2 for Brownian motion, and in section 3 for the (local time) perturbed reflecting Brownian motion. This proof hinges on the following

Proposition 5.2 : *Let* $F : C([0,1] ; E) \longrightarrow \mathbb{R}_+$ *be a measurable functional. Then :*

$$\text{(5.b)} \qquad E\left[F(X_u \; ; \; u \le 1)\, 1_{(X_1 \in I_i)}\right] = E\left[\frac{1}{\alpha_1^i}\, F\left(\frac{X_{s\alpha_1^i}}{\sqrt{\alpha_1^i}} \; ; \; s \le 1\right)\right]$$

where $(\alpha_t^i \; ; \; t \ge 0)$ *is the right-continuous inverse of* $(A_u^i \; ; \; u \ge 0)$.

To finish the proof of Proposition 5.1, we use the same arguments as in paragraph 1.4. We have : $u = \sum_{j=1}^{n} A_u^j$, for $u \ge 0$.

Hence :

$$\text{(5.c)} \qquad \alpha_t^i = t + \sum_{j\neq i} A^j_{\alpha_t^i} = t + \sum_{j \neq i} (A_\tau^j)(\ell_{\alpha_t^i}).$$

As a consequence of excursion theory, the n processes

$$\{(A_\tau^1)(t) \; ; \; (A_\tau^2)(t) \; ;\ldots; \; (A_\tau^n)(t) \; ; \; t \ge 0\}$$

are independent, and furthermore, $(\frac{1}{p_i}\,(A_\tau^i)(t) \; ; \; t \ge 0)$ is a standard one-sided stable process of index $\frac{1}{\mu}$. We then deduce from *(5.b)* and *(5.c)* that, for every measurable $f : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$:

$$\text{(5.d)} \qquad E\left[f\left(\frac{1}{\ell_1^{1/\mu}}\,(A_1^1 \; ;\ldots; \; A_1^n)\right) 1_{(X_1 \in I_i)}\right] = E\left[\left(\frac{A_\tau^i(1)}{\tau(1)}\right) f((A_\tau^1)(1),\ldots,(A_\tau^n)(1))\right].$$

The identity in law *(5.a)* follows.

We also deduce from *(5.d)*, just as in the last statement of Corollary 2.1.1. :

$$\text{(5.e)} \qquad P(X_1 \in I_i \,|\, A_1^i = a \; ; \; A_1^j \; ; \; \ell_1) = a.$$

REFERENCES

[1] **M.T. Barlow, J.W. Pitman, M. Yor** : On Walsh's Brownian motions. *Sém. Probas. XXIII, Lect. Notes in Maths. 1372, Springer (1989), p. 275-293.*

[2] **M.T. Barlow, J.W. Pitman, M. Yor** : Une extension multidimensionnelle de la loi de l'arc sinus. *Sém. Probas. XXIII, Lect. Notes in Maths. 1372, Springer (1989), p. 294-314.*

[3] **Ph. Biane, J.F. Le Gall, M. Yor** : Un processus qui ressemble au pont Brownien. *Sém. Probas. XXI, Lect. Notes in Maths. 1247, Springer (1987), p. 270-275.*

[4] **P. Carmona, F. Petit, M. Yor** : Sur les fonctionnelles exponentielles de certains processus de Lévy. *Stochastics; to appear.*

[5] **N. Ikeda, S. Watanabe** : Stochastic differential equations and Diffusion processes. *Second edition. North Holland/Kodansha (1989).*

[6] **T. Jeulin, M. Yor** : Sur les distributions de certaines fonctionnelles du mouvement brownien. *Sém. Probas. XV, Lect. Notes in Maths. 850, Springer (1981), p. 210-226.*

[7] **I. Karatzas, S.E. Shreve** : Brownian motion and stochastic calculus. *Springer (1987).*

[8] **J.F. Le Gall, M. Yor** : Enlacements du mouvement brownien autour des courbes de l'espace.
Trans. Amer. Math. Soc. 317 (1990), p. 687-722.

[9] **J.F. Le Gall, M. Yor** : Excursions browniennes et carrés de processus de Bessel.
Comptes Rendus Acad. Sci. Paris, Série I, 303 (1986), p. 73-76.

[10] **P. Lévy** : Sur certains processus stochastiques homogènes.
Compositio Math., 7 (1939), p. 283-339.

[11] **H.P. Mc Kean** : Brownian local time.
Adv. Math. 16 (1975), p. 91-111.

[12] **P.A. Meyer** : Intégrales stochastiques IV.
Sém. Probas. I, Lect. Notes in Maths. 39, Springer (1967), p. 142-162.

[13] **M. Perman, J. Pitman, M. Yor** : Size-biased sampling of Poisson point processes, and excursions.
Proc. Th. and Rel. Fields, 92, 1, 1992, p. 21-40.

[14] **F. Petit** : Sur le temps passé par le mouvement brownien au-dessus d'un multiple de son supremum, et quelques extension de la loi de l'arc sinus. *Part of a Thèse de Doctorat, Université Paris VII, February 1992.*

[15] **F. Petit** : Quelques extensions de la loi de l'arc sinus.
Comptes rendus Acad. Sci. Paris, t. 315, Série I, p. 855-858, 1992.

[16] **J.W. Pitman, M. Yor** : Arc sine laws and interval partitions derived from a stable subordinator.
Proc. London Math. Soc. (3) 65, 1992, p. 326-356.

[17] J.W. Pitman, M. Yor : Random scaling of Brownian and Bessel Bridges.
Preprint - University of California (Aug. 1992).

[18] J.W. Pitman, M. Yor : Dilatations d'espace-temps, réarrangements des trajectoires browniennes, et quelques extensions d'une identité de Knight.
Comptes Rendus Acad. Sci. Paris, t. 316, Série I (1993), p. 723-726.

[19] D. Revuz, M. Yor : Continuous Martingales and Brownian Motion.
Springer (1991).

[20] J.B. Walsh : A diffusion with discontinuous local time.
In : "Temps locaux". *Astérisque 52-53 (Société Mathématique de France), 1978, p. 37-45.*

[21] S. Watanabe : Generalized arc-sine laws for one-dimensional diffusion processes and random walks. *Preprint (April 1993).*

[22] D. Williams : Markov properties of Brownian local time.
Bull. Amer. Math. Soc. 76, 1969, p. 1035-1036.

[23] M. Yor : Some Aspects of Brownian Motion,
Part I : Some Special Functionals.
Lectures in Mathematics, ETH Zürich, Birkhaüser (1992).

[24] M. Yor : Random Brownian scaling and some absolute continuity relationships.
Preprint (July 1993). Laboratoire de Probabilités, Paris VI.