Markovian Bridges: Construction, Palm Interpretation, and Splicing

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1. Introduction.

By a Markovian bridge we mean a process obtained by conditioning a Markov process X to start in some state x at time 0 and arrive at some state z at time t. Once the definition is made precise, we call this process the (x,t,z)-bridge derived from X. Important examples are provided by Brownian and Bessel bridges, which have been extensively studied and find numerous applications. See for example [**PY1, SW, Sa, H, EL, AP, BP**]. It is part of Markovian folklore that the right way to define bridges in some generality is by a suitable Doob h-transform of the space-time process. This method was used by Getoor and Sharpe [**GS4**] for excursion bridges, and by Salminen [**Sa**] for one-dimensional diffusions, but the idea of using h-transforms to construct bridges seems to be much older. Our first object in this paper is to make this definition of bridges precise in a suitable degree of generality, with the aim of dispelling all doubts about the existence of clearly defined bridges for nice Markov processes. This we undertake in Section 2. In Section 3 we establish a conditioning formula involving bridges and continuous additive functionals of the Markov process. This formula can be found in [**RY**, Ex. (1.16) of Ch. X, p.378] under rather stringent continuity conditions. One of our goals here is to prove the formula in its "natural" setting. We apply the conditioning formula in Section 4 to show how Markovian bridges are involved in a family of Palm distributions associated with continuous additive

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functionals of the Markov process. This generalizes an approach to bridges suggested in a particular case by Kallenberg [**K1**], and connects this approach to the more conventional definition of bridges adopted here.

Our concern with these matters was prompted by an interesting splicing construction, the inverse of a path decomposition that we now describe in the context where we first encountered it. Perman, Pitman and Yor [**PPY**] considered a process $Y = (Y_s)_{0 \le s \le 1}$ constructed as follows from a standard Brownian bridge $B = (B_s)_{0 \le s \le 1}$, that is to say the $(0,1,0)$-bridge derived from a standard one-dimensional Brownian motion. Let U be a random time independent of B, with uniform distribution on $[0,1]$. Let G be the time of the last zero of B before time U, D the time of the first zero of B after time U. Delete the segment of the path of B over the excursion interval $]G,D[$, close up the gap of length $D-G$ to obtain a path of random length $1-(D-G)$, then standardize this path by Brownian scaling to obtain a random path Y parameterized by $[0,1]$. According to [**PPY**, Corollary 3.15], the law of this process Y is absolutely continuous with respect to the law of the standard Brownian bridge B, with density proportional to L_1, the local time of B at zero up to time 1. On the other hand, Aldous and Pitman [**AP**] give a decomposition of the original Brownian bridge B into three independent paths: two standard bridges obtained by Brownian scaling of the segments of B on $[0,G]$ and B on $[D,1]$, and one standard signed excursion obtained similarly from B on $[G,D]$; moreover these three paths of length 1 are independent of the triple of interval lengths $(G, D-G, 1-D)$, which has the same exchangeable Dirichlet(1/2,1/2,1/2) distribution as $(N^2/\Sigma, M^2/\Sigma, O^2/\Sigma)$, where N, M and O are independent standard normal variables, and $\Sigma = N^2 + M^2 + O^2$. Since the path of Y can be recovered from the two bridges in this decomposition and the fraction $T = \frac{G}{G+1-D}$, it follows that Y admits the following decomposition at time T:

(i) $\mathbb{P}(T \in dt) = \rho(t)dt$, where $\rho(t) = (\pi\sqrt{t(1-t)})^{-1}$ is the arc-sine density on $[0,1]$ of $N^2/(N^2+O^2)$, and

(ii) conditional on the event $\{T = t\}$, the process Y splits into independent pieces

$(Y_s)_{0\le s\le t}$ and $(Y_s)_{t\le s\le 1}$ whose conditional distributions are those of Brownian bridges over the time intervals $[0,t]$ and $[t,1]$ respectively.

Here by a Brownian bridge over $[u,v]$, say, we mean a standard Brownian motion started at 0 at time u and conditioned to return to 0 at time v.

The above observation prompts the question of what process Y is obtained by prescribing some other density $\rho(t)$ on $[0,1]$ for T as in (i) above, then splicing together two Brownian bridges of lengths T and $1-T$, as in (ii)? The law of Y can be computed explicitly:

$$\mathbb{P}(Y \in d\omega) = \int_0^1 f(t)\, dL_t(\omega) \cdot \mathbb{P}(B \in d\omega), \tag{1.0}$$

where B denotes a Brownian bridge, $L_t(\omega)$ is the Brownian local time at zero up to time t, defined for all $0 \le t \le 1$ for almost all ω relative to the law of B, and $f(t) = \rho(t)\sqrt{2\pi t(1-t)}$. Using a version of Bayes' rule one can compute the conditional distribution of T given Y:

$$\mathbb{P}(T \in dt \,|\, Y) = \frac{f(t)\, dL_t(Y)}{\int_0^1 f(s)\, dL_s(Y)}. \tag{1.1}$$

In particular, in case $\rho(t)$ is the arcsine density, $f(t) = \sqrt{2/\pi}$ is constant, the density of the law of Y relative to B reduces to $\sqrt{2/\pi} L_1$, and the conditional distribution of T given Y is proportional to the local time: $P(T \le t|Y) = L_t(Y)/L_1(Y)$. Note also that no such splicing of bridges can yield a "pure" bridge Y, for the density factor in (1.0) cannot be 1 almost surely, no matter what the choice of $\rho(t)$. Put another way, given a standard Brownian bridge B, it is impossible to find a random time T such that T falls almost surely in the zero set of B, and given T the processes B on $[0,T]$ and B on $[T,1]$ are independent bridges over these intervals.

In the main result of this paper, we apply the Palm interpretation of bridge distributions to establish a general "splicing" construction for Markov processes which includes the above example as a special case. An informal description of this result is as follows.

Let $(X_s)_{s\geq 0}$ be a time-homogeneous Markov process with state space E. Fix $\ell > 0$ and let a space-time point (Z,T) be chosen according to a suitable distribution on $E \times [0,\ell]$. Conditional on $\{Z = z, T = t\}$ let Y be the concatenation of independent X-bridges, the first an (x,t,z)-bridge, the second a $(z,\ell - t, y)$-bridge. Then the (unconditional) distribution of Y is absolutely continuous with respect to the law of an (x,ℓ,y)-bridge, and the Radon-Nikodym derivative can be written down explicitly. Moreover, the conditional distribution of (Z,T) given Y can be expressed by a formula analogous to (1.1). This result is presented more formally as Proposition 4 in Section 4. Finally, in Section 5 we record a general probabilistic interpretation of the family of Palm measures associated with a random measure, which underlies the splicing result in the context of Markovian bridges.

2. Construction of Bridges

For the rest of the paper, we will work with a right Markov process $X = (X_t)_{t\geq 0}$ with state space E and transition semigroup (P_t). Thus X is a strong Markov process with right continuous sample paths. To streamline the exposition, we assume that E is Lusinian (i.e., homeomorphic to a Borel subspace of some compact metric space), that P_t maps Borel functions to Borel functions, and that the paths of X are cadlag. This allows us to realize X as the coordinate process on the sample space Ω of all cadlag paths from $[0,\infty[$ to E. The law of X when started at x is $\mathbb{P}_x$. We write $(\mathcal{F}_t)_{t\geq 0}$ for the natural (uncompleted) filtration of X and $(\theta_t)_{t\geq 0}$ for the usual shift operators. (Certain details, in particular the completion of the σ-algebras $\mathcal{F}_t$, will be left to the reader.)

A reasonable theory of bridges requires (at a minimum) that X have transition densities; namely that

$$P_t(x,dy) = p_t(x,y)\, m(dy) \tag{2.1}$$

where m is a σ-finite measure on E. Note that this condition forbids jumps at fixed times:

$$\mathbb{P}_x(X_t = X_{t-}) = 1, \quad \forall t > 0, x \in E. \tag{2.1a}$$

To see this, consider T, the first time at which X has a jump of size bigger than ϵ; then by virtue of (2.1), $\mathbb{P}_x(T = t) = 0$ for all $t > 0$ by [**GS4**, (3.18)].

To develop things fully we need to assume a bit more. We suppose that there is a second right process $\hat{X}$ in duality with X relative to the measure m. This means that the semigroup $(\hat{P}_t)$ of $\hat{X}$ is related to (P_t) by

$$\int_E f(x) P_t g(x)\, m(dx) = \int_E \hat{P}_t f(x) g(x)\, m(dx), \tag{2.2}$$

for all $t > 0$ and all positive Borel functions f and g. For simplicity we assume that X and $\hat{X}$ have infinite lifetimes; this implies that m is an invariant measure for X (and for $\hat{X}$ as well). It is known that (2.1) and (2.2) imply that there is a version of the density $p_t(x, y)$ that is jointly measurable in (t, x, y) and such that the Chapman-Kolmogorov identity

$$p_{t+s}(x, y) = \int_E p_t(x, z) p_s(z, y)\, m(dz) \tag{2.3}$$

holds for all $s, t > 0$, and $x, y \in E$. Moreover, the dual of (2.1) is valid:

$$\hat{P}_t(x, dy) = p_t(y, x)\, m(dy). \tag{2.4}$$

See [**D, GS4, Wi, Y**].

Note, for example, that any one-dimensional regular diffusion without absorbing boundaries satisfies the above hypotheses, with the speed measure of the diffusion serving as the reference measure m ; see [**IM**, p. 149 ff.].

We now use Doob's method of h-transforms to construct bridge laws $\mathbb{P}^{\ell}_{x,y}$, which for each x and ℓ will serve as a family of regular $\mathbb{P}_x$ conditional laws for $(X_t, 0 \le t < \ell)$ given $X_{\ell-} = y$. In view of (2.1a), these $\mathbb{P}^{\ell}_{x,y}$ will serve equally well as conditional laws given

$X_\ell = y$ rather than given $X_{\ell-} = y$. But with the h-transform approach it is natural to think primarily in terms of conditioning the left limit. For background and further details, the reader can consult [**M1, F, AJ, GS1, GS4, RY**].

Fix $x, y \in E$ and $\ell > 0$ such that $0 < p_\ell(x,y) < \infty$. Using (2.3) it is a simple matter to check that the process

$$H_t = p_{\ell-t}(X_t, y), \quad 0 \le t < \ell,$$

is a (positive) martingale under $\mathbb{P}_x$. Consequently, the formula

$$Q(A) = \int_A H_t(\omega)\,\mathbb{P}_x(d\omega), \qquad A \in \mathcal{F}_t, \quad 0 \le t < \ell, \tag{2.5}$$

defines a finitely additive set function $Q = Q^\ell_{x,y}$ on the algebra $\mathcal{G} = \cup_{0\le t<\ell}\mathcal{F}_t$ such that each restriction $Q|_{\mathcal{F}_t}$ is σ-additive. We claim that Q extends to a measure on $\mathcal{F}_{\ell-}$, the σ-algebra generated by $\mathcal{G}$. This extension, when normalized by $p_\ell(x,y)$, will be the law $\mathbb{P}^\ell_{x,y}$.

We verify the claim as follows. Let Ω_ℓ be the space of right continuous paths from $[0,\ell[$ to E that have left limits on $]0,\ell[$. We can view Q as a finitely additive measure on Ω_ℓ in the obvious way. The point is that Ω_ℓ equipped with its natural filtration $(\mathcal{G}_t)_{0\le t<\ell}$ is "projectively closed" so that we may apply the projective limit theorem to conclude that Q extends to a (σ-additive) measure Q^* on the σ-algebra $\vee_{0\le t<\ell}\mathcal{G}_t$. (See [**AJ**] or the appendix in [**F**]; $(\Omega_\ell, (\mathcal{G}_t)_{0\le t<\ell})$ is a "standard system" in the terminology of Parthasarathy [**Pa**].) If we knew that Q^* gave full measure to the set of paths with left limits at ℓ, then we could identify Q^* with a measure on $(\Omega, \mathcal{F}_{\ell-})$ and this would be the desired extension. Here is the first place where the duality hypothesis (2.2) comes into play. For we can make the dual construction, obtaining a measure $\hat{Q}^*$ on Ω_ℓ corresponding to $\hat{X}$ started at y and conditioned to have left limit x at time ℓ. Let Ω^+_ℓ be the space of cadlag maps from $]0,\ell[$ to E. We can view both Q^* and $\hat{Q}^*$ as measures on Ω^+_ℓ. Using (2.2), a check of finite dimensional distributions shows that Q^* is the image of $\hat{Q}^*$ under the time-reversal mapping which sends a path $\omega \in \Omega_\ell$ to the path $(\omega((\ell-t)-))_{0<t<\ell}$.

Since $\hat{Q}^*$ is concentrated on those paths whose right limits at time 0 exist and equal y, it must be that Q^* concentrates its mass on those paths whose left limits at time ℓ exist and equal y.

Thus there is a measure $Q = Q^\ell_{x,y}$ on $(\Omega, \mathcal{F}_{\ell-})$ such that (2.5) holds. Let us define

$$\mathbb{P}^\ell_{x,y} = \begin{cases} [p_\ell(x,y)]^{-1} \, Q^\ell_{x,y}, & \text{if } 0 < p_\ell(x,y) < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

The above discussion is summarized in the following

Proposition 1. *Under the assumptions (2.1), (2.2), and (2.3), if $0 < p_\ell(x,y) < \infty$ then there is a unique probability measure $\mathbb{P}^\ell_{x,y}$ on $(\Omega, \mathcal{F}_{\ell-})$ such that*

$$\mathbb{P}^\ell_{x,y}(F) \cdot p_\ell(x,y) = \mathbb{P}_x(F \cdot p_{\ell-t}(X_t, y)) \tag{2.6}$$

for all positive $\mathcal{F}_t$-measurable functions F on Ω, for all $0 \le t < \ell$. Under $\mathbb{P}^\ell_{x,y}$, the coordinate process $(X_t)_{0 \le t < \ell}$ is a non-homogeneous strong Markov process with transition densities

$$p^{(y,\ell)}(z,s;z',t) = \frac{p_{t-s}(z,z')p_{\ell-t}(z',y)}{p_{\ell-s}(z,y)}, \qquad 0 < s < t < \ell. \tag{2.7}$$

Moreover $\mathbb{P}^\ell_{x,y}(X_0 = x, X_{\ell-} = y) = 1$. Finally, if $F \ge 0$ is $\mathcal{F}_{\ell-}$-measurable, and $g \ge 0$ is a Borel function on E, then

$$\mathbb{P}_x(F \, g(X_{\ell-})) = \int_E \mathbb{P}^\ell_{x,y}(F) \, g(y) \, p_\ell(x,y) \, m(dy), \tag{2.8}$$

a formula which holds also with X_ℓ instead of $X_{\ell-}$. Thus $(\mathbb{P}^\ell_{x,y})_{y \in E}$ is a regular version of the family of conditional probability distributions $\mathbb{P}_x(\cdot \,|X_{\ell-} = y)$, $y \in E$, equally so with X_ℓ instead of $X_{\ell-}$.

Proof. The strong Markov property of $(X_t)_{0 \le t < \ell}$ under $\mathbb{P}^\ell_{x,y}$ follows from (2.6) by optional stopping, and the reader can easily verify the formula (2.7) for the transition densities. Thanks to the monotone class theorem, in proving (2.8) it suffices to consider

$\mathcal{F}_t$-measurable F, where $0 < t < \ell$. For such F, by the Markov property at time t, (2.6), and (2.1a),

$$\begin{aligned}\mathbb{P}_x(Fg(X_{\ell-})) &= \mathbb{P}_x(F\,P_{\ell-t}\,g(X_t)) \\ &= \mathbb{P}_x(F\int p_{\ell-t}(X_t,y)g(y)\,m(dy)) = \int \mathbb{P}_x(F\,p_{\ell-t}(X_t,y))\,g(y)\,m(dy) \\ &= \int \mathbb{P}^{\ell}_{x,y}(F)p_\ell(x,y)\,g(y)\,m(dy),\end{aligned}$$

as desired. □

Remark. A few comments concerning ratios such as (2.7) are in order. First note that if $p_\ell(x,y) < \infty$, then the optional sampling theorem applied to the martingale $H_t = p_{\ell-t}(X_t,y)$ shows that $\{(z,t) : p_{\ell-t}(z,y) = \infty\}$ is $\mathbb{P}_x$-polar in the sense that $\mathbb{P}_x(p_{\ell-t}(X_t,y) = \infty$ for some $t \in]0,\ell[\,) = 0$. Because of (2.6), it is likewise true that $\{(z,t) : p_{\ell-t}(z,y) = \infty\}$ is $\mathbb{P}^{\ell}_{x,y}$-polar. Similarly, since a positive martingale "sticks" to the value 0 once that value is attained, we have $\mathbb{P}^{\ell}_{x,y}(p_{\ell-t}(X_t,y) = 0$ for some $t \in [0,\ell[\,) = 0$. These observations should comfort the reader should our subsequent treatment of ratios such as (2.7) seem cavalier.

We conclude this section by mentioning some results which flow easily from Proposition 1 under the basic hypotheses (2.1), (2.2) and (2.3).

Corollary 1. *Suppose* $0 < p_\ell(x,y) < \infty$. *The* $\mathbb{P}^{\ell}_{x,y}$*-law of the time-reversed process* $(X_{(\ell-t)-})_{0\le t<\ell}$ *is* $\hat{\mathbb{P}}^{\ell}_{y,x}$, *the law of a* (y,ℓ,x)*-bridge for the dual process* $\hat{X}$.

Corollary 2. *Suppose* $0 < p_\ell(x,y) < \infty$. *For each* $(\mathcal{F}_t)$ *stopping time* T, *a* $\mathbb{P}^{\ell}_{x,y}$ *regular conditional distribution for* $(X_{T+u}, 0 \le u < \ell - T)$ *given* $\mathcal{F}_T$ *on* $(T < \ell)$ *is provided by* $\mathbb{P}^{\ell-T}_{X_T,y}$.

Applied to the dual process after time reversal and conditioning on X_ℓ, Corollary 2 implies the following decomposition of the original Markov process X at random times τ that correspond to stopping times on the reversed time scale (first for τ with $\tau < \ell$, then for unbounded τ by an easy argument). A special case of this result is due to Kallenberg [**K1**]. See also [**Yo**] for an application to Bessel processes.

Corollary 3. *Suppose that X is a Hunt process. For each random time τ such that $(\tau \geq t)$ is in the σ-field generated by $(X_{t+u}, u \geq 0)$, a $\mathbb{P}_x$ regular conditional distribution for $(X_t, 0 \leq t < \tau)$ given $(\tau, X_{\tau-}, X_{\tau+u}, u \geq 0)$, is provided by $\mathbb{P}^{\tau}_{x, X_{\tau-}}$.*

A crucial ingredient of the proof is the "left Markov property" of X at such times τ, which are co-optional relative to the space-time process based on X. See [**GS2, GS3**] for details, especially section 6 of [**GS3**].

3. Conditioning Formula.

Recall that a continuous additive functional (CAF) of X is a continuous increasing adapted process $(A_t)_{t \geq 0}$ such that for all $s, t \geq 0$, for $\mathbb{P}_x$ a.e. $\omega \in \Omega$,

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega). \tag{3.1}$$

By a "perfection" theorem due to Walsh [**Wa**], we can assume without loss of generality that (3.1) holds for all $s, t \geq 0$ and all $\omega \in \Omega$, that $t \mapsto A_t(\omega)$ is continuous for all $\omega \in \Omega$, and that A_t is $\mathcal{F}^*_{t+}$-measurable for each $t \geq 0$, where $\mathcal{F}^*_t$ is the universal completion of $\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$. See also the appendix in [**G**] for a detailed discussion.

In the following discussion we fix a CAF $A = (A_t)$ such that

$$\mathbb{P}_x(A_t < \infty) = 1, \quad \forall t > 0, x \in E. \tag{3.2}$$

All that follows is based on a formula, due to Revuz [**R**], allowing the explicit computation of expectations involving A. The *Revuz measure* ν associated with A is defined by

$$\nu(g) = t^{-1} \mathbb{P}_m \left(\int_0^t g(X_s)\, dA_s \right), \tag{3.3}$$

where g is any positive Borel function on E. The fact that the right side of (3.3) does not depend on $t > 0$ is an easy consequence of (3.1) and the invariance of m. Because of (3.2), the measure ν is σ-finite; see [**R**, p. 509]. The role of ν is expressed in the following key formula: if f is a positive measurable function on $E\times[0,\infty[$, then for all $x\in E$ and $t>0$,

$$\mathbb{P}_x\left(\int_0^t f(X_s,s)\,dA_s\right)=\int_0^t ds\int_E \nu(dy)\,p_s(x,y)f(y,s). \tag{3.4}$$

Actually, the Laplace-transformed version of this formula (under the hypothesis of resolvent duality) was established by Revuz [**R**] for standard processes, and under weaker conditions by Meyer [**M2**]. Getoor & Sharpe [**GS4**] have proved formula (3.4) under the present hypotheses (plus standardness) by applying Revuz' result to the space-time process (X_t,t), which is in resolvent duality with $(\hat{X}_t,-t)$. Since we are restricting attention to continuous additive functionals, standardness is unnecessary. See [**AM**] and also [**GS1**].

Before proceeding we record the version of (3.4) appropriate to bridges.

Lemma 1. *Fix $\ell>0$ and $x,y\in E$ such that $0<p_\ell(x,y)<\infty$. Then for $0\le t\le\ell$ and Borel $f\ge 0$,*

$$\mathbb{P}^\ell_{x,y}\left(\int_0^t f(X_s,s)\,dA_s\right)=\int_0^t ds\int_E \nu(dz)\left[\frac{p_s(x,z)p_{\ell-s}(z,y)}{p_\ell(x,y)}\right]f(z,s). \tag{3.5}$$

Proof. It suffices to prove the lemma in case $0<t<\ell$. Using (2.6) we have

$$p_\ell(x,y)\,\mathbb{P}^\ell_{x,y}\left(\int_0^t f(X_s,s)\,dA_s\right)=\mathbb{P}_x\left(\int_0^t f(X_s,s)\,dA_s\cdot p_{\ell-t}(X_t,y)\right). \tag{3.6}$$

But for fixed t the $\mathbb{P}_x$-optional projection of the (constant in time) process $s\mapsto p_{\ell-t}(X_t,y)$, $0\le s\le t$, is the process $s\mapsto P_{t-s}(X_s,p_{\ell-t}(\cdot,y))$, $0\le s\le t$, which coincides with $p_{\ell-s}(X_s,y)$ because of the Chapman-Kolmogorov identity. Feeding this into (3.6), and then using (3.4), we arrive at the right side of (3.5), and the lemma is proved. □

We now prove our CAF conditioning formula, first for the laws $\mathbb{P}_x$.

Proposition 2. *If $H\ge 0$ is a predictable process and $f\ge 0$ is a Borel function on $E\times[0,\infty[$, then*

$$\mathbb{P}_x\left(\int_0^t H_s\,f(X_s,s)\,dA_s\right)=\mathbb{P}_x\left(\int_0^t \mathbb{P}^s_{x,X_s}(H_s)\,f(X_s,s)\,dA_s\right). \tag{3.7}$$

Proof. By the monotone class theorem we need only consider processes H of the form $H_s = 1_{]a,b]}(s) \cdot C$, where $0 \le a < b \le t$ and $C \in \mathcal{F}_a$ is bounded and positive. For such an H the left side of (3.7) becomes

$$\text{(3.8)} \qquad \begin{aligned} \mathbb{P}_x\left(C \cdot \int_a^b f(X_s, s)\, dA_s\right) &= \mathbb{P}_x\left(C \cdot \left(\int_0^{b-a} f(X_u, u+a)\, dA_u\right) \circ \theta_a\right) \\ &= \mathbb{P}_x\left(C \cdot \mathbb{P}_{X_a}\left(\int_0^{b-a} f(X_u, u+a)\, dA_u\right)\right). \end{aligned}$$

On the other hand the right side of (3.7) equals

$$\text{(3.9)} \qquad \begin{aligned} &\mathbb{P}_x\left(\int_a^b \mathbb{P}^s_{x,X_s}(C)\, f(X_s, s)\, dA_s\right) \\ &= \int_a^b ds \int_E \nu(dz) \mathbb{P}^s_{x,z}(C)\, p_s(x,z)\, f(z,s), \quad \text{by (3.4)} \\ &= \int_a^b ds \int_E \nu(dz) \mathbb{P}_x(C\, p_{s-a}(X_a, z))\, f(z,s), \quad \text{by (2.6)} \\ &= \mathbb{P}_x(C\, g(X_a)), \end{aligned}$$

where

$$g(y) = \int_a^b ds \int_E \nu(dz) p_{s-a}(y,z) f(z,s) = \mathbb{P}_y\left(\int_0^{b-a} f(X_u, u+a)\, dA_u\right).$$

Thus (3.8) and (3.9) combine to yield (3.7). □

Remark. Since the diffuse measure dA_s does not charge the countable set of discontinuities of X, formula (3.7) holds also with X_s replaced by X_{s-} (three times). This point should be borne in mind in the sequel. It is also worth noting that Proposition 2 is valid if X is a Hunt process and A is a *predictable* (but possibly discontinuous) additive functional, provided X_s is replaced by X_{s-} in (3.3) and (3.7). (See [**GS4**].) In fact, the same replacement makes (3.4) valid for predictable A, and the reader will have noticed that the continuity of A was not used explicitly in the proof of Proposition 2. Any Hunt process satisfying (2.1) (but not necessarily the duality hypothesis (2.2)) for which bridge laws $\mathbb{P}^\ell_{x,y}$ can be constructed such that the altered form of (3.7) holds for all predictable H and A, must satisfy the CMF (=Co-Markovien Forte) condition of Azéma [**A**]. If, in addition, the reference measure m is invariant, then by results in [**A**], X must have a strong Markov dual $\hat{X}$ such that (2.2) holds. In this sense our duality hypothesis is not too much to ask if we wish (3.7) to hold.

The bridge form of (3.7) follows from Proposition 2 in the same way that Lemma 1 followed from (3.4). We state the result but leave the proof to the reader.

Proposition 3. *If $H \geq 0$ is a predictable process and $f \geq 0$ is a Borel function on $E \times [0, \infty[$, and if $0 < p_\ell(x,y) < \infty$, then for $0 \leq t \leq \ell$,*

$$\mathbb{P}^\ell_{x,y}\left(\int_0^t H_s\, f(X_s, s)\, dA_s\right) = \mathbb{P}^\ell_{x,y}\left(\int_0^t \mathbb{P}^s_{x,X_s}(H_s)\, f(X_s, s)\, dA_s\right). \tag{3.10}$$

More general versions of Propositions 2 and 3, in which $\int_0^t f(X_s, s)\, dA_s$ is replaced by an arbitrary predictable additive functional of the space-time process, are also valid if X is a Hunt process. The ramified form of Propostion 3 leads to an alternate proof of Corollary 3 once one notes that for a time τ as in the corollary, the dual predictable projection of the increasing process $t \mapsto 1_{\{\tau \leq t\}}$ is an additive functional of the space-time process. (Recall from the remark following Corollary 3 that τ is co-optional for the space-time process.) In the same vein one can express the conditional distribution of $(X_t, 0 \leq t < \tau)$ given $(X_{\tau-}, \tau)$ in terms of bridge laws, provided the dual predictable projection of $t \mapsto 1_{\{0<\tau\leq t\}}$ has the form $\int_0^t \lambda_s\, dAs$, where λ is a predictable process and A is a predictable additive functional. See [**JY**, Prop. (3.13)] for a prototype result in this direction.

4. Splicing Bridges.

We are now prepared to prove the splicing theorem outlined in section 1. Fix $x, y \in E$ and $\ell > 0$ such that $0 < p_\ell(x,y) < \infty$. Also, fix a probability distribution on $E \times]0, \ell[$ of the form $\rho(z,t)\,\nu(dz)\,dt$, where $\rho(z,t) = 0$ whenever $p_t(x,z)p_{\ell-t}(z,y) = 0$, and where ν is the Revuz measure of a CAF $A = (A_t)$ as in section 3. (Revuz [**R**] has shown that a measure ν is associated with a CAF satisfying (3.2) if and only if ν charges no semipolar set and there is a strictly positive Borel function ψ on E such that $\sup_{x\in E}\int_0^\infty \exp(-t)p_t(x,y)\,\psi(y)\,\nu(dy) < \infty$ and $\inf\{t>0:\psi(X_t)\le 1/n\}\to\infty$ as $n\to\infty$, almost surely.) Consider the probability space $(\Omega^*,\mathcal{F}^*,\mathbb{P}^*)$:

$$\Omega^* = E\times]0,\ell[\times\Omega\times\Omega,$$

$$\mathcal{F}^* = \mathcal{E}\otimes\mathcal{B}(]0,\ell[)\otimes\mathcal{F}_{\ell-}\otimes\mathcal{F}_{\ell-},$$

$$\mathbb{P}^*(dz,dt,d\omega,d\omega') = \rho(z,t)\,\nu(dz)\,dt\,\mathbb{P}^t_{x,z}(d\omega)\mathbb{P}^{\ell-t}_{z,y}(d\omega').$$

(A path chosen according to $\mathbb{P}^t_{x,z}$ is continued beyond time t with the constant value z; we thereby identify $\mathbb{P}^t_{x,z}$ with a law on $\mathcal{F}_{\ell-}$ that is concentrated on $\{\omega:\omega(s)=\omega(t-),t\le s<\ell\}$.) We write $\omega^*=(z,t,\omega,\omega')$ for the generic point of Ω^*, and $(\omega/t/\omega')$ for the path obtained by splicing ω and ω' together at time t. That is,

$$(\omega/t/\omega')(s) = \begin{cases}\omega(s), & \text{if } 0\le s<t,\\ \omega'(s-t), & \text{if } s\ge t.\end{cases}$$

Notice that $\omega=(\omega/t/\theta_t\omega)$. Now define

$$Z(\omega^*)=z,\qquad T(\omega^*)=t,$$

$$Y_s(\omega^*) = X_s(\omega/t/\omega'),\quad 0\le s<\ell,$$

$$A^*_s(\omega^*) = A_s(\omega/t/\omega'),\quad 0\le s\le\ell.$$

Thus

$$\mathbb{P}^*(Z\in dz, T\in dt) = \rho(z,t)\nu(dz)\,dt,$$

and the conditional distribution of Y given $\{Z=z,\ T=t\}$ is the law $\mathbb{P}^t_{x,z}\circ\mathbb{P}^{\ell-t}_{z,y}$ on $(\Omega,\mathcal{F}_{\ell-})$ defined by

$$\mathbb{P}^t_{x,z}\circ\mathbb{P}^{\ell-t}_{z,y}(H) = \int_\Omega\int_\Omega \mathbb{P}^t_{x,z}(d\omega)\mathbb{P}^{\ell-t}_{z,y}(d\omega')H(\omega/t/\omega'). \tag{4.1}$$

In other words, under $\mathbb{P}^t_{x,z}\circ\mathbb{P}^{\ell-t}_{z,y}$ the path fragments $(Y_s)_{0\le s<t}$ and $(Y_{t+s})_{0\le s<\ell-t}$ are independent, with laws $\mathbb{P}^t_{x,z}$ and $\mathbb{P}^{\ell-t}_{z,y}$ respectively.

The following refinement of Proposition 3 is the key to our main result.

Lemma 2. *Let A be the CAF of X associated with ν as in section 3, and write*

$$K = \int_0^\ell f(X_t, t)\, dA_t, \tag{4.2}$$

where f is a positive Borel function on $E \times]0, \ell[$. If $H \geq 0$ is $\mathcal{F}_{\ell-}$-measurable, then

$$\mathbb{P}^\ell_{x,y}(HK) = \int_0^\ell dt \int_E \nu(dz) \left[\frac{p_t(x,z)p_{\ell-t}(z,y)}{p_\ell(x,y)}\right] f(z,t)\, \mathbb{P}^t_{x,z} \circ \mathbb{P}^{\ell-t}_{z,y}(H). \tag{4.3}$$

Remark. Let ξ_ℓ be the random measure on $E \times]0, \ell[$ that is the image of dA_t (a measure on $]0, \ell[$) under the mapping $t \mapsto (X_t, t)$. In the terminology of random measures, taken from Kallenberg [**K3**, Ch. 10] and reviewed in Section 5, the lemma states that under $\mathbb{P}^\ell_{x,y}$ the probabilities $\mathbb{P}^t_{x,z} \circ \mathbb{P}^{\ell-t}_{z,y}$ serve as the family of Palm distributions of ξ_ℓ. Just as this result follows from Proposition 3, it follows from Proposition 2 that for ξ the random measure on $E \times]0, \infty[$ that is the image of dA_t under the mapping $t \mapsto (X_t, t)$, under $\mathbb{P}_x$ the probabilities $\mathbb{P}^t_{x,z} \circ \mathbb{P}_z$ serve as the family of Palm distributions of ξ. In case A_t is the local time at a recurrent point in the state-space of X, the latter assertion is implicit in Kallenberg's work [**K1**].

Proof. Consider the process

$$J_t(z, \omega) = \int_\Omega \mathbb{P}^{\ell-t}_{z,y}(d\omega') H(\omega/t/\omega'), \quad 0 \leq t < \ell.$$

It is easy to check that $((\omega, t), z) \mapsto J_t(z, \omega)$ is $\mathcal{P}_\ell \otimes \mathcal{E}$-measurable, where $\mathcal{P}_\ell$ is the predictable σ-algebra restricted to $\Omega \times [0, \ell[$ and $\mathcal{E}$ is the Borel σ-algebra on E. Moreover, $J_t(X_t(\omega), \omega)$ is a version of the $\mathbb{P}^\ell_{x,y}$-optional projection of the process $t \mapsto H$. (This is easy to see if H has the form $\Pi_k f_k(X_{t_k})$; an appeal to the monotone class theorem settles the matter.) Consequently,

$$\mathbb{P}^\ell_{x,y}(HK) = \mathbb{P}^\ell_{x,y}\left(\int_0^\ell H f(X_t, t)\, dA_t\right) = \mathbb{P}^\ell_{x,y}\left(\int_0^\ell J_t(X_t, \cdot) f(X_t, t)\, dA_t\right). \tag{4.4}$$

If we now apply Proposition 3 to the last term in (4.4), then (4.3) follows. ◻

We apply Lemma 2 with a specific choice of the function f in the definition (4.2) of K:

$$f(z,t) = \rho(z,t)\left[\frac{p_\ell(x,y)}{p_t(x,z)p_{\ell-t}(z,y)}\right]. \tag{4.5}$$

Proposition 4. *Let Z, T, and Y be defined on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ as described above. If $H \geq 0$ is $\mathcal{F}_{\ell-}$-measurable, then*

$$\mathbb{P}^*(H(Y)) = \mathbb{P}^\ell_{x,y}(HK), \tag{4.6}$$

where $K = \int_0^\ell f(X_t,t)\,dA_t$ and f is given by (4.5). Moreover, the conditional distribution of (Z,T) given Y is determined by

$$\mathbb{P}^*(g(Z,T)\,|\,Y) = \frac{\int_0^\ell g(Y_t,t)f(Y_t,t)\,dA_t^*}{\int_0^\ell f(Y_t,t)\,dA_t^*}, \tag{4.7}$$

part of the assertion being that

$$\mathbb{P}^*(0 < \int_0^\ell f(Y_t,t)\,dA_t^* < \infty) = 1.$$

Proof. Let $H \geq 0$ be an $\mathcal{F}_{\ell-}$-measurable function on Ω, and let $g \geq 0$ be a Borel function on $E\times]0,\ell[$. Then by Lemma 2,

$$\begin{aligned}
&\mathbb{P}^*(H(Y)\,g(Z,T)) \\
&= \int_0^\ell dt \int_E \nu(dz)\rho(z,t)\,\mathbb{P}^t_{x,z}\circ\mathbb{P}^{\ell-t}_{z,y}(H)\,g(z,t) \\
&= \int_0^\ell dt \int_E \nu(dz)\left[\frac{p_t(x,z)p_{\ell-t}(z,y)}{p_\ell(x,y)}\right] f(z,t)\,\mathbb{P}^t_{x,z}\circ\mathbb{P}^{\ell-t}_{z,y}(H)\,g(z,t) \\
&= \mathbb{P}^\ell_{x,y}(H\tilde{K}),
\end{aligned} \tag{4.8}$$

where $\tilde{K} = \int_0^\ell g(X_t,t)f(X_t,t)\,dA_t$. The first assertion in Proposition 4 follows upon taking $g = 1$ in (4.8). Next, $A_s^* = A_s \circ Y$ is measurable over the $\mathbb{P}^*$-completion of $\sigma(Y)$, since A_s is measurable over the universal completion of $\mathcal{F}_s$. Let $K^* = \int_0^\ell f(Y_t,t)\,dA_t^*$, so that $K^* = K\circ Y$. By (4.8),

$$\mathbb{P}^*(K^* = 0) = \mathbb{P}^\ell_{x,y}(K; K = 0) = 0,$$

and

$$\mathbb{P}^*(K^* > \lambda) = \mathbb{P}^\ell_{x,y}(K; K > \lambda) \to 0 \text{ as } \lambda \to \infty,$$

since $\mathbb{P}^\ell_{x,y}(K) = \int_0^\ell dt \int_E \nu(dz)\,\rho(z,t) = 1$. Thus $\mathbb{P}^*(0 < K^* < \infty) = 1$. Now (4.7) follows easily from (4.8). □

Remark. It follows immediately from Proposition 4 that if $Q^* = (K^*)^{-1}\mathbb{P}^*$, then $Q^*(Y \in \cdot\,) = \mathbb{P}^\ell_{x,y}$, while $Q^*((Z,T) \in \cdot\,|Y) = \mathbb{P}^*((Z,T) \in \cdot\,|Y)$.

Example. Suppose that $x \in E$ is a regular point for X. That is, $\mathbb{P}_x(\inf\{t > 0 : X_t = x\} = 0) = 1$. Then by [**BG**, V,(3.13)] X admits local time at x. This is a CAF $L_t = L^x_t$ of X such that $L_t = \int_0^t 1_{\{x\}}(X_s)\,dL_s$ for all $t > 0$. The Revuz measure of L is proportional to the point mass at x, and we normalize L so that $\nu = \nu_L = \epsilon_x$. Thus, writing $p(s) = p_s(x,x)$,

$$\mathbb{P}_x(L_t) = \int_0^t p(s)\,ds, \qquad t > 0,$$

and

$$\mathbb{P}^\ell_{x,x}(L_t) = \int_0^t \left[\frac{p(s)p(\ell - s)}{p(\ell)}\right] ds, \qquad 0 < t \le \ell.$$

Let ρ be a probability density on $]0,\ell[$ and define $f(z,t) = f(t) = \rho(t)p(\ell)/[p(t)p(\ell - t)]$. Proposition 4 states that if we splice together two X-bridges (each beginning and ending at x) at an independent time T with law $\rho(t)\,dt$, then the law of the resulting process Y is absolutely continuous with respect to $\mathbb{P}^\ell_{x,x}$ with Radon-Nikodym derivative $\int_0^\ell f(t)\,dL_t$. In particular, the choice $\rho(t) = [p(t)p(\ell - t)]/[p(\ell)c(\ell)]$, where $c(\ell) = \mathbb{P}^\ell_{x,x}(L_\ell)$, leads to $f \equiv 1/c(\ell)$, so the density factor is simply $L_\ell/c(\ell)$. In case X is one-dimensional Brownian motion, these are the results presented in the introduction around formulae (1.0) and (1.1).

There is a multivariate version of the result of this section, whose formulation and proof we leave to the reader. The idea is to replace (Z,T) by $(Z_1,T_1),\dots,(Z_n,T_n)$, where $T_1 < T_2 < \cdots < T_n$. By making a suitable choice of the joint density of $(Z_1,T_1),\dots,(Z_n,T_n)$

one can obtain a spliced process whose density relative to the X-bridge is a multiple of $(A_\ell)^n/n!$.

Consider for example the setup in the introduction, with B a Brownian bridge on $[0,1]$, U uniformly distributed over $[0,1]$, independent of B, and let $]G,D[$ be the excursion interval straddling U. Using the tripartite decomposition of B at times G and D into bridge/excursion/bridge, and the fact that given G and D, U is uniform on $]G,D[$, we can use U to perform the (inverse) Vervaat shuffle on the excursion fragment [**V,Bi,BP**], thereby obtaining three adjacent bridges. The joint law of G and D is then exactly what is required to make the transformed process have unconditional law with density a constant times $(L_1)^2$ relative to the standard bridge. Compare with [**PPY**, Corollary 3.15]: a process whose law has density a constant times $(L_1)^n$ relative to that of the standard bridge B is obtained from B by length biased sampling without replacement of n excursion intervals, closing up the gaps, and standardizing.

5. A General Probabilistic Interpretation of Palm Measures

The passage from Lemma 2 to Proposition 4 illustrates a general calculation involving random measures and their Palm measures which we have found useful in other contexts (see [**PPY**, Lemma 4.1] [**PY2**, Lemma 2.2]), and which it seems worth recording in a general setting. The general result of this calculation appears in Proposition 5 below.

Let $(\Omega, \mathcal{F}, Q)$ be a probability space, $(S, \mathcal{S})$ a measurable space, and ξ a random measure on S based on $(\Omega, \mathcal{F}, Q)$. That is, $\xi(w, A)$ is a $\mathcal{F}$-measurable function of $w \in \Omega$ ($A \in \mathcal{B}(S)$ fixed) and a measure on $(S, \mathcal{B}(S))$ as a function of A ($w \in \Omega$ fixed). The intensity measure of ξ is

$$\lambda(A) = \int \xi(w, A)\, Q(dw), \qquad A \in \mathcal{B}(S), \tag{5.1}$$

which we assume to be σ-finite. Let us assume that the bimeasure

$$Q(\xi(A); B), \qquad A \in \mathcal{B}(S),\, B \in \mathcal{F},$$

admits a disintegration

$$Q(\xi(A); B) = \int_A \lambda(dx)\, Q^x(B), \tag{5.2}$$

where $(x, B) \mapsto Q^x(B)$ is a Markov kernel from $(S, \mathcal{B}(S))$ to $(\Omega, \mathcal{F})$. (For a disintegration as in (5.2) to exist, it is sufficient that Ω be a Polish space and that $\mathcal{F}$ be the corresponding Borel σ-algebra.)

The collection $\{Q^x : x \in S\}$ is the family of *Palm distributions* associated with the random measure ξ. When ξ is the random counting measure corresponding to a point process with no multiple points, Q^x may be interpreted intuitively as Q conditioned on ξ putting a point at x. See for example [**K2**]. In case ξ is a more general random measure, in particular if ξ is diffuse, the intuitive meaning of the Palm measures is less clear. Still, the following proposition offers a general probabilistic interpretation of Palm measures in terms of expanding the probability space to allow that, given ξ, a point $X \in S$ is picked at random with some density relative to ξ.

Proposition 5. *Suppose defined on a probability space $(\Omega^*, \mathcal{F}^*, P^*)$ an S-valued random variable X and an Ω-valued random variable W. If*

$$P^*(X \in dx) = f(x)\lambda(dx) \tag{5.3}$$

for some probability density f on S relative to the intensity measure λ, and

$$P^*(W \in dw | X = x) = g(w|x) Q^x(dw) \tag{5.4}$$

for some jointly measurable $g(w|x)$, then

$$P^*(W \in dw) = h(w) Q(dw), \tag{5.5}$$

$$P^*(X \in dx | W = w) = j(x|w)\xi(w, dx), \tag{5.6}$$

where

$$h(w) = \int_S f(x) g(w|x) \xi(w, dx), \tag{5.7}$$

$$j(x|w) = f(x)g(w|x)/h(w). \tag{5.8}$$

Conversely, if the joint law of W *and* X *is such that (5.5) and (5.6) hold for some probability density* h *relative to* Q, *and some jointly measurable* $j(x|w)$, *then (5.3) and (5.4) hold with*

$$f(x) = \int_\Omega h(w)j(x|w)Q^x(dw), \tag{5.9}$$

$$g(w|x) = h(w)j(x|w)/f(x). \tag{5.10}$$

The proof just uses the definition of the Palm distributions: $Q(dw)\xi(w,dx) = \lambda(dx)Q^x(dw)$, and a standard Bayes type calculation, as spelled out for instance in [**L**, Lemma 2.1].

In conclusion, we reflect that in recent years arguments involving random measures and their associated Palm measures have played an increasingly important role in the theory of Markov processes. See for instance [**AM, Fi, G**]. Just as bridge laws can be interpreted as Palm distributions, so can excursion laws. See for instance [**Pi**], where a generalization of Bismut's [**B**] decomposition of the Brownian excursion is explained in terms of Palm distributions.

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