Limit Laws for Brownian Motion Conditioned to Reach a High Level

By

Michael Klass and Jim Pitman

Technical Report No. 368
September 1992

Research supported in part by NSF grants DMS-90-07469 and DMS-91-07531

Department of Statistics
University of California
Berkeley, California 94720

Limit laws for Brownian motion conditioned to reach a high level.

by Michael Klass and Jim Pitman

Dept. Statistics, U.C. Berkeley, Ca. 94720, U.S.A.

Abstract. A functional limit theorem is presented for the behaviour of Brownian motion conditioned to reach a high level during a fixed time interval. The asymptotic behaviour of the conditioned path as the level tends to infinity is related to Williams' path decomposition at the maximum.

Keywords. Conditioned limit theorem, Brownian motion, path decomposition at the maximum.

0. Introduction.

Let $B = (B_t, t \geq 0)$ be a standard Brownian motion on the line starting at $B_0 = 0$. Let

$$M_t = \sup_{0 \leq s \leq t} B_s \; ; \quad T_{max,t} = \sup\{s \leq t : B_s = M_t\} ; \quad T_h = \inf\{t : B_t = h\}.$$

Mathew and McCormick (1992) obtain the limiting distribution as $h \to \infty$ of $h\sqrt{1 - T_{max,1}}$ conditional on the event $(T_h \leq 1)$. (Note that the events $(T_h \leq 1)$ and $(M_1 \geq h)$ are identical, due to path continuity of B). Here we consider for $h > 0$ the process $(B_t, 0 \leq t \leq 1 \mid T_h \leq 1)$, that is to say the Brownian motion on the time interval $[0, 1]$ conditioned to reach level h at some time in this interval. We extend Mathew and McCormick's result to a functional limit theorem for a rescaled version of this process as $h \to \infty$. A precursor of this type of result, in a slightly different context, was obtained by Berman (1982, Lemma 5.1).

1. Results.

As a first approximation, it is easy to see that for large h, conditioning B on the event $(T_h \leq 1)$ forces the path to stay fairly close to the line ht, $0 \leq t \leq 1$. More precisely, we have the following elementary proposition, whose proof is left to the reader:

Research supported in part by NSF grants DMS-90-07469 and DMS-91-07531

Proposition 1.1. *As $h \to \infty$,*

$$(B_t - ht, 0 \le t \le 1 \mid T_h \le 1) \xrightarrow{d} (B_t, 0 \le t \le 1 \mid B_1 = 0),$$

where the right side is standard Brownian bridge, and $\xrightarrow{d}$ denotes convergence of distributions on the path space $C\,[0,1]$ with the topology of uniform convergence.

In particular, Proposition 1.1 implies that as $h \to \infty$

(1.b) $$(B_1 - h \mid T_h \le 1) \xrightarrow{d} 0,$$

(1.c) $$(1 - T_h \mid T_h \le 1) \xrightarrow{d} 0,$$

where $\xrightarrow{d}$ now denotes convergence of one-dimensional distributions. Our concern here is a second order of approximation, starting with how to normalize the random variables in (1.b) and (1.c) to obtain non-trivial limit laws. In Section 2 we prove:

Lemma 1.2. *As $h \to \infty$*

$$(h^2(1 - T_h) \mid T_h \le 1) \xrightarrow{tv} \zeta$$

where ζ has exponential (1/2) distribution: $P(\zeta \in dz)/dz = \frac{1}{2} e^{-z/2}$, and $\xrightarrow{tv}$ denotes convergence of laws in total variation norm.

Now let

(1.d) $\zeta_h = h^2(1 - T_h)$

(1.e) $B_h(t) = h\,[B(T_h + t/h^2) - h\,],\ t \ge 0,$

By the strong Markov property of BM at time T_h, and Brownian scaling, B_h is a BM independent of T_h, hence also independent of ζ_h given $T_h \le 1$. So from Lemma 1.2 we easily obtain

Theorem 1.3. *As $h \to \infty$*

$$(B_h(t), 0 \le t \le \zeta_h \mid T_h \le 1) \xrightarrow{tv} (B_t, 0 \le t \le \zeta)$$

where ζ has exponential (1/2) law, independent of the Brownian motion B.

The limit process appearing here, Brownian motion killed at an independent exponential time with mean 2, is a diffusion with particularly simple structure. It is well known that due to its spacetime homogeneous Markov property, laws of functionals of this process such as local times, time spent above a level, time and place of the maximum, etc., assume a much simpler form than for BM stopped at a fixed time.

Since $h(B_1 - h) = B_h(\zeta_h)$ on $(T_h \le 1)$, Theorem 1.3 implies

$$(1.f) \qquad (h^2(1 - T_h), h(B_1 - h) \mid T_h \le 1) \overset{tv}{\to} (\zeta, B_\zeta).$$

In particular, the limit law for $h(B_1 - h)$ is easily identified as the bilateral exponential law. So (1.f) implies the well known fact that this is the law of B_ζ:

$$(1.g) \qquad P(B_\zeta \in dy) = \frac{1}{2} e^{-|y|} dy.$$

This amounts to the classical formula for the resolvent of the Brownian semi-group (see e.g. Itô-McKean (1965, 1.4.31). Theorem 2.1 of Mathew and McCormick (1992), with convergence in distribution strengthened to convergence in total variation, appears as the second component of the following consequence of Theorem 1.3. As $h \to \infty$, given $T_h \le 1$, the joint distribution of the four variables

$$h^2(T_{max,1} - T_h), \qquad h^2(1 - T_{max,1}), \qquad h(M_1 - h), \qquad h(M_1 - B_1),$$

converges in total variation to that of

$$T_{max,\zeta}, \qquad \zeta - T_{max,\zeta}, \qquad M_\zeta, \qquad M_\zeta - B_\zeta.$$

Here the two pairs $(T_{max,\zeta}, M_\zeta)$ and $(\zeta - T_{max,\zeta}, M_\zeta - B_\zeta)$ are independent and identically distributed, due to the path decomposition of B at its maximum on $[0, \zeta]$ (see e.g. Williams (1974), Millar (1977), Greenwood & Pitman (1980)). It is well known and easily verified that the first component of each pair has the χ_1^2 distribution of B_1^2, while the second component is exponential with rate 1.

For two more examples, let

$$A_{x,t} = \int_0^t 1(B_s > x)\, dx \qquad = \text{time } B \text{ spends above } x \text{ before } t$$

$$G_{x,t} = \sup\{s : s \le t, B_s = x\} = \text{time of last visit to } x \text{ before } t.$$

Jointly with the preceding results, we have that as $h \to \infty$, given $T_h \le 1$, the joint law of

$$h^2 A_{h,1}, \qquad h^2(1 - T_h - A_{h,1}), \qquad h^2(G_{h,1} - T_h), \qquad h^2(1 - G_{h,1})$$

converges to that of

$$A_{0,\zeta}, \qquad \zeta - A_{0,\zeta}, \qquad G_{0,\zeta}, \qquad \zeta - G_{0,\zeta}.$$

Standard arcsine laws for Brownian motion combined with the exponential distribution of ζ imply these four variables have identical χ_1^2 distribution. The first two are independent, and so are the last two, by standard algebra of beta and gamma variables.

Theorem 1.3 above describes the asymptotic behaviour of B in the interval $[T_h, 1]$ given that $T_h \le 1$. To complete the picture we now describe what happens on the same scale, considering times both before and after T_h, by consideration of a rescaled and time reversed process.

Theorem 1.4. *As $h \to \infty$, for every $T > 0$,*

$$(h\,(B\,(1 - t/h^2) - h),\, 0 \le t \le T \mid T_h \le 1) \overset{tv}{\to} (X\,(t),\, 0 \le t \le T),$$

where X is a non-Markovian process with continuous paths constructed as follows from Brownian motion B, an independent exponential (1) random variable ξ, and a further independent random sign σ which is equally likely to be ± 1: Let

$$D\,(t) = \xi + B\,(t) - t\,; \qquad \tau_0 = \inf\{t : D\,(t) = 0\},$$

$$X\,(t) = \begin{cases} \sigma D\,(t),\ 0 \le t \le \tau_0 \\ D\,(t),\ \tau_0 < t < \infty. \end{cases}$$

Remark. The process D is a BM with drift -1 started at the random level $\xi > 0$. So X starts at $X\,(0) = \sigma\xi$ which has the bilateral exponential limit distribution of $(h\,(B_1 - h) \mid T_h \le 1)$. Given $X\,(0) > 0$, $X = D$ moves as a BM with drift -1. Given $X\,(0) < 0$, X moves as a BM with drift $+1$ until time τ_0 when X first hits 0. Thereafter, X moves as a BM with drift -1.

Let $G_0 = \sup\{t : X_t = 0\}$, and let B_h and ζ_h be as in Theorem 1.3. According to Theorem 1.4

$$(B_h\,(\zeta_h - u),\, 0 \le u \le \zeta_h) \overset{tv}{\to} (X(u),\, 0 \le u \le G_0),$$

whereas Theorem 1.3 implies the same limit distribution is that of $(B\,(\zeta - u),\, 0 \le u \le \zeta)$. The fact that these are two descriptions of the same limit process amounts to:

Corollary 1.5. (Williams (1974, Theorem 4.5)). *If B is a BM and ζ is exponential* (1/2) *independent of B, and $b > 0$,*

$$(B_t,\, 0 \le t \le \zeta \mid B_\zeta = b) \overset{d}{=} (B_t + t,\, 0 \le t \le G_b)$$

where $G_b = \sup\{t : B_t + t = b\}$.

Two other consequences of Theorem 1.4, both closely related to Williams' path decompositions, are the following Corollaries.

Corollary 1.6. *As $h \to \infty$, for every $T > 0$*

$$(h\,[B\,(T_h - t/h^2) - h\,],\, 0 \le t \le T \mid T_h \le 1) \xrightarrow{tv} (D_-(G_0 + t),\, 0 \le t \le T)$$

where $(D_-(u), u \ge 0)$ is a Brownian motion with drift -1: $D_-(u) = B(u) - u$, $u \ge 0$, and $G_0 = \sup\{u : D_-(u) = 0\}$ is the last time D_- visits zero.

Remarks. (i) Williams (1974) showed the limit process appearing here as a diffusion process on $[-\infty, 0]$ and identified its generator. Intuitively, the process is "Brownian motion with drift -1 conditioned never to return to 0". See Williams' paper for a more careful account. The one-dimensional distributions and transition probability function can be found explicitly in Rogers-Pitman (1981) and Rogers (1983).

(ii) The processes in Corollary 1.6 and Theorem 1.3 describe the behaviour of B on the same spacetime scale, one looking backwards and the other forwards from time T_h, given $T_h \le 1$. Theorem 1.4 implies these processes converge jointly as $h \to \infty$ to independent limit processes. The limits of these processes are recovered from the process X in Theorem 1.4 by reading forwards and backwards from time

$$G_0 \;=\; \sup\{t : X_t = 0\} \;=\; \sup\{t : D_t = 0\}.$$

The independence assertion amounts to the last exit decomposition of D at this time.

As a final consequence of Theorem 1.4 and Williams' path decompositions, we mention the following result, where instead of centering at h and reversing we center at B_1 and reverse:

Corollary 1.7. *As $h \to \infty$, for all $T > 0$*

$$(h\,[B\,(1 - t/h^2) - B\,(1)],\, 0 \le t \le T \mid T_h \le 1) \xrightarrow{tv} (Y(t),\, 0 \le t \le T)$$

where Y is a non-Markovian process with continuous paths. Let $D_- = (D_-(t),\, t \ge 0)$, be a Brownian motion with drift -1 starting from $D_-(0) = 0$, and let $M \;=\; \sup_t D_-(t)$. The law of Y on $C[0, \infty)$ has density $\frac{1}{2} e^M$ with respect to the law of D_-.

Remark. The formula for the density amounts to the following:

(i) Conditionally given $M = m$, Y admits the same path decomposition as does a BM with drift -1 at its maximum. (This follows from Williams' results)

(ii) From (1.h) and (1.i), $P(\sup_t Y_t \in dm)/dm \;=\; e^{-m}, m > 0$, whereas $P(\sup_t D_-(t) \in dm)/dm \;=\; 2e^{-2m}, m > 0$. The ratio of these densities is $\frac{1}{2} e^m$.

2. Proofs.

Proof of Lemma 1.2. Let $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $\overline{\Phi}(h) = P(B_1 \geq h) = \int_h^\infty \phi(z)\,dz$, and recall that

(2.a) $$\overline{\Phi}(h) \sim \phi(h)/h \quad \text{as } h \to \infty .$$

By the reflection principle, the event $(T_h \leq t) = (M_t \geq h)$ has probability

$$P(T_h \leq t) = P(M_t \geq h) = 2P(B_t \geq h) = 2\overline{\Phi}(h/\sqrt{t}).$$

Also, if we let

(2.b) $$f(h,t) = P(T_h \in dt)/dt = (2\pi t^3)^{-1/2} h e^{-h^2/2t},$$

then for $0 \leq y \leq h^2$,

$$P(h^2(1 - T_h) \in dy \mid T_h \leq 1)/dy = h^{-2} f(h, 1 - y/h^2)/2\overline{\Phi}(h) \to \frac{1}{2} e^{-y/2}$$

as $h \to \infty$ by an easy evaluation of the limit using (2.b) and (2.a). □

Proof of Theorem 1.4.

Let $K = h^2$, $\tilde{B}_K(t) = hB(t/h^2)$. Then

$$(h(B(1 - t/h^2) - h), t \geq 0) = (\tilde{B}_K(K - t) - K, t \geq 0)$$
$$\stackrel{d}{=} (B(K - t) - K, t \geq 0), \text{ by Brownian scaling}.$$

Since $(T_h \leq 1) = (B_s \geq h \text{ for some } 0 \leq s \leq 1) = (\tilde{B}_K(t) \geq K \text{ for some } 0 \leq t \leq K)$,

(2.f) $$(h(B(1 - t/h^2) - h), t \geq 0 \mid T_h \leq 1) \stackrel{d}{=} (B_{K-t} - K, t \geq 0 \mid T_K \leq K).$$

So it suffices to examine the asymptotic behavior of the right hand process in (2.f) as $K \to \infty$. Due to the reflection principle, the event $T_K \leq K$ splits into two events of equal probability, namely $(T_K \leq K \text{ and } B_K \geq K) = (B_K \geq K)$, and $(T_K \leq K \text{ and } B_K < K)$. The description of the asymptotic distribution of the process in (2.f) given $(B_K \geq K)$ follows at once from the following lemma, by conditioning on B_K, and using the elementary fact that as $K \to \infty$, $(B_K - K \mid B_K \geq K) \stackrel{tv}{\to} \xi$ where ξ is exponential with rate 1. The description on the other half of the event $T_K \leq K$ follows by the reflection principle at time T_K combined with a last exit decomposition at time $\sup\{u : u \leq K, B_u = K\}$, just after which the sign of $B_K - K$ is determined. □

Lemma 2.1. Let $o(K)$ be any function of K with $o(K)/K \to 0$ as $K \to \infty$. Then for any real δ, and $T > 0$, as $K \to \infty$,

$$(B_t, 0 \le t \le T \mid B_K = \delta K + o(K)) \xrightarrow{tv} (B_t + \delta t, 0 \le t \le T),$$

where the process on the left is the initial segment of length T of a Brownian bridge of length K from 0 to $\delta K + o(K)$, and

$$(B_{K-t} - B_K, 0 \le t \le T \mid B_K = \delta K + o(K)) \xrightarrow{tv} (B_t - \delta t, 0 \le t \le T).$$

Proof. Elementary, using standard properties of Brownian bridges. In particular, the second assertion follows from the first by time reversal. □

3. Concluding Remarks. It appears that the above results remain valid with the Brownian motion B replaced by a more general diffusion with generator of the form

$$b(x)\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2},$$

for a wide class of drift coefficients $b(x)$. This substitution should be made only in the definition of the basic conditioned process: the limiting processes remain the same as in the plain Brownian case $b(x) \equiv 0$. The validity of this claim is easily checked in case of constant drift $b(x) \equiv b$. It appears also to be correct for a Bessel diffusion with dimension d, with $b(x) = (d-1)/2x$, for any real d, and more general drifts subject to very mild regularity conditions. Presumably this can be derived from the present results using the Cameron-Martin formula, but we have not checked the details.

It might also be interesting to look for analogs of the Brownian results for random walks or Lévy processes. Presumably the Brownian result can be interpreted along the lines of Donsker's invariance principle as a suitable double limit for an increasing number of steps of a random walk with bounded increments conditioned to reach an increasingly high level. But the level must not be allowed to increase too rapidly: e.g. for a simple random walk with increments + or - 1, conditioning to reach level n in n steps gives a trivial process. And for random walks with unbounded increments or Lévy processes with jumps it is to be expected that the overshoot would typically dominate behaviour of the conditioned process.

Acknowledgment. We thank Lucien Le Cam for a helpful conversation.

References

Berman, S.M. (1982), Sojourns and extremes of a diffusion process on a fixed interval. *Adv. Appl. Prob.* **14**,811-832.

Greenwood, P. and Pitman, J.W. (1980), Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. Appl. Prob.* **12**, 893-902.

Itô, K, and McKean, H.P. (1965). Diffusion processes and their sample paths. Springer.

Mathew, G. and McCormick, W.P. (1992). A conditional limit law result on the location of the maximum of Brownian motion. *Statistics and Probability Letters* **13**, 199-202.

Millar, P.W. (1977). Zero-one laws and the minimum of a Markov process. *Trans. Amer. Math. Soc.* **226**, 365-391.

Rogers, L.C.G. and Pitman, J. (1981). Markov Functions. *Annals of Probability* **9**, 573-582.

Rogers, L.C.G. (1983). Itô excursion theory via resolvents. *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **63**, 237-255.

Williams, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions. *Proc. London Math. Soc.* **28**, 738-768.