The Lévy Laplacian and the Brownian Particles

in Hilbert Spaces

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1. Introduction

Since L.Gross studied the abstract Wiener space valued Brownian motion,a lot of infinite dimensional Markove processes have appeared and become a useful tool for analysis on infinite dimensional spaces (see [1],[2],[3]).It is well known that their generators provide many elliptic differential operators on infinite dimensional spaces as is made quite clear in [1].

But this procedure can not be applied to some important cases. For example if we hope to have an analogy of Laplacian on Hilbert space H,we could not use the abstract Wiener space (E,H,μ) and the Gross Laplacian given by the E-valued Brownian motion because many regular functionals on H do not belong to its domain.So we seem to need a renormalization procedure for the Gross Laplacian to help us construct the Laplacian on H.

In [4],the authors discovered a wonderful relation between the Lévy Laplacian and the Gross Laplacian.The basic ideas of this paper are a combination of their result together with infinite dimensional stochastic processes.First we give a renormalization procedure for

E-valued Brownian motion and propose the concept of Brownian particle in Hilbert spaces,then we have the Lévy Laplacian through the stochastic calculus of Brownian particles and a double limit method (see Section 3 below).This probability approach can be applied to many infinite stochastic processes,that means we will have a variety of Levy Laplacian !

The purpose in the present work is to study the Lévy Laplacian and related problems based on this idea.This paper is organized as follows.In Section 2 we will present the concept of Brownian particle in Hilbert spaces and discuss its main properties.In Section 3 the Lévy Laplacian will be constructed by Brownian particles.The Section 4 is devoted to the harmonic functionals on Hilbert spaces.Finally we would like to give some remarks about the generalized Lévy Laplacians.

2.   The Brownian particles
in Hilbert spaces

Throughout this Section (E,H,μ) will be an abstract Wiener space in the sense of L.Gross.That is,E is a separable Banach space and $(H,\|\ \|)$ is a separable Hilbert space,H is a dense subspace of E and the inclusion map is continuous,and μ is the probability on E with the property that

$$\int_E \exp[\ i\langle\lambda,x\rangle\ ]\ d\mu(x) = \exp[-\tfrac{1}{2}\|\lambda\|^2], \qquad \lambda \in E^*,$$

where we have used the fact that the dual space E^* of E becomes a dense subspace of H when we make the natural indentification between H^* and H itself. Let (e_n) be a complete orthonormal system (CONS) of H. Now we start constructing the E-valued Brownian motion W(t) by (e_n).

Lemma 2.1. Let W(t) be the E-valued Brownian motion (see [2]), then we have the identity in law:

$$W(t) = \sum_{n,k} e_n\, a_k\,(t) G_{n,k}\ ,$$

where (a_k) is a CONS of the Sobolev space $H^1([0,1])$ and $G_{n,k}$ is a sequence of i.i.d. Gaussian random variables with mean 0 and variance 1. This series converges in $C([0,1])$ almost everywhere.

Proof. For $h \in E^*$ and $t \in [0,1]$, we have $F \in C([0,1],E\ )^*$ such that $F(u) = \langle h,u(t)\rangle$ for $u \in C([0,1],E\ )$. The functionals of the form generate a dense subset in $C([0,1],E\ )^*$. Let

$$S_N = \sum_{n,k \leq N} e_n a_k(t) G_{n,k}\ ,$$

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for N = 1,2,... .It is easy to show that

$$\lim E(\exp[i\langle F,S_N\rangle]) = \int \exp[i\langle F,x\rangle]dW,$$

where dW is the distribution of (W(t)) on C([0,1],E) and E(•) is the expectation on some probability space where ($G_{n,k}$) are defined and F is the linear functional of the form mentioned above. Because (e_n) is a CONS of H,from [2] we see

$$\sup_{0 \leqslant t \leqslant 1} \left\| \sum_{n,k} e_n a_k(t) G_{n,k} \right\|_E \leqslant M , \qquad \text{a.s.}$$

where M is a finite constant, for all N.

By using Ito-Nisio theorem for the independent symmetric random sequence in C([0,1],E),we deduce this lemma.

Q.E.D.

Let $G_n(t) = \sum_{k} a_k(t) G_{n,k}$,we know that ($G_n(t)$) is a sequence of independent Brownian motions and the identity in law

$$W(t) = \sum_{n} e_n G_n(t) .$$

Corollary 2.2. Let A is in B(H),i.e.,A is a bounded linear operator on H,then the series $\sum_{n} A e_n G_n(t)$ converges in C([0,1],E). almost everywhere.

Proof. This is immediate from Lemma 2.1. and the Gross theorem which says that $\|h\|_H = \|Ah\|_H$, $h \in H$, is a measurable norm.

<div align="right">Q.E.D.</div>

Denote the set of whole Hilbert-Schmidt operators on H by H.S. (H) and write $\text{trace } K^*K = \sum_n \|Ke_n\|^2$, $K \in$ H.S.(H).

Definition. A family of Hilbert Schmidt operators (K_ϵ , $\epsilon > 0$) is called a renormalization procedure (RP) on H if they satisfy the following

$$\lim_{\epsilon \downarrow o} \| K_\epsilon x - x \| = 0, \quad \text{for all } x \in H.$$

For a RP of H, we define the renormalization factors

$$Z(\epsilon) = (\text{ trace } K_\epsilon^* K_\epsilon)^{\frac{1}{2}}, \quad \epsilon > 0,$$

and the renormalized E-valued Brownian motions such that

$$W(\epsilon, t) = Z(\epsilon)^{-1} \sum_n K_\epsilon e_n G_n(t),$$

which is called a Brownian particle in H.

Throughout this paper we will omit ϵ in these notations when we do not need to show that K and Z and W are depending on ϵ,

and set $I_\epsilon = K_\epsilon / Z(\epsilon)$.

Note that $Z(\epsilon) \longrightarrow$ (dimension of H $)^{\frac{1}{2}}$ as $\epsilon \downarrow 0$, so when H is a finite dimensional space, the Brownian particles converges to $N^{-\frac{1}{2}}W(t)$, N = dimension of H. Obviously it is nonsense to investigate the limit when H is infinite dimensional space, but the Brownian particles behave very well.

Lemma 2.3. The Brownian particle $W(t)$ has the following properties.

a. $W(t)$ is a H-valued continuous martingale.

b. $E(\| W(t) \|_H^2) = t$.

c. For $h \in H$, the process $(h, W(t))_H$ is a Brownian motion which
   has the decomposition

$$(h,W(t)) = \sum_n (h, Ke_n) G_n(t).$$

The proof is straightforward. Now we turn to the Ito formula with respect to Brownian particles.

Lemma 2.4. Suppose that $F \in C_b^2(H)$ (in the sense of Fréchet such that its second derivative is continuous and bounded on H) and $W(t)$ ia a Brownian particle with respect to a RP of H, (K_ϵ), then we have the Ito formula

$$F(x+W(t)) = F(x) + \sum_n \int_0^t F'(x+W(s))(Ie_n)dG_n(s) +$$

$$+ \frac{1}{2} \int_0^t (F''(x+W(s))Ie_n, Ie_n) \ ds.$$

Proof. Let

$$W_N(t) = \sum_{n=1}^{N} Ie_n G_n(t), \quad Q(x_1, x_2, \ldots, x_N) = F(x + \sum_{n=1}^{N} Ie_n x_n),$$

where $(x_1, \ldots, x_N) \in R^N$. For Q use the Ito formula, we have

$$F(x+W(t)) = F(x) + \sum_{n=1}^{N} \int_0^t F'(x+W_N(s))(Ie_n) dG_n(s) +$$

$$+ \frac{1}{2} \sum_{n=1}^{N} \int_0^t (F''(x+W_N(s))Ie_n, Ie_n) ds.$$

By the facts that $I \in$ H.S.(H) and $F \in C_b^2(H)$, we can complete the proof with an approximation argument.

Q.E.D.

3. The Lévy Laplacian

In this Section we will construct the Lévy Laplacian by the Brownian particles with respect to a RP of H. from now on we set $W_N(t) = \sum_{n=1}^{N} Ie_n G_n(t), N = 1, 2, \ldots$.

Definition. Suppose that H_0 is a dense subspace and it is a

topological vector space and the inclusion map is continuous.

Let (K_ϵ) be a RP of H such that $K_\epsilon(H) \subset H_0$, $\epsilon > 0$, and $W(\epsilon, t)$ be the Brownian motion corresponding to (K_ϵ) and $W_N(\epsilon, t) = \sum_1^N I_\epsilon e_n G_n(t)$. If $F \in C_b(U)$, U is an open subset of H_0, such that for every $x \in U$ there exists $\delta > 0$ such that when $0 \leqslant t \leqslant \delta$ and $0 < \epsilon \leqslant \delta$ we have $x + W_N(\epsilon, t) \in U$ for all N and $E(F(x + W_N(\epsilon, t)))$ converges as $N \to \infty$, at this time we denote the limit by $E(F(x + W(\epsilon, t)))$ and say that F is admissible with respect to RP of H, (K_ϵ).

Remark 3.1. Let $H = L^2[0,1]$ and (K_ϵ) be the convolution operators mentioned in [4]. We can prove that the normal polynomials (see [4]) are admissible with respect to (K_ϵ).

When $F \in C_b(H)$, then F is admissible for all RP of H.

Remark 3.2. For each Hilbert space H, we have a simplest RP of H as follows. Choose a CONS (e_n) of H and set

$$K_\epsilon u = \sum_1^N (u, e_n) e_n , \quad \text{for } (N+1)^{-1} \leqslant \epsilon \leqslant N^{-1}$$

This RP of H is due to P.Lévy. Every bounded continuous functional (on H_0) is admissible with respect to it. We call this RP of H Lévy RP of H.

Remark 3.3. For every sphere $O(x,r) \subset H$, we can find a $Q \in C_b^\infty(H)$

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such that $Q(y) = 1$ when $y \in O(x,r)$ and $Q(y) = 0$ when $y \bar{\in} O(x,2r)$. So for every $F \in C_b^2(U)$, U is an open subset of H, and $x \in U$ we can find a $F_1 \in C_b^2(H)$ such that $F = F_1$ on $O(x,r) \subset U, r > 0$.

Lemma 3.4. Suppose F_1 and F_2 are in $C_b(H)$ and $F_1 = F_2$ on the sphere $O(x,r)$, then for all RP of H we have

$$\lim t^{-1} [E(F_1(x+W(t))) - E(F_2(x+W(t)))] = 0$$

Proof. Because

$$E(\|W(t)\|_H^4) \leq 3 t^2 , \qquad t > 0$$

so

$$E(F_j(x + W(t); H - O(x,r)) = O(t) , \quad \text{for} \quad j = 1,2.$$

Q.E.D.

Now we are in the position to construct the Lévy Laplacian by Brownian particles.

Definition. Suppose H_0 is a dense subspace of H and it is a topological vector space and the inclusion map is continuous. Let (K_ϵ) be a RP of H and $W(\epsilon,t)$ be its Brownian particles. If

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$F \in C_b(U)$,U is an open subset of H_0,such that F is admissible with respect to (K_ϵ).The Laplacian corresponding to (K_ϵ) is defined as follows

$$\triangle F(x) = 2 \lim_{\epsilon \to o} \lim_{t \to o} t^{-1}[E(F(x+W(\epsilon,t))-F(x)], x \in U.$$

We will see that for many important functionals this Laplacian is not anything else,but the Lévy Laplacian,so we will call it Lévy Laplacian of (K_ϵ).

Remark 3.5.   Let $F \in C_b(U)$,U is an open subset of H,From Remark 3.3. we know that for every $x \in U$ there exists $F_1 \in C_b(H)$ such that $F = F_1$ on $O(x,r), r > 0$.So we can define $\triangle F(x) = \triangle F_1(x)$,if the right hand has meaning.

Theorem 3.6.   Assume that H_0 is a dense subspace of H and it is a Hilbert space with respect to its own inner product and the inclusion map is continuous and (K_ϵ) is a RP of H and $F \in C_b^2(H_0)$ which is admissible with respect to (K_ϵ).If for every $x \in H_0$ the Fréchet second derivative $F''(x)$ corresponds to a bounded operator on H such that

$$\lim_{t \to o} \sup[\|E(F''(x+W_N(,t))-F''(x)\|_{B(H)} :N \geq 1] = 0$$

then

$$\triangle F(x) = \lim_{\epsilon \to 0} Z(\epsilon)^{-2} [\text{trace } K_\epsilon^* F''(x) K_\epsilon], \quad x \in H_0.$$

Proof. We first recall that

$$F(x+W_N(\epsilon,t)) = F(x) + \sum_1^N \int_0^t F'(x+W_N(\epsilon,s)(I_\epsilon e_n) \, dG_n(s) +$$

$$+ \frac{1}{2} \sum_1^N \int_0^t (F''(x+W_N(\epsilon,s))(I_\epsilon e_n), I_\epsilon e_n) ds$$

where $I_\epsilon = K_\epsilon / Z(\epsilon)$. Note that the second term in the right hand is martingales,hence

$$E(F(x+W_N(\epsilon,t)) = F(x) + \frac{1}{2} \sum_1^N \int_0^t (E(F''(x+W_N(\epsilon,s))(I_\epsilon e_n), I_\epsilon e_n) ds$$

where $E(F''(x+W_N(\epsilon,t))$ is the Bochner integral.

By the assumption and trace $I_\epsilon^* I_\epsilon = 1$,we have

$$E(F(x+W_N(\epsilon,t)) = F(x) + \frac{1}{2} \text{trace} K^* F''(x)K /Z(\epsilon)^2 + 0(t)$$

where $0(t)$ depends on ϵ. Because F is admissible,let $N \to \infty$,through the double limit procedure we complete the proof.

<div align="right">Q.E.D.</div>

Now we are investigating the Normal polynomials.Let H_0 be a nuclear space of all C^∞- functions with periodic boundary on T = [0,1] and assume that H_0 is a dense subspace of H and the in-

clusion map is continuous.We then have a Gelfand trip $H_0 \subset H \subset H_0^*$.
Following with [4],a finite linear combination of functionals
of the form

$$F(x) = \int_{T^n} b(t_1,t_2,\ldots,t_n)\ x(t_1)^{p_1}\ldots x(t_n)^{p_n} dt_1 dt_2 \ldots dt_n$$

where $b \in L^\infty(T^n)$ and $x \in H_0$ and $p_1,p_2,\ldots,p_n \geq 1$,is called a normal
polynomial (see [4],section 3).

Let K_ϵ, $\epsilon > 0$,be a C^∞- function satisfying the following condi-
tions:

a. $\mathrm{supp}[\ K_\epsilon\ ] \subset (-\frac{1}{2},\frac{1}{2})$,
b. $\lim\limits_{\epsilon \to 0} \int u(t)K_\epsilon(t)dt = 1$.
We define

$$K_\epsilon u(t) = \int_T K_\epsilon(t-s)u(s)ds, \qquad u \in H,$$

where we consider K as a C^∞- function on real line with period
1.(this operator K has been used in [4]).

In [4],the authors revealed a wonderful relation between the
Levy Laplacian and the Gross Laplacian by using (K_ϵ).Now we
start to explain their result. Of course (K_ϵ) is a RP of H and
this RP of H has very special properties.For example, if (e_n)

$\subset H_0$ is a CONS of H,then for every $t \in T$,

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$$Z(\varepsilon)^{-2} \sum_n K_\varepsilon e_n(t)^2 = Z(\varepsilon)^{-2} \sum_n \left[\int K_\varepsilon(t-s)e_n(s)ds \right]^2$$

$$= Z(\varepsilon)^{-2} \int K_\varepsilon(t-s)^2 \, ds = 1.$$

Remark 3.7. Every normal polynomial satisfys the assumption of Theorem 3.6. and when F is of the form

$$F(x) = b(y) \, x(y_1)^{p_1} x(y_2)^{p_2} \ldots x(y_n)^{p_n} \, dy,$$

where $y = (y_1, \ldots, y_n)$, $dy = dy_1 dy_2 \ldots dy_n$ and $b \in L^\infty(T^n)$.

In that case, F belongs to the domain of the Levy Laplacian given by (K_ε) and

$$F(x) = \sum_j \int p_j(p_j-1)x(y_j)^{p_j-2} b(y) \prod_{i \neq j} x(y_i)^{p_i} \, dy \quad (*)$$

It is just Lemma 4.2 of [4].

For simplicity we take n = 2, then

$$F(x+W_N(t)) = \int b(y) \left[\sum_{i=0}^{p_1} \binom{p_1}{i} x(y_1)^{p_1-i} W_N(t)^i \right]$$

$$\left[\sum_{j=0}^{p_2} \binom{p_2}{j} x(y_2)^{p_2-j} W_N(t)^j \right] dy,$$

Since $W_N(t)(y_j)$ is Gaussian random variable and

$$E(W_N(t)(y_1) \, W_N(t)(y_2)) = Z(\epsilon)^{-2} \sum_{n=1}^{N} (\int K_\epsilon(y_1-s_1)e_n(s_1)ds_1)(\int K_\epsilon(y_2-s_2)e_n(s_2)ds_2)$$

Through the double limit,it is easy to see F is admissible and
(*) holds.Now we examine F''(x).It is sufficient to prove that

$$\lim_{t \to 0} \sup[\, |E(\prod_{j=1}^{n} W_N(t)(y_j)^{p_j})| \; : \; N \geq 1,(y_1,y_2,\ldots,y_n)\in R^n] = 0$$

where $p_1+p_2+\ldots+p_n \geq 2$. This is immediate from the following

$$E(W_N(t)(y)^{2q}) \leq (2q)! \; t^q/2^q q!, \text{ for all N and y.}$$

Combine these facts and Theorem 3.6. we have the Lemma 4.2. of
[4].

4. The harmonic functionals
on Hilbert spaces

Now we turn to study the harmonic functionals on H .Of course,
first we should study the spherical mean.Here we prefer to pre-
sent it with probability lanquage.

Definition. Let $W(\epsilon,t)$ is a Brownian particle and $P_{\epsilon,t}$ is its
distribution on H.Suppose $F \in C_b(H)$ such that for $x \in H$ and $t > 0$

$\int F(x+y) \, dP_{\epsilon,t}(y)$ converges as $\epsilon \downarrow 0$, in this time we say F

has the spherical mean over the sphere of radius t with center

at x,and we denote the limit by MF(x,t). Let U be an open subset

of H,we say that a functional F ($\in C_b(U)$) possesses the mean

value property on U if for each $x \in U$ there exists R = R(x) > 0

such that MF(x,t) = F(x) whenever t < R.

The following result has been noted in [7],here we give it in

our setup.

Proposition 4.1. Suppose functional F is reqularly analytic

at $x_0 \in H$ (see [7]).Then for each RP of H,there exists R > 0 and

an extension F_1 of F at x_0(see Remark 3.3.) such that $MF_1(x_0,t)$ =

$F(x_0)$ whenever t < R.

Proof. For simplicity,let x_0= 0.According to [7],the regularly

analytical functional F admits an expression

$$F(x) = \sum_k < a_k, \ x^{\otimes k} > \ , \ a_k \in S^k H \ \text{ and } \sum_k \|a_k\| R^k \quad , R > 0.$$

By Remark 3.3.,we can find a functional $Q \in C_b(H)$ such that

Q(x) = 1 when $\|x\| \leq \frac{1}{4}R$ and Q(x) = 0 when$\|x\| \geq \frac{1}{2}R$.We choose

$Q(x)F(x) = F_1(x)$ as the extenssion of F.Set a family of σ-fields

such that

$$\mathcal{F}_t = \sigma[\ G_n(s) \ : \ s \leqslant t \text{ and all } n\ ].$$

Because $(G_n(t))$ is an independent stochastic process sequence, so for all n, $G_n(t)$ is a martingale with respect to $(\mathcal{F}_t)_{t \geqslant 0}$. For $R > 0$, we define stopping time τ such that

$$\tau = \inf[\ t \ : \ \|W(\epsilon,t)\|_H > \tfrac{1}{2}R\ ],$$

where $W(\epsilon,t)$ is a Brownian particle with a RP of H. We have

$$F_1(W(\epsilon,t)) = Q(W(\epsilon,t))F(W(\epsilon,t)) = F_1(W(\epsilon,t\wedge\tau))$$

$$= Q(W(\epsilon,t\wedge\tau))\sum_{k} <a_k, W(\epsilon,t\wedge\tau)^{\otimes k}> .$$

Obviously $\|W(\epsilon,t\wedge\tau)\|_H \leqslant \tfrac{1}{2}R$ and $<a_k, W(\epsilon,t\wedge\tau)^{\otimes k}>$ is a finite linear combination of functions of the form

$$H(b_1,b_2,\ldots,b_m) = (b_1,W(\epsilon,t\wedge\tau))\ldots(b_m,W(\epsilon,t\wedge\tau)),$$

where $b_j \in H$, $1 \leqslant j \leqslant m < \infty$.

Since $(b_j,W(\epsilon,t\wedge\tau))$ is martingale with respect to $(\mathcal{F}_t)$, invirtue of Ito formula, we have

$$E(H(b_1,\ldots,b_m)) = \tfrac{1}{2}\sum_{1 \leqslant i,j \leqslant m} E(\int_0^{t\wedge\tau} \prod_{r \neq i,j}(b_r,W(s\wedge\tau))(\ I_\epsilon^* b_i, I_\epsilon^* b_j)\ ds.$$

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Then

$$\left| E(\ H(b_1,\ldots,b_m))\right| \leq Z(\epsilon)^{-2}M\ \left|\sum_{1\leq i\cdot j\leq m}(K_\epsilon K_\epsilon^* b_i,b_j)\right|t,$$

where M is a constant independent from ϵ .Combining this fact with the assumption,dominated convergence theorem implies

$$\lim_{\epsilon\downarrow 0} E(\ F_1(W(\epsilon,t)) = F(0).$$

Q.E.D.

Corollary 4.2. For every RP of H regularly analytical fun-
ctionals are harmonic on H.

The purpose of studying the Lévy Laplacian aim at functionals
on infinite dimensional spaces. Naturally the harmonic functional
is very interesting object.In this subject we have

Proposition 4.3. If F C_b^2(H),then F is harmonic with respect
to any RP of H,(K_ϵ) if and only F''(x) is absolutely continuous
linear operstor on H for all x$\in$ H.

Proof. The only thing we to do is that if A$\in$B(H),then the
quadratic functional F(x) = (Ax,x) is harmonic for all RP of H
if and only if A is absolutely continuous on H.

By Theorem 3.6.,

$$\triangle F(x) = \lim_{\epsilon \downarrow 0} Z(\epsilon)^{-2}[\text{trace } K_\epsilon^* A K_\epsilon]$$

$$= \lim_{\epsilon \downarrow 0} Z(\epsilon)^{-2}[\text{trace } K_\epsilon^* (A-S) K_\epsilon],$$

where S is a finite rank operator on H.Since

$$\text{trace}[K_\epsilon^* (A-S) K_\epsilon] \leq \|A-S\|_{B(H)} \text{ trace } K_\epsilon^* K_\epsilon .$$

That means $\triangle F(x) = 0$ when A is absolutely continuous on H.

On the other hand, if $\triangle F(x) = 0$,but A was not absolutely con-
tinuous on H . In this case,from the spectral theory of self-
adjoint operators we know that there is a positive number c>0 and
an infinite dimensional subspace H_0 such that for every $y \in H_0$
$(Ay,y) \geq c > 0$.Choose a CONS of H_0 and a CONS of $H \ominus H_0$,respectively
(u_m) and (v_k).We construct a CONS of H such that

$$e_n \in (u_m) \text{ when } n \neq 2^k, k = 0,1,\ldots,$$

$$e_n = v_k , \text{ when } n = 2^k, k = 0,1,2,\ldots .$$

Using the CONS of H to construct the Levy RP of H,it is easy to

see that $\triangle F(x) \geq c > 0$.This contradiction indicates A must be

absolutely continuous on H.

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Now we come back to the Gelfand trip $H_0 \subset H \subset H_0^*$. It follows

that there exists a unique probability measure μ on H_0^* such

that

$$\exp[-\tfrac{1}{2} \| x \|_H^2 \ ] = \int_{H_o^*} \exp[i < y,x>]d\mu(y), \quad x \in H_0,$$

where $< y,x >$ stands for the canonical bilinear form on $H_0^* \times H_o$.

We will conclude this section by extending a well known fact

about the harmonic property of U-functionals (see [4] and [7]).

Theorem 4.4. Let $f(y,z) \in L^p(\ H_o^* \times H_o^*, \ d\mu(y) \times d\mu(z) \ )$,

$1 < p < \infty$. Suppose that

$$F(x) = \iint f(y,z) \ \exp[< y+iz, \ x >] \ d\mu(y) \ d\mu(z), \quad x \in H_0.$$

If (K_ϵ) is a RP of H such that $K_\epsilon H \subset H_0$ for all ϵ ,then

F is harmonic with respect to the Lévy Laplacian given by (K_ϵ),

that means $\triangle F(x) = 0$ for all $x \in H_0$.

Before giving its proof we are going to present some lemmas

which have their own interesting.

Lemma 4.5. Define

$$< y,x > = \sum_n <y,e_n>(e_n,x \ )_H \qquad \text{,for} \quad y \in H_o^*, x \in H,$$

where $(e_n) \subset H_0$ and it is a CONS of H, then the series converges in $L^2(H_e^*, d\mu(y))$ and

$$\int \exp[r < y,x >]d\mu(y) = \exp[\tfrac{1}{2}r^2 \| x \|_H^2], x \in H.$$

So we can extend F as follows

$$F(x) = \iint f(y,z) \exp[<y+iz,x>]d\mu(y) \, d\mu(z), \quad x \in H.$$

Let $W_N(\epsilon,t) = \displaystyle\sum_{n=1}^{N} I_\epsilon e_n G_n(t)$, we have

$$\lim_{N \to \infty} E(\left| F(x+W_N(\epsilon,t))-F(x+W(\epsilon,t)) \right|) = 0.$$

thats means F is admissible with respect to (K_ϵ).

Proof. Since

$$\lim_{N \to \infty} E[\int | \sum_{n=1}^{\infty} < y,e_n > (e_n,W(t)-W_N(t)) |^2 \, d\mu(y)] = 0$$

and for all $r > 1$ and all N

$$E[\iint |\exp(<y+iz, W_N(t)>)|^r d\mu(y)d\mu(z) \leq 1.$$

We have

$$\lim_{N \to \infty} E[\iint |\exp(\langle y+iz, W_N(t) \rangle) - \exp(\langle y+iz, W(t) \rangle)|^r d\mu(y) d\mu(z) = 0.$$

The proof can be completed easily.   Q.E.D.

Lemma 4.6.   Let $(e_n) \subset H_0$ and (e_n) be a CONS of H such that $K_\epsilon K_\epsilon^* e_n = \lambda_n^2 e_n$, then

$$\sum_n | \langle y+iz, K_\epsilon e_n \rangle |^2 = \sum_n \lambda_n^2 \langle y+iz, e_n \rangle^2, \qquad y,z \in H_0^*,$$

and when $r > 1$, $rt < \frac{1}{4}$, we have

$$A = \iint (\sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} \langle y+iz, e_n \rangle^2)^r \exp[-\tfrac{1}{2}rt \sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} \langle y+iz, e_n \rangle^2] d\mu(y) d\mu(z) \leq$$

$$\leq \frac{2^{2r-1} \sqrt[2]{e} \, \Gamma(r+\tfrac{1}{2})}{\sqrt{\pi}}$$

Proof.   Assume

$$B(y) = (\sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} \langle y, e_n \rangle^2) \exp[-t \sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} \langle y, e_n \rangle^2],$$

$$C(y) = \exp[-t \sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} \langle y, e_n \rangle^2],$$

where $Z(\epsilon)^2 = \text{trace } K_\epsilon^* K_\epsilon = \text{trace } K_\epsilon K_\epsilon^* = \sum_n \lambda_n^2$ and (e_n) is depending on K_ϵ.

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Since

$$A \leq 2^r [\int B(y)^r d\mu(y)][\int C(y)^r d\mu(y)],$$

and

$$\int B(y)^r d\mu(y) = \lim_{N \to \infty} [(\frac{1}{\sqrt{2\pi}})^N \prod_{n=1}^{N} (1 - \frac{2tr\lambda_n^2}{Z(\epsilon)^2})^{-\frac{1}{2}} \int \cdots \int_{R^N} (\sum_{n=1}^{N} \frac{\lambda_n^2}{Z(\epsilon)^2} (1 - \frac{2tr\lambda_n^2}{Z(\epsilon)^2})^{-1} y_n^2)^r$$

$$\exp(-\tfrac{1}{2} |y|^2) \, dy],$$

where $(y_1, \ldots, y_N) \in R^N$ and $|y|^2 = \sum_{j=1}^{N} y_j^2$, $dy = dy_1 \ldots dy_N$.

Because $1 - 2tr\lambda_n^2 / Z(\epsilon)^2 \leq \tfrac{1}{2}$ and

$$(1 - 2tr \lambda_n^2 / Z(\epsilon)^2) \geq e^{-\frac{1}{2}}$$

by Jensen"s inequality and $\sum_n \lambda_n^2 / z(\epsilon)^2 = 1$,

$$(\sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} y_n^2)^r \leq \sum_n \frac{\lambda_n^2}{Z(\epsilon)^2} y_n^{2r},$$

combining all facts, we have

$$\int B(y)^r d\mu(y) \leq 2^{2r-1} e^{\frac{1}{4}} \Gamma(r+\tfrac{1}{2}) \quad , \quad \text{for } t < \tfrac{1}{4}r^{-1}.$$

For C(y), we have $\int C(y)^r d\mu(y) \leq e^{\frac{1}{4}}$.

<div align="right">Q.E.D.</div>

We call a functional g on $H_o^* \times H_o^*$ cylindrical if g is of the form

g($\langle u_1, y \rangle, \ldots, \langle u_m, y \rangle, \langle u_1, z \rangle, \ldots, \langle u_m, z \rangle$) , $m < \infty$ and $g \in C_b(R^{2m})$ and $u_j \in H_0$, $1 \le j \le m$.

Lemma 4.7.   Suppose g is a cylidrical functional on $H_o^* \times H_o^*$ and $g \in L^p(H_o^* \times H_o^*, d\mu \times d\mu$), $1 < p < \infty$ and

$$G(x) = \iint g(y,z) \exp[\langle y+iz, x \rangle] d\mu(y) d\mu(z),$$

then G is harmonic with respect to all RP of H.

Proof.   For simplicity we assume that g is of the form

$$g(\langle e_1, y \rangle, \ldots \langle e_m, y \rangle \langle e_1, z \rangle \ldots \langle e_m, z \rangle)$$

where (e_n) is a CONS of H and ($e_n) \subset H_0$. In that case

$$G(W(t)) = \iint g(y,z) \exp[\sum_{j=1}^{m} \langle y+iz, e_j \rangle (e_j, W(t)) d\mu(y) d\mu(z)]$$

and

$$E(G(W(t))) = \iint g(y,z) \exp[-\tfrac{1}{2}t \sum_{j=1}^{m} \| K^* e_j \|^2 \langle y+iz, e_j \rangle^2 / Z(\epsilon)^2] d\mu(y) d\mu(z).$$

Now we see that

$$\triangle \tilde{G}(0) = \lim_{\epsilon \downarrow 0} \lim_{t \to 0} 2\,[E(G(W(\epsilon,t))) - G(0)]/t = 0.$$

In the similar way we can prove that $\triangle \tilde{G}(x) = 0$ for all $x \in H_0$.

Q.E.D.

The proof of Theorem 4.4. By Lemma 4.5.

$$E(F(W(\epsilon,t))) = \iint f(y,z)\exp[-\tfrac{1}{2}t\sum_n \frac{\lambda_n^2}{Z(\epsilon)^2}\langle y+iz,e_n\rangle^2]\,d\mu(y)\,d\mu(z),$$

and using Lemma 4.6. to it, we have estimate

$$t^{-1}|E(F(W(\epsilon,t))) - F(0)| \leqslant \frac{2^{2r-1}}{\sqrt{\pi}}\sqrt{e}\;\Gamma\,(r+\tfrac{1}{2})\,\|f\|_p.$$

Since the set of cylidrical functionals is dense in $L^p(H_0^* \times H_0^*, d\mu \times d\mu)$ $1 < p < \infty$, and in virtue of Lemma 4.7., we know that $\triangle F(0) = 0$. Consider $f(y,z)\exp[\langle y+iz,x_0\rangle]$ for $x_0 \in H_0$, we have $\triangle F(x_0) = 0$.

Q.E.D.

5. Some remarks of generalized Lévy Laplacian

In this section we give some generalized Lévy Laplacians. First

if $A(t,\omega)$ is bounded linear operator valued process which is adapted with respect to $(\mathscr{F}_t)$, $\mathscr{F}_t = \sigma[\ G_n(s) : s \leq t$ and for all n],and

$$E(\int \|A(s)\|^2_{B(H)}\ ds\ ) < \infty$$

we can define the stochastic integral with respect to Brownian particle $W(\varepsilon,t)$ such that

$$X(\varepsilon,t) = \int_0^t A(s)\ dW(\varepsilon,s) = \sum_n \int_0^t A(s)(I_\varepsilon e_n)dG_n(s),$$

obviously the series converges in H and

$$E(\|\ X(\varepsilon,t)\|^2) = E(\int_0^t [\text{trace } I_\varepsilon^* A(s)I_\varepsilon\ ]ds.$$

Second,if $G(t,x,\omega)$ is a jointly measurable map from $T \times H \times \Omega$ to H such that $G(0,x,\omega) = 0$ and for fixed $\omega \in \Omega$, $G(t,x,\omega)$ is continuous map from $T \times H$ to H,we set

$$X(\varepsilon,t) = G(t,W(\varepsilon,t),\omega).$$

In these cases we can define the generalized Lévy Laplacians as follows.

$$\Delta_X F(x) = 2 \lim_{\varepsilon \downarrow 0} \lim_{t \to 0} [E(x+X(\varepsilon,t))-F(x)]/t,$$

if the limit exists.

Such kind of generalized Lévy Laplacians corresponds to the variable coefficient second order differential operators in finite dimensional spaces (see [1]).

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