On Symmetric Stable Random Variables and Matrix Transposition

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Abstract: In a companion paper, the authors obtained some Fubini type identities in law for quadratic functionals of Brownian motion, and, more generally, for certain functionals of symmetric stable processes, the function: $x \to x^2$ then being replaced by: $x \to |x|^{\alpha}$.

In this paper, discrete analogues of such identities in law, which involve a sequence of independent standard symmetric stable r.v.'s of index α, are presented.

It is then shown that such identities in law characterize the symmetric α-stable distribution. Some related characterization results, either for some finite or infinite dimensional r.v.'s are also presented.

Key Words: stable random variables, matrix transposition.

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0. Introduction.

(0.1) The present work takes its origin in the simple proofs given by two of the authors of certain identities in law between some functionals of Brownian motion or Bessel processes (see [3] and [6]).

Precisely : (i) if $(B_t, t \geq 0)$ denotes a one-dimensional Brownian motion starting from 0, and $(\tilde{B}_t\ ;\ 0 \leq t \leq 1)$ a standard Brownian bridge, then :

(0.a)
$$\int_0^1 ds(B_s - G)^2 \overset{\text{(law)}}{=} \int_0^1 ds\ \tilde{B}_s^2\ , \quad \text{where} \quad G \overset{\text{def}}{=} \int_0^1 ds\ B_s$$

(see [3], where this identity in law is obtained, together with several extensions).

(ii) if, for $\delta > 0$, $(R_\delta(t), t \geq 0)$ denotes a δ-dimensional Bessel process starting from 0, then :

$$(0.b) \qquad \int_0^\infty ds\, 1_{(R_{\delta+2}(s)\le 1)} \overset{(\text{law})}{=} T_1(R_\delta),$$

where $T_1(X) \overset{\text{def}}{=} \inf\{t : X_t = 1\}$.

The identity in law *(0.b)* is due to Cielsielski-Taylor (1962) for integer dimensions ; for a detailed discussion and further extensions, see [6] and [7]. The proofs of both identities in law *(0.a)* and *(0.b)* rely essentially upon the following (Fubini type) identity in law :

$$(0.c) \qquad \int_0^\infty ds \left(\int_0^\infty \varphi(s,u)dB_u\right)^2 \overset{(\text{law})}{=} \int_0^\infty ds \left(\int_0^\infty \varphi(u,s)dB_u\right)^2,$$

where $\varphi \in L^2(\mathbb{R}_+^2, ds\, du)$.

(0.2) In the first section of the present work, we show that some discrete analogue of the identity in law *(0.c)* holds for a sequence of i.i.d. Gaussian variables, namely : if $\underline{G}_n = (G_1,\dots,G_n)$ is a random vector which consists of n independent $N(0,1)$ random variables, then the identity in law :

$$(1.a)_2 \qquad \ell_2(A\underline{G}_n) \overset{(\text{law})}{=} \ell_2(A^*\underline{G}_n)$$

holds, where A is any $n \times n$ real matrix, and $\ell_2(x) = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ denotes the euclidean norm of $x = (x_1,\dots,x_n) \in \mathbb{R}^n$.

In fact, more generally, we show that, for any $0 < \alpha \le 2$, we have :

$$(1.a)_\alpha \qquad \ell_\alpha(A\underline{C}_n^{(\alpha)}) \overset{(\text{law})}{=} \ell_\alpha(A^*\underline{C}_n^{(\alpha)}),$$

where $\underline{C}_n^{(\alpha)} = (C_1^{(\alpha)},\dots,C_n^{(\alpha)})$ now denotes a random vector, the components of which are n standard symmetric stable r.v's, with parameter α, and

$$\ell_\alpha(x) = \left(\sum_{i=1}^{n} |x_i|^\alpha\right)^{1/\alpha}.$$

(0.3) In the second section, we are interested in the study of a converse to the property *(1.a)*$_\alpha$, namely : if X_1 , X_2 ,...,X_n,... is a sequence of i.i.d. random variables which satisfies, for any $n \in \mathbb{N}$, and any $n \times n$ matrix A :

$$(1.a)'_\alpha \qquad \ell_\alpha(A\underline{X}_n) \overset{(law)}{=} \ell_\alpha(A^*\underline{X}_n),$$

where $\underline{X}_n \overset{def}{=} (X_1,\dots,X_n)$, then we show that X_1 is a symmetric stable random variable of index α.

Hence, in this sense, the property *(1.a)*$_\alpha$ characterizes the symmetric stable law of index α.

(0.4) In section 3, we consider a fixed finite dimension n, and we try to characterize the laws of n-dimensional random variables $\underline{X}_n = (X_1,\dots,X_n)$ such that *(1.a)'*$_\alpha$ is satisfied, but we do not assume any other property on the vector $\underline{X}_n$.

We obtain a complete description of such vectors for $\alpha = 2$, but our description remains incomplete for $\alpha \neq 2$.

(0.5.) In section 4, we go back to the infinite dimensional case ; if $X_1,X_2,\dots,X_n,\dots$ is a sequence of random variables which satisfies, for any $n \in \mathbb{N}$, and any $n \times n$ matrix A, the identity in law *(1.a)'*$_\alpha$ and, if moreover, $\underline{X}_\infty \overset{(law)}{=} -\underline{X}_\infty$, then, we prove that :

$$(0.d) \qquad \underline{X}_\infty \overset{(law)}{=} (HC_n^{(\alpha)}\ ;\ n \in \mathbb{N})$$

where $(C_n^{(\alpha)}\ ;\ n \in \mathbb{N})$ is a sequence of independent symmetric standard stable variables of index α, and H is a ≥ 0 r.v. which is independent of the sequence $\underline{C}_\infty^\alpha$. In fact, we may even get rid of the assumption $\underline{X}_\infty \overset{(law)}{=} -\underline{X}_\infty$, and the general result is that *(0.d)* holds up to the introduction of a

Bernoulli, ± 1 valued, random variable (see Theorem 3 below for a precise statement).

(0.6) As a conclusion of this introduction, we describe how this paper relates to its companions [3], [6] and [7] : whereas in [3] and [6], the authors presented some applications, mainly *(0.a)* and *(0.b)*, of the Fubini identity *(0.c)*, the aim of this paper, together with [7], is to unsderstand in a deeper way the role of the identity *(0.c)* :
in the present paper, we restrict ourselves to the case of a (possibly finite) sequence of variables, and, therefore, we discuss identities in law such as $(1.a)_\alpha$ and $(3.a)_\alpha$, whilst, in [7], we consider continuous time processes and, in particular, we characterize all the process $(X_t, t \geq 0)$ which satisfy :

$$(0.c)' \qquad \int_0^\infty ds \left(\int_0^\infty \varphi(s,u)dX_u\right)^2 \stackrel{(law)}{=} \int_0^\infty ds \left(\int_0^\infty \varphi(u,s)dX_u\right)^2 ,$$

for all simple functions $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$.

1. <u>The main identity in law.</u>

Let $0 < \alpha \leq 2$, and $n \in \mathbb{N} \setminus \{0\}$. We consider the application $\ell_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}_+$

defined by : $$\ell_\alpha(a) = \left(\sum_{i=1}^n |a_i|^\alpha\right)^{1/\alpha}.$$

We also consider an n-dimensional random vector

$$\underline{C}_n^{(\alpha)} = (C_1^{(\alpha)},\dots,C_n^{(\alpha)}),$$

the components of which are n independent, standard, symmetric variables, which are stable with exponent α, that is :

$$E[\exp(i\lambda C_j^{(\alpha)})] = \exp(-|\lambda|^\alpha) \qquad (\lambda \in \mathbb{R}).$$

Then, we have the :

Theorem 1 : *For any* $n \times n$ *real matrix* A, *we have :*

$$(1.a)_\alpha \qquad \ell_\alpha(A\underline{C}_n^{(\alpha)}) \overset{(law)}{=} \ell_\alpha(A^*\underline{C}_n^{(\alpha)}),$$

where A^* *is the transpose of* A.

Proof : We introduce $\tilde{\underline{C}}_n^{(\alpha)}$ an independent copy of $\underline{C}_n^{(\alpha)}$, and we write :

$$(A\underline{C}_n^{(\alpha)},\tilde{\underline{C}}_n^{(\alpha)}) = (\underline{C}_n^{(\alpha)},A^*\tilde{\underline{C}}_n^{(\alpha)}).$$

We then compute, in two different manners, the characteristic function of the above random variable ; we obtain thus :

$$E[\exp - |\lambda|^\alpha(\ell_\alpha(A\underline{C}_n^{(\alpha)})^\alpha)] = E[\exp - |\lambda|^\alpha(\ell_\alpha(A^*\tilde{\underline{C}}_n^{(\alpha)})^\alpha)]$$

for every $\lambda \in \mathbb{R}$.

Using the fact that : $\underline{C}_n^{(\alpha)} \overset{(law)}{=} \tilde{\underline{C}}_n^{(\alpha)}$ and the injectivity of the Laplace transform, we obtain $(1.a)_\alpha$. □

Remark 1 : In the case $\alpha = 2$, there is also the following alternative proof : if $\underline{G}_n = (G_1,\dots,G_n)$ is an n-dimensional random vector, with the G_i's independent, centered, each with variance equal to 1, then we have :

$$(\ell_2(A\underline{G}_n))^2 = (A\underline{G}_n,A\underline{G}_n) = (\underline{G}_n,A^*A\underline{G}_n) \overset{(law)}{=} (\underline{G}_n,AA^*\underline{G}_n),$$

since AA^* and A^*A have the same eigenvalues, with the same order of multiplicity, and the law of $\underline{G}_n$ is invariant by orthogonal transforms.
Then, the proof is ended by remarking that :

$$(\underline{G}_n,AA^*\underline{G}_n) = (\ell_2(A^*\underline{G}_n))^2.$$

Obviously, these arguments cannot be used for $\alpha \neq 2$.

2. A characterization of the symmetric stable laws.

We now consider a given application $\ell : FS \longrightarrow \mathbb{R}_+$, where FS is the set of finite sequences $a = (a_1,\dots,a_n,0,0,\dots)$ for some n, and $a_i \in \mathbb{R}$, such that the following hypotheses are satisfied :

(2.a) $\qquad \ell(a) > 0$, for every $a \neq 0$;

(2.b) $\qquad \ell(\lambda a) = |\lambda|\,\ell(a)$, for every a, and $\lambda \in \mathbb{R}$.

We also consider a real-valued random variable X and a sequence of i.i.d. random variables $X_1,X_2,\dots,X_n,\dots$, with the same distribution as X, and we write $\underline{X}_n$ for the truncated sequence $(X_1,\dots,X_n,0,0,\dots)$, which we sometimes identify with the $\mathbb{R}^n$-valued r.v : $(X_1,\dots,X_n)$.

We can now state and prove our main result.

Theorem 2 : *The following properties are equivalent :*

1) X *is a symmetric stable random variable, with parameter* α *;*

2) there exists $\ell : FS \longrightarrow \mathbb{R}_+$ *which satisfies the properties (2.a) and (2.b) and such that, for every* $n \in \mathbb{N}^*$, *and every* $n \times n$ *real matrix* A, *we have :*

(2.c) $\qquad \ell(A\underline{X}_n) \overset{(law)}{=} \ell(A^*\underline{X}_n)$;

3) there exists an application $\tilde{\ell} : FS \longrightarrow \mathbb{R}_+$ *such that :*

(2.ã) $\qquad \tilde{\ell}(a) > 0$, *for every* $a \neq 0$,

and :

(2.d) $\qquad$ *for every* $a = (a_1,\dots,a_n,0,0,\dots)$, $\left|\sum_{i=1}^{n} a_i X_i\right| \overset{(law)}{=} \tilde{\ell}(a)|X_1|$.

Moreover, when 1) is satisfied, the applications ℓ *and* $\tilde{\ell}$ *are given by :*

$$\ell(a) = c\,\ell_\alpha(a), \text{ for some } c > 0, \text{ and } \tilde{\ell}(a) = \ell_\alpha(a).$$

Remark 2 : In the statement of Theorem 2, we have tried to make some minimal hypothesis about the application $\ell : FS \longrightarrow \mathbb{R}_+$, namely *(2.a)* and *(2.b)*. However, even these hypotheses may be superfluous, as the following seems to suggest : if X is a symmetric stable random variable with parameter α, and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a Borel function, then : $\ell = f o \ell_\alpha$ obviously satisfies *(2.c)*. Such applications ℓ may well be the largest class of applications from FS to $\mathbb{R}$ which satisfy *(2.c)*. □

Proof of Theorem 2 : a) From Theorem 1, we already know that 1) $\Longrightarrow$ 2), with $\ell = \ell_\alpha$.

To prove that 2) $\Longrightarrow$ 3), we remark that, if we take $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ 0 \end{pmatrix}$, and

$A = \begin{pmatrix} a_1 & 0\dots0 \\ . & 0\dots0 \\ \vdots & \vdots \quad \vdots \\ a_n & 0\dots0 \end{pmatrix}$, so that : $A^* = \begin{pmatrix} a_1 \dots a_n \\ 0 \dots 0 \\ 0 \dots 0 \\ 0 \dots 0 \end{pmatrix}$, we obtain, from our hypothesis *(2.c)* : $\ell(aX_1) \overset{(law)}{=} \ell\begin{pmatrix} \sum a_i X_i \\ 0 \\ 0 \\ \vdots \end{pmatrix}$,

and now, using *(2.b)*, we have :

$$(2.e) \qquad |X_1| \; \ell(a) \overset{(law)}{=} \left|\sum_{i=1}^{n} a_i X_i\right| \ell\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, 3) is satisfied with : $\tilde{\ell}(a) = \ell(a)/\ell\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

b) It now remains to prove that 3) $\Rightarrow$ 1), and that, when 1) is satisfied, ℓ and $\tilde{\ell}$ are determined as announced in the last statement of the Theorem.

Indeed, let us assume for one moment that we have proved the implication : 3) $\longrightarrow$ 1), so that X_1 is symmetric, stable, with exponent α.

Then, we deduce from *(2.d)* that : $\tilde{\ell}(a)|X_1| \overset{(law)}{=} \ell_\alpha(a)|X_1|$, so that : $\tilde{\ell} = \ell_\alpha$, and we deduce from *(2.e)* that :

$$\ell(a) = c\ell_\alpha(a), \text{ with } c = \ell\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

c) We now prove that 3) $\Longrightarrow$ 1).

To help the reader with the sequel of the proof, we first assume that X_1 is symmetric ; then, we deduce from *(2.d)* that we have, by taking $a \equiv \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ (1 is featured here n times) :

$$X_1 \overset{(law)}{=} \frac{1}{\lambda_n}\left(\sum_{i=1}^{n} X_i\right).$$

This implies (see Feller [4], p. 166) that $\lambda_n \sim n^{1/\alpha}$, for some $0 < \alpha \leq 2$, and that X_1 is symmetric, stable, with exponent α.

d) Now, we give the complete proof without assuming a priori that X_1 is symmetric.

By taking $a = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$ the (2n)-dimensional vector, with its n first components equal to 1, and the n last ones equal to -1, we obtain, from our hypothesis *(2.d)*, that :

$$|X_1| \overset{(law)}{=} \frac{1}{\lambda_n}\left|\sum_{i=1}^{n} (X_i - X'_i)\right|$$

and, consequently, the (symmetric) law of $\frac{1}{\lambda_n} \sum_{i=1}^{n} (X_i - X'_i)$ does not depend on n. Consequently, just as above in c), we obtain that $X_1 - X'_1$ is (symmetric) stable with some exponent α.

e) It now remains to show that X_1 is symmetric.

To do this, we shall use the hypothesis *(2.d)*, with $a_1 = (1,1)$, and $a_2 = (1,-1)$. Thus, we deduce from *(2.d)* that :

$$|X_1 + X_2| \overset{(\text{law})}{=} \mu |X_1|, \text{ and } \quad |X_1 - X_2| \overset{(\text{law})}{=} \nu |X_1|. \tag{2.f}$$

This identity *(2.f)* is equivalent to :

$$\varepsilon(X_1 + X_2) \overset{(\text{law})}{=} \mu\varepsilon X_1 \quad \text{and : } \quad X_1 - X_2 \overset{(\text{law})}{=} \nu\varepsilon X_1 \ , \tag{2.f'}$$

where ε is a symmetric Bernoulli variable, which is independent of the pair (X_1, X_2).

We define $\phi(t) \equiv E(\exp i(t\, X_1))$, and we remark that, since we now know $X_1 - X_2$ to be (symmetric) stable, with exponent α, we have :

$$|\phi(t)| = \exp(-c|t|^{\alpha}), \text{ for some } c. \tag{2.g}$$

Hence, the identity *(2.f')* may be written as the following pair of identities :

$$\frac{1}{2}\,(\phi(t) + \overline{\phi(t)}) = \frac{1}{2}\,((\phi(\tfrac{t}{\mu}))^2 + (\overline{\phi(\tfrac{t}{\mu})})^2), \tag{2.h}$$

and

$$\frac{1}{2}\,(\phi(t) + \overline{\phi(t)}) = |\phi(\tfrac{t}{\nu})|^2 = \exp(-2c|\tfrac{t}{\nu}|^{\alpha}), \text{ from } (2.g). \tag{2.i}$$

Now, we remark that the right-hand side of *(2.h)* is (obviously) equal to :

$$\frac{1}{2}(\phi(\tfrac{t}{\mu}) + \overline{\phi(\tfrac{t}{\mu})})^2 - |\phi(\tfrac{t}{\mu})|^2,$$

and then, using *(2.g)* and *(2.i)*, the identity *(2.h)* may now be written as :

$$\exp\left(-2c\left|\frac{t}{\nu}\right|^{\alpha}\right) = \frac{1}{2}\left\{2\ \exp\ -\ 2c\left|\frac{t}{\mu\nu}\right|^{\alpha}\right\}^{2} - \exp(-2c\left|\frac{t}{\mu}\right|^{\alpha}),$$

which, if we write $s = 2c|t|^{\alpha}$, $m = \frac{1}{\mu^{\alpha}}$, $n = \frac{1}{\nu^{\alpha}}$, is equivalent to :

$$2\ \exp(-2mns) = \exp(-ms) + \exp(-ns),\ \text{for all}\quad s \geq 0.$$

From the injectivity of Laplace transforms (for instance !), we now deduce that : $2mn = m = n$, so that : $\mu = \nu$, and we now deduce from *(2.f)* that :

(2.j) $$|X_1-X_2| \overset{(law)}{=} |X_1+X_2|.$$

This relation *(2.j)* implies, by Lemma 1 below, that X_1 is symmetric, and the proof of our Theorem 2 is finished. □

It may be helpful to isolate the following characterization of a symmetric random variable.

Lemma 1 : *A real-valued random variable* X *is symmetric if, and only if :*

(2.k) $$|X+X'| \overset{(law)}{=} |X-X'|,$$

where X' *is an independent copy of* X.

Proof : All we need to show is that, if *(2.k)* is satisfied, then X is symmetric.
Consider a symmetric Bernoulli random variable ε, which is independent of the pair (X,X'). Then, *(2.k)* is equivalent to :

(2.ℓ) $$X-X' \overset{(law)}{=} \varepsilon(X+X'),$$

and, if we note : $z = E[e^{itX}]$, we have, from *(2.ℓ)* :

$$|z|^2 = \frac{1}{2}\{z^2 + \bar{z}^2\},$$

which is equivalent to : $\mathrm{Im}(z) = 0$; hence, $E[e^{itX}]$ is real, and X is symmetric. □

3. The finite dimensional study.

Let $n \in \mathbb{N}$, $n \geq 1$, and $0 < \alpha \leq 2$. In this section, we should like to characterize the n-dimensional random variables $\underline{X}_n = (X_1,\dots,X_n)$ which satisfy :

$$(1.a)_\alpha \qquad \ell_\alpha(A\underline{X}_n) \overset{(\text{law})}{=} \ell_\alpha(A^*\underline{X}_n).$$

The difference with the study made in the previous sections is that we do not assume here that the components $X_1, X_2, \dots, X_n$ are independent, nor that they are identically distributed.

Our first result in this study is the following

Proposition 1 : *The* n-*dimensional r.v.* $\underline{X}_n = (X_1,\dots,X_n)$ *satisfies* $(1.a)_\alpha$ *if, and only if, for any* $(a_i)_{1\leq i\leq n} \in \mathbb{R}^n$, *we have*

$$(3.a)_\alpha \qquad \left|\sum_{i=1}^n a_i X_i\right| \overset{(\text{law})}{=} \left(\sum_{i=1}^n |a_i|^\alpha\right)^{1/\alpha} |X_1|.$$

Proof : 1) Using arguments similar to those in the proof of Theorem 1, it is easily seen that $\underline{X}_n$ satisfies $(1.a)_\alpha$ if, and only if, for every $n \times n$ matrix A, we have :

$$(3.b) \qquad (\underline{C}_n^{(\alpha)}, A\underline{X}_n) \overset{(\text{law})}{=} (\underline{C}_n^{(\alpha)}, A^*\underline{X}_n),$$

where the vector $\underline{C}_n^{(\alpha)}$ is assumed to be independent of $\underline{X}_n$.

Letting A vary among all $n \times n$ matrices, we obtain that *(3.b)* is equivalent to :

(3.c) $$\left(C_i^{(\alpha)}X_j\ ;\ 1 \le i \le j \le n\right) \overset{(law)}{=} \left(X_i C_j^{(\alpha)}\ ;\ 1 \le i,j \le n\right).$$

Now, if two vectors (a_i) and (a'_i) satisfy : $\sum_{i=1}^{n} |a_i|^{\alpha} = \sum_{i=1}^{n} |a'_i|^{\alpha}$, then, the variables :

$$\sum_{i=1}^{n} a_i C_i^{(\alpha)} \quad \text{and} \quad \sum_{i=1}^{n} a'_i C_i^{(\alpha)}$$

have the same law. We then deduce from *(3.c)* that :

$$\left\{\left(\sum_{i=1}^{n} a_i X_i\right) C_j\ ;\ j \le n\right\} \overset{(law)}{=} \left\{\left(\sum_{i=1}^{n} a'_i X_i\right) C_j\ ;\ j \le n\right\}$$

which is equivalent to : $\left|\sum_{i=1}^{n} a_i X_i\right| \overset{(law)}{=} \left|\sum_{i=1}^{n} a'_i X_i\right|$.

Consequently, we have obtained that $(3.a)_\alpha$ is satisfied.

2) Conversely, we aim to show that if $(3.a)_\alpha$ is satisfied, then so is $(1.a)_\alpha$. We remark the following equivalences, with the help of our above notations for *(3.b)* :

$$(1.a)_\alpha \iff (\underline{C}_n^{(\alpha)}, A\underline{X}_n) \overset{(law)}{=} (\underline{C}_n^{(\alpha)}, A^{*}\underline{X}_n), \text{ for every } n \times n \text{ matrix } A$$

$$\iff (A^{*}\underline{C}_n^{(\alpha)}, \underline{X}_n) \overset{(law)}{=} (A\underline{C}_n^{(\alpha)}, \underline{X}_n), \text{ for every matrix } A.$$

$$\iff (3.d) : (A^{*}\underline{C}_n^{(\alpha)}, \tilde{\underline{X}}_n) \overset{(law)}{=} (A\underline{C}_n^{(\alpha)}, \tilde{\underline{X}}_n), \text{ for every } A,$$

where we have denoted : $\tilde{\underline{X}}_n = \tilde{\varepsilon}\underline{X}_n$, with $\tilde{\varepsilon}$ a symmetric Bernoulli variable which is independent of the pair of n-dimensional variables $\underline{C}_n^{(\alpha)}$ and $\underline{X}_n$.

Now, the property $(3.a)_\alpha$ is equivalent to :

$$(3.a)_\alpha^{\sim} \qquad \sum_{i=1}^{n} a_i \tilde{X}_i \overset{(law)}{=} \left(\sum_{i=1}^{n} |a_i|^{\alpha}\right)^{1/\alpha} \tilde{X}_1 ,$$

and we shall deduce *(3.d)*, hence $(1.a)_\alpha$, from $(3.a)_\alpha^{\sim}$.

Indeed, we have :

$$(A^{*}\underline{C}_n^{(\alpha)},\tilde{\underline{X}}_n) \overset{(\text{law})}{=} \ell_\alpha(A^{*}\underline{C}_n^{(\alpha)})\tilde{X}_1 \qquad \text{(from } (3.a)^{\sim}_\alpha\text{)}$$

$$\overset{(\text{law})}{=} \ell_\alpha(A\underline{C}_n^{(\alpha)})\tilde{X}_1 \qquad \text{(since } \underline{C}_n^{(\alpha)} \text{ satisfies } (1.a)_\alpha\text{)}$$

$$\overset{(\text{law})}{=} (A\underline{C}_n^{(\alpha)},\tilde{\underline{X}}_n) \qquad \text{(from } (3.a)^{\sim}_\alpha\text{)}.$$

Hence, we have shown *(3.d)*, and the proof is finished. □

In the case $\alpha = 2$, we have the following characterization of all vectors $\underline{X}_n$ which satisfy $(1.a)_2$.

<u>**Proposition 2**</u> : *An* n*-dimensional random variable* $\underline{X}_n = (X_1,\dots,X_n)$ *satisfies* $(1.a)_2$ *if, and only if, it may be represented (possibly on a larger probability space than the original one) as :*

$$(3.e) \qquad \underline{X}_n = \varepsilon\rho\,\underline{U}_n \ ,$$

where ρ *is a r.v. which takes its values in* $\mathbb{R}_+$, $\underline{U}_n$ *is uniformly distributed on the unit sphere* S_{n-1} , ε *takes only the values* +1 *and* -1, *and* ρ *and* $\underline{U}_n$ *are independent* (*but no stochastic relationship between* ε *and the couple* (ρ,U_n) *is assumed*).

<u>Proof</u> : As we have already seen, $\underline{X}_n$ satisfies $(1.a)_\alpha$ if, and only if, it satisfies :

$$(3.a)^{\sim}_\alpha \qquad \sum_{i=1}^n a_i\tilde{X}_i \overset{(\text{law})}{=} \left(\sum_{i=1}^n |a_i|^\alpha\right)^{1/\alpha} \tilde{X}_1 \ ,$$

where : $\tilde{\underline{X}}_n = \varepsilon\underline{X}_n$, with ε a Bernoulli random variable which is independent of $\underline{X}_n$.

Thus, $\tilde{\underline{X}}_n$ satisfies $(3.a)_2^{\sim}$ if, and only if, its law is rotationnally invariant. Hence, we can write :

$$(3.f) \qquad \tilde{\underline{X}}_n = \rho \underline{U}_n ,$$

where ρ and $\underline{U}_n$ satisfy the properties stated in the Proposition.

Finally, since we have : $\tilde{\underline{X}}_n = \varepsilon \underline{X}_n$, we deduce from *(3.c)* that :

$$(3.e) \qquad \underline{X}_n = \varepsilon\rho\, \underline{U}_n . \qquad \square$$

In the case : $0 < \alpha < 2$, we have not been able to decide whether or not every n-dimensional variable $\underline{X}_n$ which satisfies $(3.a)_\alpha$ may be written in the form :

$$(3.e)_\alpha \qquad \underline{X}_n = \varepsilon\rho\, \underline{U}_n^{(\alpha)},$$

where ρ is ≥ 0, independent of $\underline{U}_n^{(\alpha)}$, a vector which is assumed to have a "universal" distribution depending only on n and α.

However, we are able to exhibit a number of examples of variables $\underline{X}_n$ which satisfy $(1.a)_\alpha$ [or, equivalently, $(3.a)_\alpha$].

In order to do this, it is of interest to introduce the class of variables $\underline{T}_n = (T_1,\ldots,T_n)$, all components of which take their values in $\mathbb{R}_+$, and which satisfy :

$$(3.a)^{+}_{\alpha/2} \qquad \sum_{i=1}^{n} a_i T_i \overset{(law)}{=} \left(\sum_{i=1}^{n} a_i^{\alpha/2}\right)^{2/\alpha} T_1 ,$$

for all $a_i \geq 0$, $1 \leq i \leq n$.

We can now state and prove the following

Proposition 3 : *Consider two independent vectors* $\underline{\xi}_n = (\xi_1,\ldots,\xi_n)$ *and*

$\underline{T}_n = (T_1,\dots,T_n)$ *which satisfy respectively* $(3.a)_2$ *and* $(3.a)^+_{\alpha/2}$. *Then, the random vector :*

$$\underline{X}_n = (\sqrt{T_j}\,\xi_j \ ;\ 1 \le j \le n)$$

satisfies $(3.a)_\alpha$.

<u>Proof</u> : We remark that, by conditioning first with respect to $\underline{T}_n$, we have for all $a_j \in \mathbb{R}$, $1 \le j \le n$:

$$\Big|\sum_{j=1}^{n} a_j \sqrt{T_j}\xi_j\Big| \overset{(\text{law})}{=} \Big(\sum_{j=1}^{n} a_j^2\, T_j\Big)^{1/2} |\xi_1| \qquad (\text{since } \underline{\xi}_n \text{ satisfies } (3.a)_2)$$

$$\overset{(\text{law})}{=} \Big(\sum_{j=1}^{n} |a_j|^\alpha\Big)^{2/\alpha} T_1^{1/2}\,|\xi_1| \quad (\text{since } \underline{T}_n \text{ satisfies } (3.a)^+_{\alpha/2})$$

$$\overset{(\text{law})}{=} \Big(\sum_{j=1}^{n} |a_j|^\alpha\Big)^{2/\alpha} |X_1|.$$

Hence, $\underline{X}_n$ satisfies $(3.a)_\alpha$. □

In fact, the same arguments allow us to obtain the following generalization of Proposition 3.

<u>Proposition 3'</u> : *Let* $0 < \alpha \le \gamma \le 2$, *and consider two independent vectors* $\underline{\xi}_n = (\xi_1,\dots,\xi_n)$ *and* $\underline{T}_n = (T_1,\dots,T_n)$ *which satisfy respectively* $(3.a)_\gamma$ *and* $(3.a)^+_{\alpha/\gamma}$.

Then, the random vector $\underline{X}_n = (T_j^{1/\gamma}\xi_j \ ;\ 1 \le j \le n)$ *satisfies* $(3.a)_\alpha$.

In order to obtain a better understanding of the class of vectors $\underline{X}_n$ which satisfy either $(3.a)_\alpha$, for $0 < \alpha \le 2$, or $(3.a)^+_\alpha$, for $0 < \alpha \le 1$, we find it interesting to introduce the following

Definition : *An* $\mathbb{R}_+$*-valued random variable* ρ *is called a simplifiable r.v. if the identity in law :*

$$\rho X \overset{(law)}{=} \rho Y,$$

where X, *resp :* Y, *is an* $\mathbb{R}_+$ *valued random variable which is assumed to be independent of* ρ, *implies :* $X \overset{(law)}{=} Y$.

The interest of this definition in our study shows up in the following

Lemma 2 : *1) If* $\underline{T}_n = \rho\, \underline{S}_n$ *satisfies* $(3.a)^+_\alpha$ *, for some* $\alpha \le 1$, *and if* ρ *is a simplifiable random variable which is independent of* $\underline{S}_n$ *, then* $\underline{S}_n$ *satisfies* $(3.a)^+_\alpha$ *;*

2) A similiar statement holds with $\underline{X}_n = \rho\, \underline{Y}_n$ *which is assumed to satisfy* $(3.a)_\alpha$ *, for some* $\alpha \le 2$.

The proof of this lemma is obvious from the definition of a simplifiable variable, and the properties $(3.a)^+_\alpha$ and $(3.a)_\alpha$.

As an application, we remark that, if $\underline{U}_n$ and $\underline{U}'_n$ are two independent n-dimensional random variables which are uniformly distributed on the unit sphere S_{n-1} then :

$$(3.g) \qquad U_n/\underline{U}'_n \equiv \left(\frac{U_i}{U'_i}\ ;\ 1 \le i \le n\right) \qquad \text{satisfies } (3.a)_1\ ,$$

$$\text{and} \qquad 1/(U'_n)^2 \equiv \left(\frac{1}{(U'_i)^2}\ ;\ 1 \le i \le n\right) \qquad \text{satisfies } (3.a)_{1/2}\ .$$

The property *(3.g)* may be proven as follows :

if $\underline{G}_n = (G_1,\dots,G_n)$ and $\underline{G}'_n = (G'_1,\dots,G'_n)$ are two independent n-dimensional centered Gaussian vectors, each component of which has variance 1, then $|\underline{G}_n|, |\underline{G}'_n|$, $\underline{U}_n \equiv \underline{G}_n/|\underline{G}_n|$, $\underline{U}'_n \equiv \underline{G}'_n/|\underline{G}'_n|$ are independent, and $|\underline{G}_n|$, hence : $|\underline{G}'_n|$, is simplifiable ; likewise, the second assertion in *(3.g)* is proven by remarking that :

$$\underline{T}_n \equiv \left(\frac{1}{G_i^2} \; ; \; 1 \le i \le n\right)$$

is an n-dimensional vector which consists of independent one-sided stable $(\frac{1}{2})$ random variables, hence $\underline{T}_n$ satisfies $(3.a)^{+}_{1/2}$.

Consequently, since $\underline{T}_n = \frac{1}{|G_n|^2}\left(\frac{1}{(\underline{U}_n)^2}\right)$, and $\frac{1}{|\underline{G}_n|^2}$ is simplifiable, then $\frac{1}{(\underline{U}_n)^2}$ satisfies $(3.a)_{1/2}$.

In the preceding discussion, we asserted that certain random variables are simplifiable ; these assertions are justified by the

Lemma 3 : *1) If* ρ *is a simplifiable random variable, then :*

(i) $P(\rho > 0) = 1$ *; (ii) for any* $m \in \mathbb{R}$, ρ^m *is simplifiable.*

2) A gamma distributed random variable is simplifiable.

3) A strictly positive random variable ρ *is simplifiable if, and only if, the characteristic function of* $(\log \rho)$ *does not vanish on any interval of* $\mathbb{R}$.

Proof : The proof of this lemma is elementary ; hence, we leave it to reader.

Now, we can state and prove the following converse of Proposition 3.

Proposition 4 : *Consider two independent vectors* $\underline{\xi}_n = (\xi_1,\dots,\xi_n)$ *and* $\underline{T}_n = (T_1,\dots,T_n)$ *such that :*

(i) $\underline{\xi}_n$ *satisfies* $(3.a)_2$ *, and (ii)* $\underline{X}_n \equiv (\sqrt{T_j}\,\xi_j\ ;\ 1 \le j \le n)$ *satisfies* $(3.a)_\alpha$.

Then, if moreover $|\xi_1|$ *is a simplifiable variable, the sequence* $\underline{T}_n$ *satisfies* $(3.a)^+_{\alpha/2}$.

Proof : From our hypothesis on $\underline{X}_n$, we have, for any $(a_j)_{j\le n} \in \mathbb{R}^n$:

$$(3.h)\qquad \Big|\sum_{j=1}^{n} a_j \sqrt{T_j}\,\xi_j\Big| \overset{(law)}{=} \left(\sum_{j=1}^{n} |a_j|^\alpha\right)^{2/\alpha} \sqrt{T_1}\,|\xi_1|.$$

From our hypothesis on $\underline{\xi}_n$, the left-hand side of *(3.h)* is equal in law to :

$$\left(\sum_{j=1}^{n} a_j^2\, T_j\right)^{1/2} |\xi_j|.$$

Hence, we have :

$$(3.h')\qquad \left(\sum_{j=1}^{n} a_j^2\, T_j\right)^{1/2} |\xi_1| \overset{(law)}{=} \left(\sum_{j=1}^{n} |a_j|^\alpha\right)^{2/\alpha} \sqrt{T_1}\ |\xi_1|$$

which, since $|\xi_1|$ is a simplifiable variable, implies that $\underline{T}_n$ satisfies $(3.a)^+_{\alpha/2}$. □

4. The general infinite dimensional study.

The aim of this section is to bridge the gap which exists between section 2, where we consider a sequence $X_1,\dots,X_n,\dots$ of i.i.d. random variables, and section 3 where we consider a finite dimensional sequence $X_1,\dots,X_n$, for which we make no a priori independence, nor distributional identity property assumption.

In this section, we consider an infinite sequence $\underline{X}_\infty = (X_1,\dots,X_n,\dots)$ such that for any $n \in \mathbb{N}^*$, the finite sequence $\underset{-n}{X} = (X_1,\dots,X_n)$ satisfies $(1.a)_\alpha$, for some α, with $0 < \alpha \le 2$.

Thanks to the infinite dimensionality of the sequence $\underset{-\infty}{X}$, we obtain a characterization result which completes Theorem 2.

Theorem 3 : *The following properties are equivalent :*

1) for any $n \in \mathbb{N}^*$, $\underset{-n}{X}$ *satisfies :*

$$(1.a)_\alpha \qquad \ell_\alpha(A\underset{-n}{X}) \overset{(law)}{=} \ell_\alpha(A^*\underset{-n}{X}), \text{ for every matrix } A,$$

2) for any $n \in \mathbb{N}^*$, *and* $(a_i)_{i\le n} \in \mathbb{R}^n$, $\underset{-n}{X}$ *satisfies :*

$$(3.a)_\alpha \qquad \left|\sum_{i=1}^{n} a_i X_i\right| \overset{(law)}{=} \left(\sum_{i=1}^{n} |a_i|^\alpha\right)^{1/\alpha} |X_1| ,$$

3) there exist ε, H *and* $\underset{-\infty}{C}^{(\alpha)} \equiv (C_n^{(\alpha)}\ ;\ n \in \mathbb{N})$ *such that*

$$(4.a) \qquad \underset{-\infty}{X} \overset{(law)}{=} (\varepsilon H\, C_n^{(\alpha)}\ ;\ n \in \mathbb{N})$$

where ε *takes the values* ± 1, H *is an* $\mathbb{R}_+$*-valued random variable,* $\underset{-\infty}{C}^{(\alpha)}$ *is a sequence* $(C_n^{(\alpha)}, n \in \mathbb{N})$ *of independent symmetric standard stable* (α) *random variables, and* H *and* $\underset{-\infty}{C}^{(\alpha)}$ *are independent (but no distributional relationship is assumed about* ε *with respect to the pair* $(H, C_\infty^{(\alpha)})$*).*

Proof : a) Proposition 1 ensures the equivalence between properties 1) and 2) ; moreover, since $\underset{-n}{C}^{(\alpha)}$ satisfies $(3.a)_\alpha$ for any n, it is immediate that, if $\underset{-\infty}{X}$ satisfies *(4.a)*, then it satisfies $(3.a)_\alpha$ for any $n \in \mathbb{N}$. Hence, it remains to show that 2) implies 3).

b) We first introduce (if necessary on an enlarged probability space) a symmetric Bernoulli random variable ε which is assumed to be independent of $\underset{-\infty}{X}$. Call $\underset{-\infty}{\tilde{X}} = \varepsilon \underset{-\infty}{X} \equiv (\varepsilon X_n\ ;\ n \in \mathbb{N})$. Then, we deduce from 2) that $\underset{-n}{\tilde{X}}$

satisfies :

$$(3.a)^{\sim}_n \qquad \sum_{i=1}^{n} a_i\tilde{X}_i \overset{(law)}{=} \ell_\alpha(a)\tilde{X}_1 \ ,$$

for any $a = (a_1,\dots,a_n) \in \mathbb{R}^n$.

In particular, the sequence $\tilde{X}_{\underline{\infty}}$ is exchangeable. Consequently, from Neveu ([5], Exercice IV.5.2, p. 137), or Chow-Teicher ([2], Theorem 2, section 7.2), there exists a sub σ-field $\mathcal{G}$ such that, conditionnally on $\mathcal{G}$, the variables $(\tilde{X}_n \ ; n \in \mathbb{N})$ are i.i.d ; here, we may take for $\mathcal{G}$ **the σ-field of** symmetrical events in $\sigma\{\tilde{X}_{\underline{\infty}}\}$, or the asymptotic σ-field $\bigcap_n \sigma\{\tilde{X}_m \ ; m \geq n\}$.

We now show that the conditional distribution of $\tilde{X}_1$ given $\mathcal{G}$ is the symmetric stable law of index α.

Indeed, from $(3.a)^{\sim}_n$, we have :

$$(4.d) \qquad \Phi_n(a_1,\dots,a_n) \overset{def}{=} E[\exp i \sum_{j=1}^{n} a_j\tilde{X}_j] = \Phi_1(\ell_\alpha(a)).$$

Now, if we denote by $\phi_\omega(a) = E[\exp(ia\ \tilde{X}_1)|\mathcal{G}](\omega)$ the characteristic function of $\tilde{X}_1$ given $\mathcal{G}$, then Bretagnolle, Dacunha-Castelle and Krivine ([1], p. 234-235) show that, as a consequence of *(4.b)*, one has :

$$\phi_\omega(a) = \exp((-K(\omega)|a|^\alpha),$$

for some $\mathcal{G}$-measurable $\mathbb{R}_+$-valued r.v. K.

The proposition *(4.a)* now follows easily. □

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