Symmetric Stable Processes and Fubini's Theorem

By

C. Donati-Martin[1], S. Song[2] and M. Yor[3]

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[1]Université de Provence - U.R.A. 225 - 3, place Victor Hugo
13331 Marseille cedex 3

[2]Institut Für Angewandte Mathematik - Im Neuenheimer Feld 294
D - 6900 Heidelberg - Germany

[3]Université P. et M. Curie - Laboratoire de Probabilités
Tour 56 3éme Etage - 4, Place Jussieu - 75252 Paris cedex 05

Department of Statistics
University of California
Berkeley, California 94720

Abstract:

In the first half of this paper, a Fubini type identity in law - which was previously developed by two of the authors - between quadratic functionals of Brownian motion is extended in two directions:

- an analogue of this identity in law holds when Brownian motion is replaced by a symmetric stable process of any order $\alpha \in (0,2)$, provided the function: $x \to x^2$ is replaced by: $x \to |x|^\alpha$;

- such Fubini type identities in law yield, as a particular case, an identity in law which resembles the integration by parts formula; as a consequence, some extensions of the Ciesielski-Taylor identities in law are obtained.

The second half of the paper is devoted to showing that such Fubini type identities in law "nearly" characterize the symmetric stable processes.

A characterization result of lesser scope is obtained for two particular classes of processes which satisfy the integration by parts identity in law: the class of Gaussian processes on one hand, and the class of squares of Gaussian processes on the other hand.

Key Words: Fubini's theorem, integration by parts, stable processes, Ray-Knight theorems, Ciesielski-Taylor identities, diffusions.

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C. Donati-Martin(1) , *S. Song* (2) *and M. Yor*(3)

(1) Université de Provence - U.R.A. 225 - 3, Place Victor Hugo
13331 Marseille Cédex 3.

(2) Institut für Angewandte Mathematik - Im Neuenheimer Feld 294
D - 6900 Heidelberg - Germany.

(3) Université P. et M. Curie - Laboratoire de Probabilités - Tour 56 -
3ème Etage - 4, place Jussieu - 75252 PARIS CEDEX 05.

0. Introduction.

(0.1) We first recall the following identities in law :

(0.a)
$$\int_0^1 ds \left(B_s - \int_0^1 dt\, B_t\right)^2 \overset{(law)}{=} \int_0^1 ds\, \tilde{B}_s^2$$

where B is a real valued Brownian motion, and $\tilde{B}$ is a standard Brownian Bridge, indexed by $[0,1]$,

and :

(0.b)
$$\int_0^\infty ds\, 1_{(R_{d+2}(s)\le 1)} \overset{(law)}{=} T_1(R_d),$$

where R_d (resp. R_{d+2}) is a Bessel process of dimension d (resp. $(d+2)$) starting from 0, and $T_1(R_d)$ is the first hitting time of 1 by R_d.

The identity in law *(0.a)* has been obtained independently by Chiang - Chow - Lee [4] and Chan - Jansons - Rogers [3], using the - by now, classical -

diagonalization procedure due to Paul Lévy [11] to compute Laplace, or Fourier, transforms of quadratic functionals of Brownian motion.

The identity in law *(0.b)* has been obtained by Ciesielski and Taylor [5] for integer dimensions, and then extended to any dimension $d > 0$ by Getoor and Sharpe [9]. Biane [1] subsequently obtained a wide class of identities in law which extend *(0.b)* to suitably related functionals of pairs of diffusions, which satisfy a certain duality assumption. However, in all cases, the proofs of such results by these authors rely exclusively upon the computation of Laplace transforms of the two functionals.

(0.2) In [6] and [13], the authors have obtained the identities in law *(0.a)* and *(0.b)* as a consequence of the following Fubini **type theorem** : if B and C are two independent real-valued Brownian motions, then :

$$(0.c) \qquad \int_0^1 dB_u \int_0^1 dC_s\ \varphi(u,s) \overset{a.s.}{=} \int_0^1 dC_s \int_0^1 dB_u\ \varphi(u,s),$$

for any $\varphi \in L^2([0,1]^2, du\ ds)$,

or rather as a consequence of its corollary :

$$(0.d) \qquad \int_0^1 du \left(\int_0^1 dB_s\ \varphi(u,s)\right)^2 \overset{(law)}{=} \int_0^1 du \left(\int_0^1 dB_s\ \varphi(s,u)\right)^2 .$$

(0.3) The aim of this paper is to prove some extensions of this Fubini theorem, and to present in a unified manner a large class of identities such as *(0.a)* and *(0.b)*.

We now describe the organization of the paper.

In the first section, we present an identity in law which resembles the classical integration by parts formula, from which we then deduce, with the help of the Ray - Knight theorems for Bessel local times (see [13]) some interesting extensions of the Ciesielski - Taylor identities *(0.b)*.

In the second section, we extend Fubini's theorem in several directions : we first replace in *(0.c)* the independent Brownian motions B and C respectively by two symmetric stable process $C^{(\alpha)}$, and $C^{(\beta)}$, with respective indexes α et β, which allows us to obtain some interesting extensions of *(0.d)*, relating the laws of certain integrals of $C^{(\alpha)}$ on one hand, and $C^{(\beta)}$ on the other hand ; in a second direction, we show that it is possible to polarize the identity in law *(0.d)* with two independent Brownian motions B and C, which, as a consequence, allows us to extend the identity in law *(0.d)* into an infinite dimensional identity in law between two sequences of random variables which are associated to $\varphi(s,u)$ on one hand, and to $\varphi(u,s)$ on the other hand.

In section 3, we are interested in the converse of the property *(0.d)* for the Brownian motion (or the identity in law *(2.d)* for stable processes ; see section 2) ; namely, we want to characterize the processes $(X_t, 0 \le t \le 1)$ which satisfy the identity in law :

$$(0.e) \qquad \int_0^1 du \, \left| \int_0^1 dX_s \, \varphi(u,s) \right|^\alpha \overset{(law)}{=} \int_0^1 du \, \left| \int_0^1 dX_s \, \varphi(s,u) \right|^\alpha$$

for any bounded simple function $\varphi : [0,1]^2 \longrightarrow \mathbb{R}$, and $0 < \alpha \le 2$.
We show that X_t is "nearly" a symmetric stable process with index α (cf. Theorems 4 and 5).
We also prove a precise characterization of a stable symmetric process combining a Fubini-type identity in law in the discrete case ($(1.a)_\alpha$ in [14]) and in the continuous case : this is the aim of Theorem 6.

The last section is devoted to studying the processes $(X_t, t \ge 0)$ which satisfy the following identity in law, which resembles an integration by part formula :

$$(0.f) \qquad -\int_a^b dx \, f'(x) \, X_{g(x)} + f(b) \, X_{g(b)} \overset{(law)}{=} g(a) \, X_{f(a)} + \int_a^b dx \, g'(x) \, X_{f(x)}$$

for any functions $f,g : [a,b] \longrightarrow \mathbb{R}^+$ two C^1-functions with f decreasing and g increasing.

In section 1, we proved that $X_t = |C_t^{(\alpha)}|^\alpha$ satisfies *(0.f)* where $C_\cdot^{(\alpha)}$ is a stable symmetric process with index α.

We obtain a complete description of processes satisfying *(0.f)* when we restrict our study to two classes : the first one is the class of Gaussian processes, the second one is the class of square of Gaussian processes.

To conclude this introduction, we now explain how this paper relates to its companion [14] : in both papers, we try to gain an insight into the identity *(0.c)* and more precisely, we are interested in converse problems i.e. to characterize a Brownian motion (or a stable process) via some identities in law like the Fubini-type identity *(0.c)*.

In [14], we restrict our attention to the case of a sequence of random variables X which satisfy a discrete Fubini type identity, namely :

$$\|A\underline{X}_n\|_\alpha \overset{(law)}{=} \|A^*\underline{X}_n\|_\alpha$$

where $\underline{X}_n$ is a n sample of X, A is a $n \times n$ matrix, A^* is the transpose of A and $\|x\|_\alpha = (\Sigma|x_i|^\alpha)^{1/\alpha}$ ($0 < \alpha \le 2$) for $x \in \mathbb{R}^n$. In this paper, we consider continuous processes and we discuss identities in law such as *(0.d)* (or *(2.d)*) and *(1.d)*.

1. Fubini's theorem and some identities in law :

(1.1) We give some applications of the identity in law *(0.d)*, whose proof follows immediately from Fubini's theorem *(0.c)*. First, we recall, following [6], how *(0.a)* is obtained as a corollary of *(0.d)*.

Proposition 1 : *Let* $f : [0,1] \longrightarrow \mathbb{R}$ *be a* C^1 *function such that* $f(1) = 1$. *Then, we have :*

$$\text{(1.a)} \qquad \int_0^1 ds\left(B_s - \int_0^1 dt\ f'(t)B_t\right)^2 \overset{(law)}{=} \int_0^1 ds(B_s - f(s)B_1)^2$$

and

$$(1.a') \qquad \int_0^1 ds\left(B_s - \int_0^1 dt\ B_t\right)^2 \overset{(law)}{=} \int_0^1 ds\ \tilde{B}_s^2 \ ,$$

where $\tilde{B}$ *is a standard Brownian bridge.*

Proof : The identity in law *(1.a)* follows from *(0.d)*, where we take :

$$\varphi(s,u) = (1_{(u\le s)} - (f(1) - f(u)))\ 1_{((s,u)\in[0,1]^2)} .$$

The identity in law *(1.a')* is a particular case of *(1.a)*, with $f(u) = u$. □

As a second application of *(0.d)*, or rather of a discrete version of *(0.d)*, we now prove a striking identity in law *(1.b)* which resembles the integration by parts formula.

Theorem 1 : *Let* $(C_t^{(\alpha)}, t \ge 0)$ *be a symmetric stable process with index* α $(0< \alpha \le 2)$*, starting from* 0.

Let $0 \le a \le b < \infty$*, and* $f,g : [a,b] \longrightarrow \mathbb{R}_+$ *be two right-continuous functions, with* f *decreasing and* g *increasing ; then, we have :*

$$(1.b) \qquad \int_{]a,b]} df(x)\,|C^{(\alpha)}_{g(x_-)}|^\alpha + f(b)\,|C^{(\alpha)}_{g(b)}|^\alpha$$

$$\overset{(law)}{=} g(a)\,|C^{(\alpha)}_{f(a)}|^\alpha + \int_{]a,b]} dg(x)\,|C^{(\alpha)}_{f(x)}|^\alpha$$

and

$$(1.b') \qquad \int_{]a,b]} - df(x)\,|C^{(\alpha)}_{g(x)}|^\alpha + f(b)\,|C^{(\alpha)}_{g(b)}|^\alpha$$

$$\overset{(law)}{=} g(a)\,|C^{(\alpha)}_{f(a)}|^\alpha + \int_{]a,b]} dg(x)\,|C^{(\alpha)}_{f(x_-)}|^\alpha .$$

First, we recall the discrete version of the Fubini-type identity in law *(0.d)*

Theorem ([14]) : *Let* $0 < \alpha \le 2$ *and* $\underline{C}_n^{(\alpha)} = (C_1^{(\alpha)},\dots,C_n^{(\alpha)})$ *be a* n*-dimensional random vector whose components are* n *independents standard, symmetric, stable variables with index* α.

Then, for any $n \times n$ *real matrix* A, *we have :*

$$\|A\underline{C}_n^{(\alpha)}\|_\alpha \overset{(law)}{=} \|A^* \underline{C}_n^{(\alpha)}\|_\alpha \tag{1.c}$$

where A^* *is the transpose of* A, *and*

$$\|x\|_\alpha = \left(\sum_{i=1}^n |x_i|^\alpha\right)^{1/\alpha} \qquad \text{for} \quad x \in \mathbb{R}^n.$$

Proof of Theorem 1 : a) First, let us state a corollary of *(1.c)*.

Corollary 1 : *Let* $(Y_1,\dots,Y_n)$ *and* $(Z_1,\dots,Z_n)$ *be two symmetric stable* n*-dimensional vectors with index* α $\left(\textit{i.e.}\ \sum_{i=1}^n a_k Y_k \text{ is a symmetric stable } (\alpha) \text{ r.v. for all } (a_k) \in \mathbb{R}^n\right)$ *such that :*

i) $Y_1, Y_2-Y_1,\dots,Y_n-Y_{n-1}$ *are independent,*

ii) $Z_n, Z_n-Z_{n-1},\dots,Z_2-Z_1$ *are independent.*

Define the increasing sequence $(\delta_k, 0 \le k \le n)$ *by :*

$$\delta_o = 0 \ ;\ E[\exp(i\lambda Y_k)] = \exp(-\delta_k|\lambda|^\alpha) \qquad \forall\lambda \in \mathbb{R},\ \ 1 \le k \le n,$$

and the decreasing sequence $(\gamma_k, 1 \le k \le n+1)$ *by :*

$$E[\exp(i\lambda Z_k)] = \exp(-\gamma_k|\lambda|^\alpha), \qquad 1 \le k \le n \ ;\ \gamma_{n+1} = 0.$$

Then, we have :

$$(1.d)\qquad -\sum_{i=1}^{n} (\gamma_{i+1}-\gamma_i)|Y_i|^\alpha \overset{(law)}{=} \sum_{i=1}^{n} (\delta_i-\delta_{i-1})|Z_i|^\alpha.$$

<u>Proof</u> : We apply *(1.c)* to the matrix $A = (a_{ij}, 1 \le i \le j \le n)$ with $a_{ij} = \alpha_i \beta_j 1_{j\le i}$; we then obtain :

$$\sum_{i=1}^{n} |\alpha_i|^\alpha \, \Big|\sum_{j\le i} \beta_j C_j^{(\alpha)}\Big|^\alpha \overset{(law)}{=} \sum_{i=1}^{n} |\beta_i|^\alpha \, \Big|\sum_{j\ge i} \alpha_j C_j^{(\alpha)}\Big|^\alpha$$

where $C_1^{(\alpha)},\dots,C_n^{(\alpha)}$ are n independent, symmetric standard stable r.v.'s with index α.

Now, we remark that the two sequences

$$Y_i = \sum_{j\le i} \beta_j C_j^{(\alpha)} \quad \text{and} \quad Z_i = \sum_{j\ge i} \alpha_j C_j^{(\alpha)}, \ 1 \le i \le n,$$

satisfy the independence properties stated in the Corollary.

Finally, it remains to define $\delta_i = \sum_{j\le i} |\beta_j|^\alpha$ and $\gamma_i = \sum_{j\ge i} |\alpha_j|^\alpha$ to obtain the identity *(1.d)*. □

b) i) We apply *(1.d)* with

$$Y_i = C_{g(t_i)}^{(\alpha)}, \ Z_i = C_{f(t_i)}^{(\alpha)}, \ \gamma_i = f(t_i), \ \delta_i = g(t_i), \ 1 \le i \le n.$$

It follows :

$$-\sum_{i=1}^{n} (f(t_{i+1}) - f(t_i))|C_{g(t_i)}^{(\alpha)}|^\alpha + f(t_n)|C_{g(t_n)}^{(\alpha)}|^\alpha$$

$$\overset{(law)}{=} g(t_1)|C_{g(t_1)}^{(\alpha)}|^\alpha + \sum_{i=2}^{n} (g(t_i) - g(t_{i-1}))|C_{f(t_i)}^{(\alpha)}|^\alpha.$$

When the mesh of the subdivision of $[a,b]$ tends to zero, we obtain

$$\int_{]a,b]} - df(x)\,|C^{(\alpha)}_{g(x_-)-}|^{\alpha} + f(b)\,|C^{(\alpha)}_{g(b)}|^{\alpha}$$

(1.e)

$$\overset{(law)}{=} g(a)\,|C^{(\alpha)}_{f(a)}|^{\alpha}\ g(a)\ |C^{(\alpha)}_{f(a)}|^{\alpha} + \int_{]a,b]} dg(x)\,|C^{(\alpha)}_{g(x)_-}|^{\alpha}$$

where $C^{(\alpha)}_{t-} = \lim_{s\uparrow t} C^{(\alpha)}_{s}$

(we recall that the process $(C^{(\alpha)}_{t}, t \geq 0)$ *is* **right-continuous and has** left limits).

Now, we replace the two functions f and g by $f+\varepsilon$ and $g+\varepsilon$ $(\varepsilon > 0)$ in *(1.e)*, and letting $\varepsilon \searrow 0$, we obtain *(1.b)*.

ii) We apply *(1.d)* with

$$Y_i = C^{(\alpha)}_{g(t_{i+1})}\ ,\ Z_i = C^{(\alpha)}_{g(t_i)}\ ,\ \gamma_i = f(t_i)\ ,\ \delta_i = g(t_{i+1}),\ 1 \leq i \leq n.$$

It follows :

$$- \sum_{i=1}^{n-1} (f(t_{i+1}) - f(t_i))\,|C^{(\alpha)}_{g(t_{i+1})}|^{\alpha} + f(t_n)\,|C^{(\alpha)}_{g(t_{n+1})}|^{\alpha}$$

$$\overset{(law)}{=} g(t_2\,|C^{(\alpha)}_{f(t_1)}|^{\alpha} + \sum_{i=2}^{n} (g(t_{i+1}) - g(t_i))\,|C^{(\alpha)}_{f(t_i)}|^{\alpha}$$

with the subdivision $\tau_n = (a = t_1 < t_2 <\ldots< t_n = b < t_{n+1})$.

As in i), when the mesh of τ_n tends to zero, we obtain *(1.b')*. $\square$

(1.2) We now show how the Ciesielski - Taylor identities *(0.b)*, or rather a family of extensions of those idnetities, are deduced from the identity in law *(1.b)*, in the particular case $\alpha = 2$.

We consider a transient diffusion on $(0,\infty)$, whose infinitesimal generator is given by

$$\Gamma = \frac{1}{2}\frac{d}{dm}\frac{d}{ds}\,, \quad \text{with} \quad s(0) = -\infty,\ s(\infty) = 0. \tag{1.f}$$

s, resp. m, is a scale function, resp. speed the measure of X. We consider the particular case where

$$\Gamma = \frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}\,. \tag{1.g}$$

By developing *(1.f)*, it follows that *(1.g)* is satisfied if

$$m'(x)\, s'(x) = 1 \quad \text{and} \quad b(x) = -\frac{1}{2}\frac{s''(x)}{s(x)}\,. \tag{1.h}$$

Following Biane [1], we introduce two functions v and n defined by

$$v = -\frac{1}{s} \quad \text{and} \quad dn = s^2\, dm\ ; \tag{1.i}$$

then, b can be written as :

$$b(x) = \frac{v'(x)}{v(x)} - \frac{1}{2}\frac{v''(x)}{v'(x)}\,. \tag{1.i'}$$

Then, we have the following representation of X : there exists a 3-dimensional Bessel process R_3 such that :

$$v(X_t) = R_3\left(\int_0^t ds\ v'^2(X_s)\right). \tag{1.j}$$

$(L_t^a(X), a > 0, t \geq 0)$ denotes the family of local times of X, which we define via the density of occupation formula : for any Borel function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$\int_0^t f(X_s)ds = \int_0^\infty f(a)\ L_t^a(X)da.$$

Using the representation *(1.j)*, we can extend the version presented in [13] of the Ray - Knight theorem in the following

Proposition 2 : *Let* $c > 0$, v *defined by (1.i) and*

$$k_c(x) = \frac{1}{v(x)} - \frac{1}{v(c)}\ , \quad 0 \le x \le c\ ;$$

then, we have

1) $(L^a_\infty(X),\ a \ge 0) \overset{(law)}{=} \left(\frac{1}{v'(a)}\ |B_{v(a)}|^2\ ,\ a \ge 0\right)$

2) $(L^a_{T_c}(X),\ 0 \le a \le c) \overset{(law)}{=} \left(\frac{1}{|k'_c(a)|}\ |B_{k_c(a)}|^2\ ,\ 0 \le a \le c\right).$

The proof of 1) follows easily from the identity in law for $(L^a_\infty(R^3), a \ge 0)$ given in [13], and the representation *(1.j)*.

For 2), we use *(1.j)*, the identity in law for $(L^a_{T_1}(R^3), a \le 1)$ given in [13], in terms of a Brownian Bridge $\tilde{B}$ and the classical identity

$$(\tilde{B}_v, v \in [0,1]) \overset{(law)}{=} \left(v\ B_{(\frac{1}{v}-1)}\ ,\ v \in [0,1]\right). \qquad \square$$

In [1], Biane associates to pair (v,n) defined by *(1.i)* a new diffusion $\hat{X}$, with infinitesimal generator $\hat{\Gamma} = \frac{1}{2}\frac{d}{dv}\frac{d}{dn} = \frac{1}{2}\frac{d^2}{dx^2} + \hat{b}(x)\frac{d}{dx}$.

Thanks to the formula *(1.i)*, we have

$$(1.k) \qquad v'(x)\ n'(x) = 1 \quad \text{and} \quad \hat{b}(x) = -\frac{1}{2}\frac{n''(x)}{n'(x)}\ .$$

As in *(1.i')*, we can write $\hat{b}$ as $\hat{b}(x) = \frac{\hat{v}'(x)}{\hat{v}(x)} - \frac{1}{2}\frac{\hat{v}''(x)}{\hat{v}'(x)}$ with $\hat{v}(x) = \frac{-1}{n(x)}$

where n is any function whose derivative is $n'(x) = \frac{1}{v'(x)}$.

(For $\hat{\Gamma}$, n' has the same rôle as s' for Γ).

We are now ready to prove the following extension of the Ciesielski - Taylor identities in law.

Theorem 2 : *Let* $0 \le a \le b \le c$ *and* $X, \hat{X}$ *defined as above. We then have :*

$$\int_0^\infty ds\, 1_{a \le X_s \le b} + (n(c) - n(b))\, v'(b)\, L_\infty^b(X) \overset{(law)}{=} v(a) n'(a)\, L^a_{T_c}(\hat{X}) \tag{1.ℓ}$$

$$+ \int_0^{T_c(\hat{X})} 1_{a \le \hat{X}_s \le b}\, ds.$$

Proof : We first remark that from Proposition 2, the identity in law *(1.ℓ)* is equivalnt to

$$\int_a^b \frac{B^2_{v(x)}}{v'x)} dx + (n(c) - n(b))\, B^2_{v(b)} \tag{1.m}$$

$$\overset{(law)}{=} v(a) n'(a) |\hat{k}'_c(a)|^{-1} B^2_{\hat{k}_c(a)} + \int_a^b \frac{B^2_{\hat{k}_c(x)}}{|\hat{k}'_c(x)|} dx$$

where

$$\hat{k}_c(x) = \frac{1}{\hat{v}(x)} - \frac{1}{\hat{v}(c)} = n(c) - n(x) \tag{1.n}$$

(here, we have taken B to be a real valued Brownian motion).

Now, *(1.m)* is a particular case of the identity *(1.b)* with $g(x) = v(x)$, $f(x) = \hat{k}_c(x)$, since, from *(1.n)* and *(1.k)*, we have : that $|f'(x)| = n'(x) = \frac{1}{v'(x)}$. □

Corollary : *In the particular case where* X *is a* (d+2)*-dimensional Bessel process,* $v(x) = x^d$ and $\hat{X}$ *is a* d*-dimensional Bessel process, then (1.ℓ) takes the particular form*

$$\int_0^\infty ds\, 1_{a \le R_{d+2}(s) \le b} + \left(b^{d-1} \int_b^c \frac{dx}{x^{d-1}} \right) L_\infty^b(R_{d+2})$$

$$\stackrel{(law)}{=} \frac{a}{d} L_{T_c}^a(R_d) + \int_0^{T_c(R_d)} 1_{a \le R_d(s) \le b}\, ds$$

where $(R_d(t), t \ge 0)$ *denotes the* d*-dimensional Bessel process starting from* 0.

2. Some extensions of Fubini's theorem, and more applications.

(2.1) We first present a fairly general extension to symmetric stable variables of the identity in law $(1.a)_\alpha$ obtained in Theorem 1 in [14].

Proposition 3 : *Let* $0 \le \alpha \le \beta \le 2$. *Consider, for* $\theta = \alpha$, *or* β,

$\underline{X}_n^{(\theta)} = (X_1^{(\theta)}, \ldots, X_n^{(\theta)})$ *an* n*-dimensional random vector, whose*

components are independent, symmetric, stable variables, with exponent θ,

that is : $\quad E[\exp(i\lambda X_j^{(\theta)})] = \exp - |\lambda|^\theta \qquad (\lambda \in \mathbb{R}).$

Then, for any $n \times n$ *real matrix* A, *we have :*

$$\text{(2.a)} \qquad \| A \underline{X}_n^{(\beta)} \|_\alpha \, (T_\gamma)^{1/\beta} \stackrel{(law)}{=} \| A^* \underline{X}_n^{(\alpha)} \|_\beta \,,$$

where $\gamma = \alpha/_\beta$, T_γ *is a one-sided stable random variable with exponent* γ

(that is : $E[\exp(-\lambda T_\gamma)] = \exp(-\lambda^\gamma) \qquad (\lambda \ge 0)$),

which is assumed to be independent of $\underline{X}_n^{(\beta)}$ *, and* $\|\underline{x}\|_\theta = \left(\sum_{i=1}^{n} |x_i|^\theta\right)^{1/\theta}$.

In the particular case $\alpha = \beta$*, we have :* $T_\gamma = 1$ *a.s., hence the identity (2.a) takes the simpler form :*

(2.b) $$\|A\underline{X}_n^{(\alpha)}\|_\alpha \overset{(\text{law})}{=} \|A^*\underline{X}_n^{(\alpha)}\|_\alpha.$$

which is the identity $(1.a)_\alpha$ *obtained in* [14].

<u>Proof</u> : We may consider that $\underline{X}_n^{(\alpha)}$ and $\underline{X}_n^{(\beta)}$ are independent.

From the a.s. identity :

$$(A\underline{X}_n^{(\beta)}, \underline{X}_n^{(\alpha)}) = (\underline{X}_n^{(\beta)}, A^*\underline{X}_n^{(\alpha)}),$$

we deduce, by taking the characteristic functions of both sides that, for any $\lambda \in \mathbb{R}$:

$$E\left[\exp - |\lambda|^\beta (\|A^*\underline{X}_n^{(\alpha)}\|_\beta)^\beta\right] = E\left[\exp - |\lambda|^\alpha (\|A\underline{X}_n^{(\beta)}\|_\alpha)^\alpha\right].$$

Taking $\mu = |\lambda|^\alpha$ as a new variable, we obtain :

$$E\left[\exp - \mu^{1/\gamma}(\|A^*\underline{X}_n^{(\alpha)}\|_\beta)^\beta\right] = E\left[\exp - \mu(\|A\underline{X}_n^{(\beta)}\|_\alpha)^\alpha\right].$$

The right-hand side is equal to :

$$E\left[\exp - \left\{\mu^{1/\gamma}(\|A\underline{X}_n^{(\beta)}\|_\alpha)^\beta\right\}^\gamma\right] = E\left[\exp - \mu^{1/\gamma}(\|A\underline{X}_n^{(\beta)}\|_\alpha)^\beta\, T_\gamma\right]$$

Hence, we have obtained that :

$$(\|A\underline{X}_n^{(\beta)}\|_\alpha)^\beta\, T_\gamma \overset{(\text{law})}{=} (\|A^*\underline{X}_n^{(\alpha)}\|_\beta)^\beta,$$

which is equivalent to *(2.a)*. $\square$

(2.2) We now derive an infinite dimensional analogue of *(2.a)* and *(2.b)* ; to this end, we consider $(C_t^{(\alpha)}, t \geq 0)$ and $C_t^{(\beta)}, t \geq 0)$ two stable symmetric processes, with respective indexes α and β.

The a.s. identity *(0.c)* extends easily as follows :

$$\int_0^1 dC_u^{(\alpha)} \int_0^1 dC_s^{(\beta)} \, \varphi(u,s) = \int_0^1 dC_s^{(\beta)} \int_0^1 dC_u^{(\alpha)} \, \varphi(u,s),$$

for any bounded Borel function $\varphi : [0,1]^2 \longrightarrow \mathbb{R}$,

and arguments analogous to those we have just used lead us to the following

Proposition 4 : *Let* $0 < \alpha \leq \beta \leq 2$ *; define* $\gamma = \frac{\alpha}{\beta}$.

Let $X_{\beta,\alpha} = \int_0^1 du \, \left| \int_0^1 dC_s^{(\beta)} \, \varphi(u,s) \right|^\alpha$ *and* $\tilde{X}_{\alpha,\beta} = \int_0^1 du \left| \int_0^1 dC_s^{(\alpha)} \, \tilde{\varphi}(u,s) \right|^\beta$.

Then, we have :

$$\text{(2.c)} \qquad (X_{\beta,\alpha})^{1/\gamma} \, T_\gamma \overset{\text{(law)}}{=} \tilde{X}_{\alpha,\beta} \, ,$$

where T_γ *has the same meaning as in Proposition 3, and is assumed to be independent of* $X_{\beta,\alpha}$.

In the particular case where $\alpha = \beta$, *we obtain the identity in law :*

$$\text{(2.d)} \qquad \int_0^1 du \, \left| \int_0^1 dC_s^{(\alpha)} \, \varphi(u,s) \right|^\alpha \overset{\text{(law)}}{=} \int_0^1 du \, \left| \int_0^1 dC_s^{(\alpha)} \, \tilde{\varphi}(u,s) \right|^\alpha.$$

We now give some interesting application of the identity in law *(2.c)*.

Theorem 3 : *We use the same notation as in Proposition 4, and we write :*

$$G^{(\alpha)} = \int_0^1 du\; C_u^{(\alpha)} .$$

Then, we have :

$$\text{(2.e)} \qquad \left(\int_0^1 du\; |C_u^{(\beta)}|^{\alpha}\right)^{\beta/\alpha} T_\gamma \overset{(law)}{=} \int_0^1 du\; |C_u^{(\alpha)}|^{\beta}$$

$$\text{(2.f)} \qquad \left(\int_0^1 du\; |C_u^{(\beta)} - G^{(\beta)}|^{\alpha}\right)^{\beta/\alpha} T_\gamma \overset{(law)}{=} \int_0^1 du\; |C_u^{(\alpha)} - uC_1^{(\alpha)}|^{\beta}$$

$$\text{(2.g)} \qquad \left(\int_0^1 du\; |C_u^{(\beta)} - uC_1^{(\beta)}|^{\alpha}\right)^{\beta/\alpha} T_\gamma \overset{(law)}{=} \int_0^1 du\; |C_u^{(\alpha)} - G^{(\alpha)}|^{\beta}$$

In the particular case $\beta = 2$, *we obtain :*

$$\text{(2.g')} \qquad \left(\int_0^1 du\; |\tilde{B}_u|^{\alpha}\right)^{2/\alpha} T_{\alpha/2} \overset{(law)}{=} \int_0^1 du\; (C_u^{(\alpha)} - G^{(\alpha)})^2$$

where $\tilde{B}_u \equiv B_u - uB_1$, $u \le 1$, *is a standard Brownian bridge.*

(in the even more particular case $\alpha = \beta = 2$, *this is precisely (0.a)).*

Proof : 1) The identity in law *(2.e)* follows from *(2.c)*, when we take

$\varphi(u,s) = 1_{(s\le u)}$, together with the identity in law :

$$\text{(2.h)} \qquad (C_s^{(\alpha)}\, ;\, s \le 1) \overset{(law)}{=} (C_1^{(\alpha)} - C_{1-s}^{(\alpha)}\, ;\, s \le 1).$$

2) The identity in law *(2.f)* follows from *(2.c)*, when we take :

$\varphi(u,s) = 1_{(s\le u)} - (1-s)$, and we use again the identity in law *(2.h)*.

3) We then obtain *(2.g)* by exchanging the rôles played by φ and $\tilde{\varphi}$ to prove *(2.f)*. □

Theorem 3 may be used to interpret some Laplace transforms computations made for a particular symmetric stable process, say Brownian motion, or the Cauchy process for instance, in terms of the other symmetric stable processes.
As an example, we write the identities in law *(2.e)* for $\beta = 2$, and *(2.g')*, in terms of Laplace transforms ; we obtain :

$$(2.i)_\alpha \qquad E\left[\exp - \lambda^{2/\alpha} \int_0^1 du\, (C_u^{(\alpha)})^2\right] = E\left[\exp - \lambda \int_0^1 du\, |B_u|^\alpha\right]$$

and

$$(2.j)_\alpha \qquad E\left[\exp - \lambda^{2/\alpha} \int_0^1 du\, (C_u^{(\alpha)} - G^{(\alpha)})^2\right] = E\left[\exp - \lambda \int_0^1 du\, |\tilde{B}_u|^\alpha\right].$$

In particular, in the case $\alpha = 1$, we obtain the following identities, where we have simply written $(C_u, u \geq 0)$ for the standard symmetric Cauchy process,

and $G = \int_0^1 du\, C_u$:

$$(2.i)_1 \qquad E\left[\exp\left(- \lambda^2 \int_0^1 du\, C_u^2\right)\right] = E\left[\exp\left(- \lambda \int_0^1 du\, |B_u|\right)\right]$$

and

$$(2.j)_1 \qquad E\left[\exp\left(- \lambda^2 \int_0^1 du\, (C_u - G)^2\right)\right] = E\left[\exp\left(- \lambda \int_0^1 du\, |\tilde{B}_u|\right)\right]$$

Now, the right-hand sides of $(2.i)_1$ and $(2.j)_1$ have been computed respectively by Kac [10] (see also Erdös-Kac [7]) and by Shepp [12] (see also Biane-Yor [2], p. 75) ; here are the results of those computations

(2.k) $$E\left[\exp\left(-\lambda\int_0^1 du\ |B_u|\right)\right] = \sum_{i=1}^{\infty} c_i \exp(-\lambda^{1/3} k_i)$$

for certain constants c_i and k_i, related to the Bessel functions $J_{1/3}$ and $J_{-1/3}$, and

(2.ℓ) $$E\left[\exp\left(-\lambda|N|^3\int_0^1 du\ |\tilde{B}_u|\right)\right] = 2\left(\frac{K_{1/3}}{K_{2/3}}\right)\left(\frac{1}{3\lambda}\right),$$

where N is a reduced, centered, Gaussian variable, which is independent of $\tilde{B}$.

With the help of the identities *(2.i)*$_1$ and *(2.j)*$_1$, we now deduce from *(2.k)* and *(2.ℓ)* that :

(2.i') $$E\left[\exp\left(-\lambda\int_0^1 du\ C_u^2\right)\right] = \sum_{i=1}^{\infty} e_i \exp(-\lambda^{1/3} k_i)$$

and

(2.j') $$E\left[\exp\left(-\lambda^2 N^6\int_0^1 du\ (C_u-G)^2\right)\right] = 2\left(\frac{K_{1/3}}{K_{2/3}}\right)\left(\frac{1}{3\lambda}\right),$$

where N is a reduced, centered, Gaussian variable, which is independent of C.

(2.3) Polarization of the identity (0.d) with two independent Brownian motion.

In this section, we always represent the function $\varphi(s,u)$ by $\tilde{\varphi}(u,s)$, and $\tilde{B}$ and B' are two independent Brownian motions, starting from 0.

We first give a general result

Proposition 5 : *Let* $\varphi \in L^2([0,1]^2, du\ ds)$ *; we have the following identities in law :*

$$\int_0^1 du \left(\int_0^1 dB_s\, \varphi(u,s)\right)\left(\int_0^1 dB'_s\, \varphi(u,s)\right)$$

(2.m)

$$\overset{(law)}{=} \frac{1}{2}\int_0^1 du \left\{\left(\int_0^1 dB_s\, \varphi(u,s)\right)^2 - \left(\int_0^1 dB'_s\, \varphi(u,s)\right)^2\right\}$$

and

$$\int_0^1 du \left(\int_0^1 dB_s\, \varphi(u,s)\right)\left(\int_0^1 dB'_s\, \varphi(u,s)\right)$$

(2.n)

$$\overset{(law)}{=} \int_0^1 du \left(\int_0^1 dB_s\, \tilde{\varphi}(u,s)\right)\left(\int_0^1 dB'_s\, \tilde{\varphi}(u,s)\right).$$

Proof : 1) From the identity $4ab = (a+b)^2 - (a-b)^2$, we deduce

$$2ab = \left(\frac{a+b}{\sqrt{2}}\right)^2 - \left(\frac{a-b}{\sqrt{2}}\right)^2 .$$

Now, using that $\left(\frac{B_s+B'_s}{\sqrt{2}}, \frac{B_s-B'_s}{\sqrt{2}}, s \geq 0\right)$ is a planar Brownian motion, we obtain *(2.m)*.

2) *(2.n)* is then a consequence of *(2.m)* and *(0.d)*. □

We give two applications of Proposition 5.

Proposition 6 : *Let* $\lambda > 0$.

1) For λ *small enough, we have :*

(2.o)
$$E\left[\exp\left(\lambda^2 \int_0^1 ds\, B_s\, B'_s\right)\right] \overset{(i)}{=} E\left[\exp\left(\frac{\lambda^4}{2}\int_0^1 du \left(\int_u^1 ds\, B_s\right)^2\right)\right]$$

$$\overset{(ii)}{=} ((\cos \lambda)(\text{ch } \lambda))^{-1/2}.$$

2) We write $G_s = \frac{1}{s}\int_0^s du\ B_u$ *and* $G = G_1$ *; we define in the same way* G'_s *and* G' *with* B'*. Then, we have :*

$$(2.p) \qquad \int_0^1 ds (B_s - G)(B'_s - G') = \int_0^1 ds\ (B_s - G_s)(B'_s - G'_s).$$

If $(\tilde{B}_s, s \le 1)$ *and* $(B'_s, s \le 1)$ *denote two independent standard Brownian bridges, we have, for* λ *small enough*

$$E\left[\exp \lambda^2 \int_0^1 ds (B_s - G)(B'_s - G')\right] \overset{(i')}{=} E\left[\exp \lambda^2 \int_0^1 ds\ \tilde{B}_s\ \tilde{B}'_s\right]$$

$$\overset{(ii')}{=} \left(\frac{\lambda^2}{(\mathrm{sh}\ \lambda)(\sin\ \lambda)}\right)^{1/2} .$$

The formula *(2.o)* has been obtained by Bearman [0].

Proof : 1) i) is a consequence of the representation of $\int_0^1 ds\ B_s\ B'_s$ as a stochastic integral with respect to dB'_s , that is :

$$\int_0^1 dB'_u \left(\int_u^1 ds\ B_s\right).$$

The identity ii) follows from *(2.m)* and the well-known formula

$$E\left[\exp \frac{\lambda^2}{2} \int_0^1 du\ B_u^2\right] = (\cos\ \lambda)^{-1/2} .$$

2) ii') follows from (2.m) and the well-known formula

$$E\left[\exp \frac{\lambda^2}{2} \int_0^1 du\ \tilde{B}_u^2\right] = \left(\frac{\lambda}{\sin \lambda}\right)^{1/2}.$$

On the other hand, the a.s. identity *(2.p)* is a consequence of the polarization of Hardy's identity (see section 3 of [13]) :

$$\int_0^1 ds\ (f(s) - \bar{f})^2 = \int_0^1 ds\ \left(f(s) - \frac{1}{s}\int_0^s f(u)du\right)^2,$$

where $\bar{f} = \int_0^1 ds\ f(s)$; we apply this polarization to $f(s) = B_s$ and $g(s) = B'_s$.

i') follows from *(2.n)* with $\varphi(u,s) = 1_{s\le u} - (1-s)$. □

Now, we extend the identity *(0.d)* into an identity in law between two sequences of r.v.'s ; we recall some notation used in section 4 of [6] :

$$H(s,h) = \int_0^1 du\ \varphi(u,s)\ \varphi(u,h)$$

and $K(s,h) = \int_0^1 du\ \tilde{\varphi}(u,s)\ \tilde{\varphi}(u,h) = \int_0^1 du\ \varphi(s,u)\ \varphi(h,u).$

In the following, we consider the kernel $M^{(n)}$ $(M = K \text{ or } H)$ defined by induction :

$$M^{(n)}(s,h) = \int_0^1 du\ M^{(n-1)}(s,u)\ M(u,h).$$

If we look at the polarized form *(2.n)* of the identity in law *(0.d)*, we notice that the left-hand side of *(2.n)* is equal to :

$$\int_0^1 dB'_h \int_0^1 dB_s \; H(s,h),$$

whilst the right-hand side is equal to :

$$\int_0^1 dB'_h \int_0^1 dB_s \; K(s,h).$$

Thus, we can deduce from *(2.n)* - in the same way as *(0.d)* follows from *(0.c)* - the identity in law :

(2.q)
$$\int_0^1 dh \left(\int_0^1 dB_s \; H(s,h) \right)^2 \overset{(law)}{=} \int_0^1 dh \left(\int_0^1 dB_s \; K(s,h) \right)^2 .$$

Using the same transformations as before, we obtain a polarized version of *(2.q)* with two independent Brownian motions B and B', and we write it as an identity in law between two stochastic integrals with respect to B'. Then, taking the conditional variance, it follows :

(2.r)
$$\int_0^1 dh \left(\int_0^1 dB_s \; H^{(2)}(s,h) \right)^2 \overset{(law)}{=} \int_0^1 dh \left(\int_0^1 dB_s \; K^{(2)}(s,h) \right)^2 .$$

Proceeding on with this method, we obtain, more generally, the following identities for any $(m,n) \in (\mathbb{N}^*)^2$:

(2.s)$_{m,n}$
$$\int_0^1 dh \left(\int_0^1 dB_s \; H^{(m)}(s,h) \right) \left(\int_0^1 dB_s \; H^{(n)}(s,h) \right)$$
$$\overset{(law)}{=} \int_0^1 dh \left(\int_0^1 dB_s \; K^{(m)}(s,h) \right) \left(\int_0^1 dB_s \; K^{(n)}(s,h) \right)$$

and some analogous identities when we replace

$$\int_0^1 dB_s\, H^{(n)}(s,h), \text{ resp. } \int_0^1 dB_s\, K^{(n)}(s,h) \quad \text{by} \quad \int_0^1 dB'_s\, H^{(n)}(s,h),$$

resp. $\displaystyle\int_0^1 dB'_s\, K^{(n)}(s,h)$.

With this method, we have not been able to draw the conclusion that the identities in law $(2.s)_{m,n}$, indexed on $(m,n) \in \mathbb{N} \times \mathbb{N}^*$, hold simultaneously. To prove this result, which is nonetheless true, we now use the spectral method, developed in section 4 of [3].

There exists an orthonormal basis $(\varphi_j) \in L^2([0,1])$ of eigenvectors of the kernel H, with eigenvalues (μ_j), that is :

$$H\varphi_j(s) \equiv \int_0^1 ds\, H(s,u)\, \varphi_j(u) = \mu_j\, \varphi_j(s)$$

and $H(s,u) = \sum_j \mu_j\, \varphi_j(s)\, \varphi_j(u)$.

In the same way, there exists an orthonormal basis $(\tilde{\varphi}_j)$ of eigenvectors of K, with the <u>same eigenvalues</u> (μ_j) i.e.

$$K\tilde{\varphi}_s = \mu_j\, \tilde{\varphi}_j \quad \text{and} \quad K(s,u) = \sum_j \mu_j\, \tilde{\varphi}_j(s)\, \tilde{\varphi}_j(u).$$

Thus, the left-hand side of $(2.s)_{m,n}$ is equal to :

$$\sum_j \mu_j^{m+n} \left(\int_0^1 dB_s\, \varphi_j(s)\right)^2.$$

We deduce, from the identity in law between the two sequences

$$\left(N_j = \int_0^1 dB_s\, \varphi_j(s),\ j \geq 1\right) \quad \text{and} \quad \left(\tilde{N}_j = \int_0^1 dB_s\, \tilde{\varphi}_j(s),\ j \geq 1\right),$$

the following improvement of $(2.s)_{m,n}$.

Proposition 7 : *We have the following identity in law :*

$$(2.t)\left(\int_0^1 dB_s \int_0^s dB_u\ H^{(m)}(s,u)\ ;\ m \geq 1\right) \overset{(\text{law})}{=} \left(\int_0^1 dB_s \int_0^s dB_u\ K^{(m)}(s,u)\ ;\ m \geq 1\right)$$

Remark : The identity *(2.t)* is really an improvement of all the identities $(2.s)_{m,n}$ and *(0.d)* simultaneously ; in fact, thanks to Itô's formula, we have the a.s. identities :

$$\int_0^1 du\left(\int_0^1 dB_s\ \varphi(u,s)\right)^2 = 2\int_0^1 dB_s \int_0^s dB_u\ H(s,u) + \int_0^1 du \int_0^1 ds\ \varphi^2(s,u)$$

and for $m,n \geq 1$:

$$\int_0^1 du\left(\int_0^1 dB_s\ H^{(m)}(u,s)\right)\left(\int_0^1 dB_s\ H^{(n)}(u,s)\right) = 2\int_0^1 dB_s \int_0^s dB_u\ H^{(m+n)}(s,u)$$
$$+ \int_0^1 du \int_0^1 ds\ H^{(m)}(s,u)\ H^{(n)}(s,u)$$
$$= 2\int_0^1 dB_s \int_0^s dB_u\ H^{(m+n)}(s,u) + \int_0^1 du\ H^{(m+n)}(u,u)$$

with analogous formulas when φ is replaced by $\tilde{\varphi}$ and H is replaced by K.

3. A characterization of the stable processes.

(3.1) We present a characterization of the processes $(X_t\ ;\ 0 \leq t \leq 1)$, starting from 0, which satisfy the identity in law :

$$(0.d)_{simple} \qquad \int_0^1 du \left(\int_0^1 dX_s\ \varphi(u,s)\right)^2 \overset{(law)}{=} \int_0^1 du\left(\int_0^1 dX_s\ \varphi(s,u)\right)^2 ,$$

for any bounded simple function $\varphi : [0,1]^2 \longrightarrow \mathbb{R}$.

In this paragraph, we shall use the notation : $(HW_t \ ; \ 0 < t \le 1)$ to denote the product of a standard one-dimensional Brownian motion $(W_t \ ; \ 0 < t \le 1)$, and of a random variable H such that $|H|$ and $(W_t \ ; \ 0 \le t \le 1)$ are independent (the same convention applies to $(H'W_t \ ; \ 0 \le t \le 1)$, or $(H_* W_t \ ; \ 0 \le t \le 1)$, and so on...).

We may now state and prove the following

Theorem 4 : *Let* $(X_t \ ; \ 0 \le t \le 1)$ *be a real valued process starting from* 0. *The following properties are equivalent :*

1) X *satisfies the identities* $(0.d)_{simple}$;

2) there exist a random variable ε, *which takes only the values* -1 *and* +1, *and a real valued random variable* H *such that :*

$$(3.a) \qquad (\varepsilon X_t \ ; \ 0 \le t \le 1) \overset{(law)}{=} (HW_t \ ; \ 0 \le t \le 1).$$

3) there exist a Bernoulli random variable ε_* , *that is :*

$$P(\varepsilon_* = +1) = P(\varepsilon_* = -1) = 1/_2 ,$$

which is independent of X, *and a positive random variable* H_* *such that :*

$$(3.a)_* \qquad (\varepsilon_* X_t \ ; \ 0 \le t \le 1) \overset{(law)}{=} (H_* W_t \ ; \ 0 \le t \le 1).$$

Comments : 1) Particular cases of processes X which satisfy the equivalent properties of Theorem 4 are the processes $(X_t, t \le 1)$ such that :

$$(3.a') \qquad (X_t \; ; \; 0 \le t \le 1) \overset{(law)}{=} (H'W_t \; ; \; 0 \le t \le 1).$$

The natural question now arises whether any process which satisfies *(3.a)* does in fact satisfy *(3.a')*.

2) Another natural question is the following : given a process X which satisfies *(3.a)*, or even *(3.a')*, to find a "direct" procedure to construct a Bernoulli variable ε_* such that $(3.a)_*$ is satisfied. Here is an example of such a situation : consider the process

$$X_t = \mathrm{sgn}(W_1)W_t \,, \qquad 0 \le t \le 1,$$

which appears thus under the form *(3.a')*.

Since $W \overset{(law)}{=} -W$, we see that $\varepsilon_* \equiv \mathrm{sgn}(W_1)$ is independent of $(X_t, t \le 1)$, and we have, obviously :

$$\varepsilon_* X_t = W_t \qquad (0 \le t \le 1) \; ;$$

hence, the property $(3.a)_*$ is satisfied, and we remark that the process X is not a Brownian motion, since $X_1 = |W_1|$.

3) Comparing again the identities in law *(3.a')* and $(3.a)_*$, it would be interesting, in the light of the above example, to know whether, if we have :

$$X_t = \eta W_t \,, \qquad 0 \le t \le 1,$$

where η takes only the values $+1$ and -1, and is measurable with respect to the σ-field given by W_t , $0 \le t \le 1$,
then, there exists a Bernoulli variable ε_* , which is also measurable with respect to W_t , $0 \le t \le 1$, but independent of X_t , $0 \le t \le 1$, and such that :

$$(3.a)_* \qquad (\varepsilon_* X_t, t \le 1) \overset{(law)}{=} (H_* W_t, t \le 1), \quad \text{with} \quad H_* \ge 0.$$

It is now immediate that we must have : $H_* = 1$, a.s., and therefore, the question is :

can we find ε_* such that : $(\varepsilon_* X_t \, ; \, t \le 1)$ is a Brownian motion ?

4) It may also be interesting to find out some examples of processes X which satisfy the equivalent properties of Theorem 4, and which are Brownian motions. In this direction, we have the following elementary lemma, the proof of which is left to the reader.

Lemma 1 : *Let* $(W_t, t \ge 0)$ *be a Brownian motion starting from* 0*, and let* ε_* *be a symmetric Bernoulli variable.*

We assume that ε_* *and* $X_t \equiv \varepsilon_* W_t$, $t \ge 0$*, are independent.*

The following properties are equivalent :

(i) X_t , $t \ge 0$, *is a Brownian motion ;*

(ii) ε_* *is independent of* W.

(In fact, what is important here is only the property of the law of W to be symmetric, that is : $W \overset{(law)}{=} -W$).

Proof of Theorem 4 : a) Property 3) obviously implies 2) ;

b) If X satisfies 2), then we have, for a simple function φ :

$$\int_0^1 du \left(\int_0^1 dX_s \; \varphi(u,s) \right)^2 \overset{(law)}{=} H^2 \int_0^1 du \left(\int_0^1 dW_s \; \varphi(u,s) \right)^2$$

and

$$\int_0^1 du \left(\int_0^1 dX_s \; \tilde{\varphi}(u,s) \right)^2 \overset{(law)}{=} H^2 \int_0^1 du \left(\int_0^1 dW_s \; \tilde{\varphi}(u,s) \right)^2 \; ;$$

it now follows from the independence of H^2 and **W**, and from the fact that **W** satisfies *(0.d)*, that X satisfies 1).

c) We now prove that 1) implies 3).

Hence, we assume that $(0.d)_{simple}$ is satisfied.

Then, introducing a Brownian motion $(B_t \,;\, 0 \le t \le 1)$ which is independent of X, we remark that $(0.d)_{simple}$ is equivalent to :

$$\text{(3.b)} \qquad \int_0^1 dB_u \int_0^1 dX_s\, \varphi(u,s) \overset{(law)}{=} \int_0^1 dB_u \int_0^1 dX_s\, \tilde{\varphi}(u,s) \;;$$

for any simple function φ.

However, interverting the order of the integrals and then exchanging the names of the variables s and u on the right-hand side, we obtain :

$$\int_0^1 dB_u \int_0^1 dX_s\, \varphi(s,u) = \int_0^1 dX_s \int_0^1 dB_u\, \varphi(s,u) = \int_0^1 dX_u \int_0^1 dB_u\, \varphi(u,s),$$

and so, *(3.b)* may be written as :

$$\text{(3.b')} \qquad \int_0^1 dB_u \int_0^1 dX_s\, \varphi(u,s) \overset{(law)}{=} \int_0^1 dX_u \int_0^1 dB_s\, \varphi(u,s).$$

Now, both sides of *(3.b')* are linear in φ, and so *(3.b')* is equivalent to the identity in law between processes :

$$\text{(3.c)} \qquad (B_u X_s \,;\, u \le 1, s \le 1) \overset{(law)}{=} (X_u B_s \,;\, u \le 1, s \le 1).$$

In particular, we have :

$$\text{(3.c')} \qquad (X_u B_1 \,;\, u \le 1) \overset{(law)}{=} (B_u X_1 \,;\, u \le 1).$$

We now write : $\varepsilon_* = \mathrm{sgn}(B_1)$, and $\tilde{X}_u = \varepsilon_* X_u$.

Hence, we deduce from *(3.c')* that :

$$(3.d) \qquad (|B_1|\tilde{X}_u \ ; \ u \le 1) \overset{(law)}{=} (|X_1|W_u \ ; \ u \le 1),$$

where $W_u \equiv \mathrm{sgn}(X_1)B_u$ is a Brownian motion which is independent of X_1.
From *(3.d)*, we deduce that : $B_1^2\langle\tilde{X}\rangle_1 \overset{(law)}{=} X_1^2$,
where we have denoted :

$$\langle\tilde{X}\rangle_1 = \underset{n\to\infty}{\text{P-lim}}\ \Sigma_{\tau_n}(\tilde{X}_{t_{i+1}}-\tilde{X}_{t_i})^2 = \underset{n\to\infty}{\text{P-lim}}\ \Sigma_{\tau_n}(X_{t_{i+1}}-X_{t_i})^2 = \langle X\rangle_1$$

($\tau_n \equiv (0 = t_o <\ldots< t_{p_n} = 1)$, $n \in \mathbb{N}$, is a sequence of subdivisions of $[0,1]$, the mesh of which goes to 0, as $n \longrightarrow \infty$).

Hence, if we write : $H_* = \sqrt{\langle X\rangle_1}$, we obtain :

$$|X_1| \overset{(law)}{=} |B_1|H_* \ , \text{ with } H_* \ge 0, \text{ and independent of } B_1.$$

Going back to *(3.d)*, we now have the following identity in law :

$$(3.e) \qquad (|B_1|\tilde{X}_u \ ; \ u \le 1) \overset{(law)}{=} (|B_1|H_*W_u \ ; \ u \le 1)$$

where, on the left-hand side, $|B_1|$ and $\tilde{X}$ are independent, whereas, on the right-hand side, $|B_1|$, H_* and W are independent.

It is now easily deduced, by a Laplace transform argument, that *(3.e)* implies:

$$(3.f) \qquad (\tilde{X}_u \ ; \ u \le 1) \overset{(law)}{=} (H_*W_u \ ; \ u \le 1),$$

which is precisely the identity $(3.a)_*$. □

(3.2) Our aim is now to characterize the processes $(X_t \ ; \ 0 \le t \le 1)$ starting from 0, which satisfy the identity :

(Here, we assume : $0 < \alpha \leq 2$).

$(0.d)^{\alpha}_{simple}$ $$\int_0^1 du \left| \int_0^1 dX_s \, \varphi(u,s) \right|^{\alpha} \overset{(law)}{=} \int_0^1 du \left| \int_0^1 dX_s \, \varphi(s,u) \right|^{\alpha},$$

for any bounded simple function $\varphi : [0,1]^2 \longrightarrow \mathbb{R}$.

In order to state our answer to this problem, we find it convenient to introduce the

Definition : *An $\mathbb{R}_+$-valued random variable Y is said to be simplifiable if the following property holds :*

if $YX \overset{(law)}{=} YZ$, with X and Z taking their values in $\mathbb{R}_+$, and X, resp : Z, independent of Y, then : $X \overset{(law)}{=} Z$.

The following lemma gives some interesting examples of simplifiable variables.

Lemma 2 : *1) If Y takes its values in $\mathbb{R}_+ \setminus \{0\}$, and if the characteristic function of $(\log Y)$ has only isolated zeros, then Y is simplifiable ;*

2) If $Y = |C^{(\alpha)}|$, where $C^{(\alpha)}$ is a symmetric stable random variable, then Y is simplifiable.

In the following, we make the convention, similar to that preceding the statement of Theorem 6, that the product $(HC^{(\alpha)}_t, t \leq 1)$ denotes the product of a variable H and a stable process $C^{(\alpha)}$, such that $|H|$ and $C^{(\alpha)}$ are independent.

We may now state and prove the

Theorem 5 : *Let $(X_t \,;\, 0 \leq t \leq 1)$ be a real valued process starting from 0. The following properties are equivalent :*

1) X *satisfies the identities* $(0.d)^{\alpha}_{simple}$;

2) there exists a simplifiable variable Y, *a random variable* ε *which takes only the values* $+1$ *and* -1, *and a real-valued random variable* H *such that :*

$$(Y\varepsilon X_t \ ; \ 0 \le t \le 1) \overset{(law)}{=} (HC^{(\alpha)}_t \ ; \ 0 \le t \le 1),$$

where, on the left-hand side, Y *and* X *are assumed to be independent ;*

3) there exists a symmetric stable variable $\tilde{Y}_{\alpha}$, *of index* α, *which is independent of* X, *and a positive random variable* H_* *such that :*

$$(\tilde{Y}_{\alpha} X_t \ ; \ 0 \le t \le 1) \overset{(law)}{=} (H_* \, C^{(\alpha)}_t \ ; \ 0 \le t \le 1).$$

Proof : a) The property 3) obviously implies 2), thanks to the second part of Lemma 2.

b) If X satisfies 2), then, we have, for a simple function φ :

$$Y^{\alpha} \int_0^1 du \, \left|\int_0^1 dX_s \, \varphi(u,s)\right|^{\alpha} \overset{(law)}{=} |H|^{\alpha} \int_0^1 du \, \left|\int_0^1 dC^{(\alpha)}_s \, \varphi(u,s)\right|^{\alpha}$$

and

$$Y^{\alpha} \int_0^1 du \, \left|\int_0^1 dX_s \, \tilde{\varphi}(u,s)\right|^{\alpha} \overset{(law)}{=} |H|^{\alpha} \int_0^1 du \, \left|\int_0^1 dC^{(\alpha)}_s \, \tilde{\varphi}(u,s)\right|^{\alpha}.$$

It now follows from the independence of $|H|$ and $C^{(\alpha)}$, and from the fact that $C^{(\alpha)}$ satisfies $(0.d)^{\alpha}$, that :

$$Y\left(\int_0^1 du \, \left|\int_0^1 dX_s \, \varphi(u,s)\right|^{\alpha}\right)^{1/\alpha} \overset{(law)}{=} Y\left(\int_0^1 du \, \left|\int_0^1 dX_s \, \tilde{\varphi}(u,s)\right|^{\alpha}\right)^{1/\alpha}.$$

Finally, since Y is simplifiable, we obtain that X satisfies $(0.d)^{\alpha}_{simple}$.

c) We now assume that $(0.d)^{\alpha}_{simple}$ is satisfied, and we prove that 3) holds.

Just as in part 3) of the proof of Theorem 4, we arrive without any new difficulty to :

$$(3.c')_{\alpha} \qquad (X_u\, C_1^{(\alpha)} \ ; \ u \le 1) \overset{(law)}{=} (C_u^{(\alpha)}\, X_1 \ ; \ u \le 1),$$

where $C^{(\alpha)}$ and X are independent (on both sides) ;

thus, defining $\tilde{C}_u^{(\alpha)} = \mathrm{sgn}(X_1)C_u^{(\alpha)}$, we obtain a symmetric stable process which is independent of X_1, and it is now clear that 3) is satisfied, with $\tilde{Y}_{\alpha} = C_1^{(\alpha)}$, and $H_* = |X_1|$. □

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