Invariant Directional Orderings

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Abstract

Statistical concepts of order permeate the theory and practice of statistics. The present paper is concerned with a large class of directional orderings of univariate distributions. (What do we mean by saying that a random variable Y is larger than another random variable X?) Attention is restricted to pre-orders that are invariant under monotone transformations; this includes orderings such as monotone likelihood ratio, hazard ordering, and stochastic ordering. Simple characterizations of these orderings are obtained in terms of a maximal invariant. It is shown how such invariant pre-orderings can be used to generate concepts of Y_2 being further to the right of X_2 than Y_1 is of X_1.

1. Introduction.

The concept of statistical order which compares two random variables or distribution functions has many uses in both applied and theoretical statistics. In reliability theory, the concepts of increasing failure rate, increasing failure rate on the average, and new better than used, compare distribution functions F and G in terms of the function $\phi = G^{-1}F$. (Van Zwet (1964), Barlow and Proschan (1975), Loh (1984).) The monotonicity of the power of monotone rank tests is considered by Lehmann (1959) and Doksum (1969). Comparisons of distributions based on tail heaviness or skewness were treated respectively by Rojo (1988, 1992) and MacGillivray (1986). For some extensions to the multivariate case see for example Lehmann (1952, 1955), Whitt (1982), Keilson and Sumita (1983), Karlin and Rinott (1983), and to the comparison of stochastic processes Pledger and Proschan (1973), and Whitt (1981).

In this paper, attention will be restricted to univariate directional orderings which are concerned with the question of whether one distribution is in some sense to the right of the other. In this connection many different ordering concepts have been proposed, of which we mention the following:

(i) *Stochastic Ordering*. The random variable Y is said to be stochastically larger than X if their distribution functions G and F satisfy

$$F(x) \geq G(x) \quad \text{for all} \quad x, \tag{1.1}$$

and this will be denoted by $F \underset{st}{\leq} G$.

(ii) *Monotone Likelihood Ratio (MLR)*. When F and G have densities f and g with respect to some common dominating measure μ, and

$$g/f \quad \text{is nondecreasing}, \tag{1.2}$$

the pair (F, G) is said to have monotone likelihood ratio, which we denote by $F \underset{mlr}{\leq} G$.

(iii) *Hazard ordering*. A cdf F is said to be smaller than G in the hazard ordering, ($F \underset{h}{\leq} G$) if

$$\frac{g(x)}{1-G(x)} \leq \frac{f(x)}{1-F(x)} \quad \text{for all} \quad x. \tag{1.3}$$

Here $f(x)/[1-F(x)]$ is the "mortality" of a subject at time x given that it has survived to this time. If the densities f and g exist, condition (1.3) is easily seen to be equivalent to

$$[1-F(x)]/[1-G(x)] \quad \text{is nonincreasing}. \tag{1.4}$$

(iv) *Restricted Definitions*. It is not always appropriate to require that the comparisons (i)-(iii) hold for all values of x. For example, in a comparison of lengths of lives of women and men one may want to restrict the comparison to childhood or perhaps to ages past childbearing. We consider two cases of such restrictions:

(a) Definition (i)-(iii) for all x exceeding a specified x_0, and

(b) Definitions (i)-(iii) for all sufficiently large x.

(v) *Comparisons based on a single functional.* One population of heights, ages, incomes, etc. is frequently considered to be larger than another such population if it is larger "on the average", or if the median of the first population exceeds that of the second. More generally a rather weak comparison of two distributions can be defined in terms of some measure μ of location, i.e. by seeing whether $\mu(F)$ is larger or smaller than $\mu(G)$.

Examples (i)-(v) seem to constitute a rather haphazard collection. The purpose of the present paper is to present a more systematic way of defining and studying such orderings, and to obtain simple characterizations of the orderings discussed above subject only to very general restrictions on the distribution functions F and G such as continuity or being strictly increasing. Possible approaches are suggested by two well known rather obvious characterizations of stochastic ordering, namely that (1.1) holds if and only if

(1.5) $$F^{-1}[G(x)] \leq x \text{ for all } x$$

or if

(1.6) $E_F\,\psi(X) \leq E_G\,\psi(X)$ for all nondecreasing functions ψ, where here and below $F^{-1}(u) = \inf\{t: F(t) \geq u\}$.

Natural extensions of these characterizations are:

1) Let $F \leq G$ when $\phi = F^{-1}G$ is a member of some specified class of functions. For discussion of such an approach see for example Oja (1981) and Loh (1984).

2) Let $F \leq G$ when

$$E_F[\psi(X)] \leq E_G[\psi(X)]$$

for all ψ belonging to some specified class. (See for example Whitt (1980)).

Unfortunately it turns out that neither monotone likelihood ratio nor hazard ordering can be characterized in either of these ways. We shall here develop an alternative approach which will provide a fairly simple characterization of a large class of directional orderings including (i)-(iii).

2. Invariant directional pre-orderings

In this section we restrict attention to the class of distributions **F** defined as follows:

(2.1) $\mathbf{F} = \{F: F \text{ is continuous and strictly increasing on } (-\infty, \infty)\}$.

We shall later weaken this restriction.

A pre-order of the distributions $F \in \mathbf{F}$ is a set S of ordered pairs (F, G) in $\mathbf{F} \times \mathbf{F}$ satisfying,

(2.2) $$(F,F) \in S \text{ for all } F \in \mathbf{F}, \text{ and}$$

(2.3) $$(F,G) \in S, \ (G,H) \in S \text{ implies } (F,H) \in S.$$

When $(F,G) \in S$, we say that $F \leq_S G$ or $X \leq_S Y$ where X, Y are random variables with distributions F and G respectively.

Definition. The pre-order S is invariant (under monotone transformations) if

(2.4) $$X \leq_S Y \text{ implies } \psi(X) \leq_S \psi(Y) \text{ for all } \psi \in \Psi,$$

where Ψ is the class of all strictly increasing continuous functions ψ with $\psi(-\infty) = -\infty$, $\psi(+\infty) = +\infty$.

It is easy to see that (i)-(v) in section 1 all satisfy (2.2) and (2.3). Examples (i)-(iii) and (ivb) also satisfy (2.4) but (iva) does not.

When X is distributed according to F, the distribution of $\psi(X)$ is $F\psi^{-1}$, and in terms of S condition (2.4) becomes

(2.5) $$(F,G) \in S \Rightarrow (F\psi^{-1}, G\psi^{-1}) \in S \text{ for all } \psi \in \Psi.$$

To see the simplification resulting from (2.5) consider the orbits under the group of transformations $\psi \in \Psi$ in the space $\mathbf{F} \times \mathbf{F}$ of pairs (F, G):

(2.6) $$O(F,G) = \{(F\psi^{-1}, G\psi^{-1}): \psi \in \Psi\}.$$

It follows from the definition of the orbits and (2.5) that $(F,G) \in S$ if and only if the whole orbit $O(F,G)$ is contained in S. Therefore, to characterize an invariant pre-ordering S, it is sufficient to list the totality of orbits in S, and for that purpose, it is convenient to have available a suitable labeling of the orbits, that is, a maximal invariant under Ψ. It is easy to see that such a labeling is provided by

(2.7) $$k(u) = GF^{-1}(u), \quad 0 \leq u \leq 1.$$

On the one hand k is invariant under Ψ since $G\psi^{-1}(F\psi^{-1})^{-1} = GF^{-1}$; on the other hand, if $G_1F_1^{-1} = G_2F_2^{-1}$, we have $F_1^{-1}F_2 = G_1^{-1}G_2 = \psi$, and thus $F_1 = F_2\psi^{-1}$ and $G_1 = G_2\psi^{-1}$ so that (F_1, G_1) and (F_2, G_2) lie on the same orbit.

For any $F, G \in \mathbf{F}$, it is seen that $k = GF^{-1}$ has the following properties:

(2.8) $k(0) = 0$ and $k(1) = 1$, and on $(0,1)$ k is continuous and strictly increasing.

While MLR and hazard ordering cannot be characterized in terms of $F^{-1}G$, they have very simple characterizations in terms of $k = GF^{-1}$.

Theorem 1.

i) $F \underset{\text{mlr}}{\leq} G \iff k(\cdot)$ is convex;

ii) $F \underset{h}{\leq} G \iff 1 - k^{-1}(1-u)$ is star-shaped;

iii) $F \underset{\text{st}}{\leq} G \iff k(u) \leq u$ for all $0 < u < 1$

Proof: i) follows easily by noting that the derivative of k is $k' = gF^{-1} / fF^{-1}$ and recalling that k is convex if and only if k' is nondecreasing. ii) Note that $(1 - k^{-1}(1-u))$ is star-shaped if and only if $(1 - FG^{-1}(t))/(1-t)$ is nonincreasing in t, or equivalently, if $\bar{F}/\bar{G}$ is nonincreasing, where $\bar{F}$ denotes $1 - F$. Then it is easy to verify that the latter condition is equivalent to (1.3). The proof of iii) is trivial.

Theorem 2. Inclusion relationships among i), ii) and iii) are given in the following diagram

$$
\begin{array}{ccccc}
k(\cdot)\ \text{convex} & \iff \text{m.l.r.} \iff & 1-k^{-1}(1-u)\ \text{convex} & & \\
\Updownarrow & & \Updownarrow & & \\
k(\cdot)\ \text{star-shaped} & & 1-k^{-1}(1-u)\ \text{star-shaped} & \iff & F \leq_h G \\
\Updownarrow & & \Updownarrow & & \\
G/F\ \text{nondecreasing} & & \bar{G}/\bar{F}\ \text{nondecreasing} & & \\
& \searrow\!\!\!\searrow \quad k(u) \leq u. \quad \swarrow\!\!\!\swarrow & & & \\
& \Updownarrow & & & \\
& F \leq_{st} G & & &
\end{array}
$$

Proof. The nontrivial aspects follow from the well-known fact that for functions going through the origin, convexity implies star-shapedness.

3. Characterization of sets of k-functions defining pre-orders orders

It was pointed out in section 2 that the partial order (iva) is not invariant. On the other hand, for example, (iva) applied to (i) has a k-analogue

$$k(u) \geq u \quad \text{for all} \quad u \geq u_0, \tag{3.1}$$

which clearly is invariant. It is easy to translate (3.1) into a condition on the pair

(F, G); the condition is simply

$$G^{-1}(u) \geq F^{-1}(u) \text{ for all } u \geq u_0. \tag{3.2}$$

This example shows that it is sometimes convenient to start with a pre-order of the k-functions and from it derive the equivalent ordering rather than the other way around. To be able to do so we need to know when the set

$$S = \{(F,G) : k(F,G) \in \mathbf{K}\}$$

is a pre-order. Clearly the set S will satisfy (2.2) if and only if the function $k(u) = u$ for all $0 < u < 1$ is in **K**. To satisfy (2.3), **K** must satisfy

$$GF^{-1} \in \mathbf{K}, \quad HG^{-1} \in \mathbf{K} \quad \Rightarrow \quad HF^{-1} \in \mathbf{K}.$$

But if $GF^{-1} = k_1$ and $HG^{-1} = k_2$ then $HF^{-1} = k_2(k_1)$ so that (2.3) will hold provided **K** is closed under composition.

Thus **K** corresponds to a pre-order provided it contains the identity function and is closed under composition.

As an application consider the problem of finding a pre-order that is stronger than monotone likelihood ratio. In k-space, a natural strengthening of convexity is to require that

$$k^{(i)}(u) \geq 0 \text{ for all } 0 < u < 1 \text{ and all } i = 1, \ldots, n. \tag{3.3}$$

The identity function obviously satisfies (3.3) and it is not difficult to show that the class **K** defined by (3.3) is closed under composition. The conditions on (F, G) corresponding to (3.3) do not appear to have any simple interpretation. As an illustration of (3.3) consider the following examples.

Example 3.1. Let $k(u) = u^n$, $n \geq 3$. Then $k^{(i)}(u) \geq 0$, $0 \leq u \leq 1$ for $i = 1,2,3, \ldots, n$. Moreover, it is easy to see that $k(u) = u^n$ implies that $G(t) = (F(t))^n$ and hence the nth order statistic $X_{(n)}$ in a random sample of size n from F satisfies (3.3) so that $X < X_{(n)}$ in the sense of (3.3).

Example 3.2. In the normal location case, say with $F(x) = \Phi(x)$, $G(x) = \Phi(x - \theta)$, $\theta > 0$, we have MLR so that k satisfies (3.3) with $n = 2$; it does not however satisfy (3.3) with $n = 3$. A location family satisfying (3.3) for all n is given in the following example.

Example 3.3. Consider the extreme value location family of distribution with densities given by

$$f_\theta(x) = e^{x-\theta} e^{-e^{x-\theta}}. \tag{3.4}$$

For $\theta > 0$, it is easy to see that

$$k(u) = F_\theta F_0^{-1}(u) = 1 - (1-u)^\lambda, \tag{3.5}$$

where $\lambda = e^{-\theta} < 1$. It then follows that $k(u)$ satisfies (3.3) for every positive integer n.

Example 3.4. Consider the family of exponential distributions with density $\frac{1}{a}e^{-x/a}$. To see that this is ordered in the sense of (3.3) for all n, we need only note that if X is distributed according to (3.4), then the distribution of e^X is exponential with scale parameter $a = e^\theta$. Since any invariant order is unaffected by a monotone transformation, (3.3) follows from Example 3.3.

4. Extension to a larger class of distribution functions

At the beginning of Section 2 we restricted attention to the class **F** of distribution functions which are continuous and strictly increasing on $(-\infty, \infty)$. This is too restrictive and we shall now extend the theory of the preceding sections to the class $\mathbf{F}^*$ of distributions F whose support is an interval (which may change with F and which may be finite, semi-infinite, or infinite), on which F is assumed to be continuous and strictly increasing.

If the support of $F \in \mathbf{F}^*$ is the interval (a, b), the quantile function F^{-1} is continuous and strictly increasing on the open interval $(0, 1)$ and satisfies

$$\lim_{u\to 0} F^{-1}(u) = a, \quad \lim_{u\to 1} F^{-1}(u) = b$$

and

$$F^{-1}(0) = -\infty, \quad F^{-1}(1) = b.$$

The nature of $k = GF^{-1}$ depends on the relative positions of the supports (a, b) of G and (c, d) of F. Seven different situations are possible:

(i) *a=c, b=d.* In this case k increases strictly and continuously from $k(0) = 0$ to $k(1)=1$ as it does when $a = c = -\infty$, $b = d = \infty$. No changes are needed.

The behavior in cases (ii)-(v) is shown in the following table

Case	k increases strictly and continuously
(ii) $a < c < d < b$	$(0, G(c))$ to $(1, G(d))$
(iii) $c < a < b < d$	$(F(a), 0)$ to $(F(b), 1)$
(iv) $a < c < b < d$	$(0, G(c))$ to $(F(b), 1)$
(v) $c < a < d < b$	$(F(a), 0)$ to $(1, G(d))$

Finally, in case (vi) in which $b \le c$, we have $k(u) = 1$ for $0 < u \le 1$ and $k(0) = 0$, and case (vii) in which $d \le a$, $k(u) = 0$ for $0 \le u < 1$ and $k(1) = 1$.

The type remains invariant under the transformations $\psi \in \Psi$ defined in (2.4). So does the nature of the carriers C and D of F and G, ie. whether each of C and D is finite, semi-infinite with a finite right end point or left end point, or infinite in both directions. As a result, although k remains invariant under the transformations of Ψ - it no longer is maximal invariant. Consider for example the pairs (F, G) with $F = G$. They all correspond to the single function $k(u) = u$ for all $0 < u < 1$, but they do not constitute a single orbit since the cases $C = D =$ finite, $C = D =$ infinite, etc. all have different orbits. Thus the maximal invariant must take account of the nature of C and D; in addition, with respect to the types (vi) and (vii) defined in the preceding paragraph, it must distinguish between the types

(via) $b < c$ and (vib) $b = c$

and

(viia) $d < a$ and (viib) $d = a$,

which can also not be transformed into each other.

If the assumptions made at the beginning of this section are further weakened, the complexity of the maximal invariant will increase even further.

There is an alternative way of handling a weakening of the assumptions such as that at the beginning of Section 2, and this is to enlarge the group Ψ of transformations defined in (2.4). However, this approach also runs into difficulties and we shall therefore here consider no further extensions.

5. Ordering the orbits

An invariant pre-order S allows us to go a step further. We can not only say of certain pairs (F, G) - the pairs in S - that $F \le G$, but also for certain quadruples $(F_1, G_1; F_2, G_2)$ that, according to the ordering $\underset{S}{\le}$, G_2 is further away from F_2 than G_1 is from F_1.

To fix ideas, consider distribution functions F_1 and G_1, with $F_1 \underset{S}{\leq} G_1$ and define $F_2 = F_1\psi^{-1}$ and $G_2 = G_1\psi^{-1}$ for $\psi \in \Psi$. Since (F_1, G_1) and (F_2, G_2) are in the same orbit, we say F_1 and G_1 are at the same distance from each other as F_2 and G_2. That is, all pairs on the same orbit are equally distant. On the other hand, if $F_1 \underset{S}{\leq} G_1 \underset{S}{\leq} G_2$, we say that G_2 is further to the right of F_1 than G_1 is. This last situation implies that $G_1 = F_1\psi^{-1}$ and $F_1\psi^{-1} \underset{S}{\leq} G_2$ for some monotone ψ. Extending this argument to the case of the quadruple $(F_1, G_1; F_2, G_2)$ with $(F_i, G_i) \in S$, $i = 1,2$, we say that G_2 is further to the right of F_2 than G_1 is of F_1 if $F_2 = F_1(\psi^{-1})$ and $G_1(\psi^{-1}) \underset{S}{\leq} G_2$ for some $\psi \in \Psi$. In term of $k_1 = G_1F_1^{-1}$ and $k_2 = G_2F_2^{-1}$, this is equivalent to requiring that $k_2(k_1^{-1}) \in \mathbf{K}_S$, and this motivates the following definition.

Definition. Let (F_1, G_1), $(F_2, G_2) \in S$ then, G_2 in said to be further to the right of F_2 than G_1 is of F_1 if

$$(5.1) \qquad (k_1, k_2) \in S,$$

where $k_i = G_iF_i^{-1}$, $i = 1,2$.

Since k_1, k_2 are distribution functions, (5.1) is equivalent to the condition that $k_2(k_1^{-1}) \in \mathbf{K}_S$ that is, $k_1 \underset{S}{\leq} k_2$. It also follows trivially that $k_1 \underset{S}{\leq} k_2$ if and only if $k_2 = k_3(k_1)$ for some $k_3 \in \mathbf{K}_S$. Thus, for example, when S represents stochastic ordering, G_2 is further to the right of F_2 than G_1 is of F_1 if $G_2F_2^{-1}(u) \leq G_1F_1^{-1}(u)$, $0 < u < 1$. If S represents monotone likelihood ratio ordering, G_2 will be further to the right of F_2 than G_1 is of F_1 if $G_2F_2^{-1}(u) = h(G_1F_1^{-1}(u))$ for some convex function h. This is illustrated by the following two examples.

Example 5.1. Consider the location family $F(x - \theta)$, and define $G_1 = (F(x - \theta_1)$ and $G_2 = F(x - \theta_2)$ with $\theta_2 > \theta_1 > \theta$. Then $F \underset{st}{\leq} G_1$ and $F \underset{st}{\leq} G_2$. Moreover, $k_1 \underset{st}{\leq} k_2$, where $k_1 = G_1F^{-1}$ and $k_2 = G_2F^{-1}$ so that G_2 is further to the right of F than G_1 is.

Example 5.2. Consider the family of beta distributions F_α, with densities

$$f_\alpha(x) = \alpha x^{\alpha-1}, \quad \alpha > 0, \quad 0 < x < 1.$$

then $F_{\alpha_1} \underset{mlr}{\leq} F_{\beta_1}$ and $F_{\alpha_2} \underset{mlr}{\leq} F_{\beta_2}$ when $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. Moreover, when $\dfrac{\beta_2}{\alpha_2} > \dfrac{\beta_1}{\alpha_1}$, $F_{\beta_2}F_{\alpha_2}^{-1}F_{\alpha_1}F_{\beta_1}^{-1}$ is convex and hence F_{β_2} is further to the right of F_{α_2} than F_{β_1} is of F_{α_1}.

6. Distances between ordered distributions

The qualitative distance relation, introduced by (5.1) on pairs of distribution function $(F,G) \in S$ by comparing their corresponding k-functions raises the question of whether there exists a metric $d(F,G)$ which is consistent with the approach using k functions. To be consistent, such a metric must satisfy the following two conditions:

(i) Since the k-functions are invariant under monotone transformations, the distance $d(F,G)$ must also be invariant under monotone transformations. Formally, we shall require that

$$d(F,G) = d(F\psi^{-1}, G\psi^{-1}) \quad \text{for all} \quad \psi \in \Psi,$$

(ii) and that if $k_i = G_i F_i^{-1} \in \mathbf{K}_S$, $i = 1,2$ with $k_1 \underset{S}{<} k_2$, then

$$d_S(F_1, G_1) \le d_S(F_2, G_2).$$

Condition (ii) immediately implies the additional desirable requirement that if the ordering S_1 is stronger than S_2, then

$$d_{S_1}(F_1, G_1) \le d_{S_1}(F_2, G_2) \Rightarrow d_{S_2}(F_1, G_1) \le d_{S_2}(F_2, G_2).$$

A metric that satisfies (i) and (ii) above when S is stochastic ordering — and hence also for any stronger ordering — is provided by the supnorm d_0. Since

$$d_0(F,G) = \sup_x |G(x) - F(x)| = \sup_{0 \le u \le 1} |GF^{-1}(u) - u|$$
$$= \sup_{0 \le u \le 1} |k(u) - u|,$$

d_0 is invariant under monotone transformations. Also, if $k_1, k_2 \in \mathbf{K}_S$ with $k_1(u) \le k_2(u)$ for all u, then

$$d_0(F_1, G_1) \le d_0(F_2, G_2).$$

Example 6.1. Let $G(x) = F(x - \theta)$, $\theta > 0$ so that $F \underset{st}{\le} G$. Then

$$d_0(F,G) = \sup_{0 \le u \le 1} \{u - F[F^{-1}(u) - \theta]\}.$$

If F is unimodal with a density f which is symmetric about t_0, the supremum is attained at $t = t_0 + \frac{\theta}{2}$, and hence

$$d_0(F,G) = F(t_0 + \frac{\theta}{2}) - F(t_0 - \frac{\theta}{2}).$$

Having defined a distance function consistent with stochastic ordering, we next consider the orderings

S_1: hazard ordering

S_2: the ordering in which $F \leq_S G$ if $k(\cdot)$ is star-shaped

S_3: MLR ordering

and for each $i = 1,2,3$ define a distance function d_i consistent with S_i as follows:

$$d_1(F,G) = \sup_x |\ln(\overline{G}(x)/\overline{F}(x))|$$

$$d_2(F,G) = \sup_x |\ln(G(x)/F(x))|$$

$$d_3(F,G) = \sup_x |\ln(g(x)/f(x))|.$$

It is clear that d_1, d_2, and d_3 satisfy the invariance condition (i) given above. That they also satisfy condition (ii) follow from the following theorem.

Theorem 3. If $k_1, k_2, k_2(k_1^{-1}) \in \mathbf{K}_{S_i}$, then

$$d_i(F_1,G_1) \leq d_i(F_2,G_2), \qquad i = 1,2,3.$$

Proof: Clearly, the membership of k_1, k_2 and $k_2(k_1^{-1})$ in $\mathbf{K}_{S_i}$ $i = 1,2$ or 3, implies that

$$k_1(u) \leq u, \quad k_2(u) \leq u, \quad \text{and} \quad k_1(u) \geq k_2(u). \tag{6.1}$$

We now proceed by cases.

(i) For the S_1 ordering, (6.1) implies that

$$d_1(F_j,G_j) = \ln \sup_x \frac{\overline{G}_j(x)}{\overline{F}_j(x)} = \ln \sup_{0\leq u\leq 1} \frac{\overline{G}_j\overline{F}_j^{-1}(u)}{u}, \qquad j = 1,2, \tag{6.2}$$

and that $\dfrac{\overline{k}_2}{\overline{k}_1}$ is nondecreasing. This latter implication is equivalent to

$$\frac{\overline{G}_2\overline{F}_2^{-1}(u)}{\overline{G}_1\overline{F}_1^{-1}(u)} \quad \text{is nonincreasing in } u. \tag{6.3}$$

Therefore, $\dfrac{\overline{G}_2\overline{F}_2^{-1}(u)}{u} \geq \dfrac{\overline{G}_1\overline{F}_1^{-1}(u)}{u}$ and the result for S_1 follows from (6.2).

(ii) For the S_2 ordering, condition (6.1) implies that

$$d_2(F_j,G_j) = \ln \sup_{0\leq u\leq 1} \frac{u}{G_jF_j^{-1}(u)}, \tag{6.4}$$

and the fact that $k_2(k_1^{-1}) \in \mathbf{K}_{S_2}$ implies that $\dfrac{k_2}{k_1}$ is nondecreasing and hence

$$\frac{u}{G_2F_2^{-1}(u)} \geq \frac{u}{G_1F_1^{-1}(u)}, \qquad 0 < u < 1 \tag{6.5}$$

the result then follow from (6.4) and (6.5).

(iii) In the case of S_3, (6.1) together with the condition that k_j' is nondecreasing for j = 1,2, imply that

$$d_3(F_j, G_j) = \max\{-\ln k_j'(0^+),\ \ln k_j'(1^-)\}. \tag{6.6}$$

But $k_1(u) \geq k_2(u)$, together with $k_j(0) = 0$ and $k_j(1) = 1$, $j = 1,2$, imply that $k_1'(0^+) \geq k_2'(0^+)$ and $k_1'(1^-) \leq k_2'(1^-)$, and now the result follows from (6.6).

Example 6.2. Let $\overline{G}(x) = e^{-x}$ and $\overline{F} = \frac{1}{2}[e^{-4x} + e^{-x}]$, $x > 0$. Then $F \underset{mlr}{\leq} G$ and we have

$$d_1(F, G) = \sup_x |\ln \frac{\overline{G}(x)}{\overline{F}(x)}| = \ln 2$$

and

$$d_3(F, G) = \sup_x |\ln \frac{g(x)}{f(x)}| = \ln(5/2).$$

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