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ABSTRACT. We consider two independent Dawson-Watanabe super-Brownian motions, Y^1 and Y^2. These processes are diffusions taking values in the space of finite measures on $\mathbb{R}^d$. We show that if $d \leq 5$ then with positive probability there exist times t such that the closed supports of Y_t^1 and Y_t^2 intersect; whereas if $d > 5$ then no such intersections occur. For the case $d \leq 5$, we construct a continuous, non-decreasing measure-valued process $L(Y^1, Y^2)$, the "collision local time", such that the measure defined by $[0,t] \times B \mapsto L_t(Y^1, Y^2)(B)$, $B \in \mathcal{B}(\mathbb{R}^d)$, is concentrated on the set of times and places at which intersections occur. We give a Tanaka-like semimartingale decomposition of $L(Y^1, Y^2)$. We also extend these results to a certain class of coupled measure-valued processes. This extension will be important in a forthcoming paper where we use the tools developed here to construct coupled pairs of measure-valued diffusions with "point interactions". In the course of our proofs we obtain smoothness results for the random measures Y_t^i that are uniform in t. These theorems use a nonstandard description of Y^i and are of independent interest.

1 Introduction

There has been considerable recent interest in measure-valued critical branching Markov processes or superprocesses (see for example Dawson-Iscoe-Perkins (1989), Dynkin (1990b) Fitzsimmons (1988) and Le Gall (1989) to name only a few references). These processes arise as limits of systems of particles undergoing random migration (which in this paper we will take to be Brownian motion in $\mathbb{R}^d$) and random critical branching. The independence of the individual particles makes these models mathematically tractable. At the same time from the point of view of potential applications it is desirable to introduce interactions between colliding particles. In this paper we show collisions between two potentially interacting populations occur typically if $d \leq 5$ and not if $d > 5$ and in the former case construct random measures (collision local times) which measure the number and locations of the collisions. The properties of this local time derived in this work will be used in subsequent work to construct, and in some cases characterize, models in which pointwise interactions occur.

$M_F(\mathbb{R}^d)$ is the set of finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with the weak topology, $b\mathcal{B}(\mathbb{R}^d)$ (respectively C_b) denotes the set of bounded measurable (respectively bounded continuous) functions from $\mathbb{R}^d$ to $\mathbb{R}$, and we write $\nu(\varphi)$ for $\int \varphi d\nu$ and $\nu_1 \leq \nu_2$ if $\nu_2 - \nu_1$ is a measure ($\nu_i \in M_F(\mathbb{R}^d)$). Here then is our central object of interest.

Definition. Let X^1 and X^2 be cadlag $M_F(\mathbb{R}^d)$-valued stochastic processes. If $\epsilon > 0$ and $\varphi \in b\mathcal{B}(\mathbb{R}^d)$, let

$$L_t^\epsilon(X^1, X^2)(\varphi) = \int_0^t \int \int p_\epsilon(x_1 - x_2)\varphi((x_1 + x_2)/2) X_s^1(dx_1) X_s^2(dx_2) ds$$

where $p_\epsilon(x)$ is the Brownian transition density. The *collision local time of* (X^1, X^2) is a cadlag, non-decreasing $M_F(\mathbb{R}^d)$-valued process, $L_t(X^1, X^2)$ such that $L_t^\epsilon(X^1, X^2)(\varphi) \xrightarrow{P} L_t(X^1, X^2)(\varphi)$ as $\epsilon \downarrow 0$ for each $\varphi \in C_b$ and $t \geq 0$.

If $L_t(X^1, X^2)$ exists, it clearly is unique up to evanescent sets. Note that in the definition of $L_t^\epsilon(X^1, X^2)$ $\varphi((x_1 + x_2)/2)$ could be replaced by $\varphi(x_i)$ ($i = 1$ or 2) without altering the definition because of the uniform continuity of φ on compacts. We remark that in other studies of "local time" like objects for measure-valued processes (for example, Adler-Lewin (1990)), the local time is constructed as an L^2 limit rather than a limit in probability. We could also construct the collision local time as an L^2 limit under suitable hypotheses, but these hypotheses would be too restrictive for the future work mentioned above.

For X^1 and X^2 as above, the graph of X^i is the random space-time set

$$G(X^i) = \{(t,x) : t > 0, x \in S(X_t^i)\} \in \mathcal{B}((0,\infty) \times \mathbb{R}^d),$$

where $S(\nu)$ denotes the closed support of the measure ν. The closed graph of X^i is

$$\bar{G}(X^i) = \cup_{\delta>0}\mathrm{cl}(([\delta,\infty) \times \mathbb{R}^d) \cap G(X^i)).$$

Note that $\bar{G}(X^i)$ is the closure of $G(X^i)$ in $(0,\infty) \times \mathbb{R}^d$. If $L_t(X^1,X^2)$ exists let $L(X^1,X^2)$ denote the random measure on $\mathcal{B}((0,\infty) \times \mathbb{R}^d)$ given by

$$L(X^1,X^2)((0,t] \times B) = L_t(X^1,X^2)(B).$$

Hence $S(L(X^1,X^2))$ is the closed support of $L(X^1,X^2)$ in $(0,\infty) \times \mathbb{R}^d$. It is easy to show from the above definition that

$$S(L(X^1,X^2)) \subset \bar{G}(X^1) \cap \bar{G}(X^2) \tag{1.1}$$

and hence the collision local time is supported on the space-times set of collisions between the two "populations".

We introduce some notation to describe super-Brownian motion with immigration. Let P^x denote d-dimensional Wiener measure starting at x, and let P_t and A (on $\mathcal{D}(A)$) denote the Brownian semigroup and infinitesimal generator on the Banach space C_l of continuous functions with a finite limit at infinity. If $\psi \in bp\mathcal{B}(\mathbb{R}^d)$ (the non-negative functions in $\mathcal{B}(\mathbb{R}^d)$), let $U_t\psi(x)$ denote the unique solution of

$$U_t\psi = P_t\psi - \int_0^t P_s((U_{t-s}\psi)^2/2)ds \tag{1.2}$$

(see Fitzsimmons (1988, Proposition 2.3) or Pazy (1983)). Let

$$\begin{aligned} M_{LF} &= M_{LF}([0,\infty) \times \mathbb{R}^d) \\ &= \{\mu : \mu \text{ a measure on } [0,\infty) \times \mathbb{R}^d,\ \mu([0,T) \times \mathbb{R}^d) < \infty \\ &\qquad \text{for all } T > 0,\ \mu(\{t\} \times \mathbb{R}^d) = 0 \text{ for all } t \geq 0\} \end{aligned}$$

with the topology of weak convergence on $[0,T] \times \mathbb{R}^d$ for all $T > 0$. If $s \geq 0$, $m \in M_F(\mathbb{R}^d)$ and $\mu \in M_{LF}$ we consider the following time-inhomogeneous martingale problem on a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ (unless otherwise indicated all filtrations are right-continuous):

$$(M_{s,m,\mu})\quad Y_t(\phi) = m(\varphi) + Z_t(\varphi) + \textstyle\int_s^t Y_r(A\varphi)dr + \int_s^t \int \varphi(x)\mu(dr,dx), t \geq s, \varphi \in \mathcal{D}(A)$$

$$Y_t = 0 \text{ for } t < s$$

$$\{Z_t(\varphi) : t \geq s\} \text{ is a continuous } \mathcal{F}_t-\text{martingale with } Z_s = 0 \text{ and}$$

$$\langle Z(\varphi)\rangle_t = \textstyle\int_s^t Y_r(\varphi^2)dr.$$

A process Y on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a solution of $(M_{s,m,\mu})$ if it is a continuous, adapted $M_F(\mathbb{R}^d)$-valued process satisfying $(M_{s,m,\mu})$.

Theorem 1.1 *(a) There is a solution of $(M_{s,m,\mu})$ and the law, $\mathbb{Q}_{s,m,\mu}$, of any solution of $(M_{s,m,\mu})$ (on $C([0,\infty), M_F(\mathbb{R}^d))$) is unique.*

(b) $\mathbb{Q}_{s,m,\mu}(\exp(-Y_t(\varphi)) = \exp\{-m(U_{t-s}\varphi) - \int_s^t \int U_{t-r}\varphi(x)d\mu(r,x)\}$ *for all $s \leq t$ and $\varphi \in bp\mathcal{B}(\mathbb{R}^d)$.*

(c) If $\Omega^o = C([0,\infty), M_F(\mathbb{R}^d))$, $\mathcal{F}^o = \mathcal{B}(\Omega^o)$ and $\mathcal{F}^o[s,t+] = \cap_n \sigma(Y_r : s \leq r \leq t+n^{-1})$, where $Y_r(w) = w(r)$ on Ω^o, then $(\Omega^o, \mathcal{F}^o, \mathcal{F}^o[s,t+], Y_t, \mathbb{Q}_{s,m,\mu})$ is an inhomogeneous Borel strong Markov process (IBSMP) with continuous paths.

(d) The mapping $(s,m,\mu) \to \mathbb{Q}_{s,m,\mu}$ is Borel measurable, and if $A \in \mathcal{F}^o[s,t+]$ then $\mu \to \mathbb{Q}_{s,m,\mu}(A)$ is $\cap_n \sigma(\mu(A)$: $A \in \mathcal{B}([s,t+n^{-1}] \times \mathbb{R}^d\})$-measurable.

Remarks. 1. An IBSMP is an inhomogeneous strong Markov process with a Borel semigroup $Q^{\mu}_{s,t}f(m) = \mathbb{Q}_{s,m,\mu}(f(Y_t))$ (see Dawson-Perkins (1990, Definition 2.1.0)). It is easy to show that $Q^{\mu}_{s,t}$ extends to an inhomogeneous strongly continuous (in $t \geq s$) semigroup on $C_\ell(M_F(\bar{\mathbb{R}}^d))$ where $\bar{\mathbb{R}}^d$ is the one-point compactification of $\mathbb{R}^d$. The same results held when A is the generator of a Feller process on a locally compact second countable space.

2. When $\mu = 0$ we of course get super-Brownian motion. This is a homogeneous Markov process and we let $\mathbb{Q}^m = \mathbb{Q}_{0,m,0}$ (see Ethier-Kurtz (1986, Ch.9)). For general μ the existence of a unique Markov process satisfying (b) is a special case of Dynkin (1990a, Theorem 1.1). It is easy to prove (a) using the martingale techniques of Roelly-Coppoletta (1986) and Fitzsimmons (1988, 1989). The strong Markov property in (c) then follows as usual and the measurability required in (d) is clear from (b). We leave the details as an exercise.

We will also need a bivariate version of Theorem 1.1. The proof is the same as the one omitted above and carries over to the general Feller setting without change.

Theorem 1.2 *Assume Y^i is a solution of (M_{s,m^i,μ^i}) $(i = 1,2)$ on some $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Assume also that $\langle Z^1(\varphi_1), Z^2(\varphi_2)\rangle_t \equiv 0$ for all $\varphi_i \in D(A)$ (Z^i is the martingale measure in (M_{s,m^i,μ^i})). Then the law of (Y^1, Y^2) is $\mathbb{Q}_{s,m^1,\mu^1} \times \mathbb{Q}_{s,m^2,\mu^2}$.*

Thus (Y^1, Y^2) are independent $(\mathcal{F}_t)$-super-Brownian motions if they satisfy the hypotheses of Theorem 1.2 and $\mu_1 = \mu_2 = 0$. For $d \leq 3$, Evans-Perkins (1989) showed that in this case $L_t(Y^1, Y^2)$ exists and is absolutely continuous in t (see Remarks 5.12 (4)). In Section 3 we prove (Theorem 3.6) that if $d \geq 6$, then the closed supports of Y^1 and Y^2 do not intersect for all $t > 0$ and hence if the collision local time exists it must be zero. In Section 5 we will show (Proposition 5.11) these supports do intersect at some $t > 0$ with positive probability if $d \leq 5$ by proving the existence of a non-trivial collision local time. To establish the non-existence of collisions in the critical case when $d = 6$, we introduce (in Section 3) the random measure (on $[0, \infty) \times \mathbb{R}^d$)

$$V(A) = \int_0^\infty \int 1_A(s,x) Y_s(dx) ds$$

(Y a super-Brownian motion) and show it is comparable to the restricted Hausdorff measure of Taylor and Watson (1985) on the graph of Y (Theorem 3.1). The arguments here are similar to those used by Dawson-Iscoe-Perkins (1989) (hereafter abbreviated D.I.P. (1989)) to analyze the "range" of Y. Routine extensions of these arguments give the non-existence of higher-order collisions in the critical cases (see Remarks 3.7).

Recently Dynkin (1990c) has completely characterized those sets which intersect with $\bar{G}(Y)$ with positive probability (his results also apply to α-stable branching mechanisms). These precise results, in conjunction with an elementary estimate such as Proposition 3.3, would also establish the above results on existence and non-existence of collisions, and would do so in greater generality. The results on the Taylor-Watson restricted Hausdorff measure are of independent interest and constitute a useful probabilistic tool especially when used in conjunction with Dynkin's results, just as the Taylor-Watson results complement the classical parabolic capacity results for Brownian motion.

In Section 4 (see Theorem 4.7) upper bounds are obtained for $\sup_x Y_t(B(x,r))$ as $r \downarrow 0$ (Y super-Brownian motion). (Here $B(x,r)$ denotes the open ball and $\bar{B}(y,r)$ will denote the closed ball.) These results are used in Section 5 but the reason for deriving a very precise asymptotic bound for small t is its use in future work where it will be used to explicitly describe the unique law of a pair of interacting populations.

Section 5 contains the main result of this paper, a Tanaka formula for $L_t(X^1, X^2)(\varphi)$ when $d \leq 5$. Here X^1, X^2 are continuous, adapted $M_F(\mathbb{R}^d)$-valued processes on some $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}')$ such that

$$X_t^i(\varphi) = m^i(\varphi) + \int_0^t X_s^i(A\varphi)ds + Z_t^i(\varphi) - A_t^i(\varphi), \qquad \varphi \in \mathcal{D}(A), \ \ i = 1,2,$$

(M_{m^1,m^2}) $\quad Z_t^i(\varphi)$ $(i = 1,2)$ are continuous $\mathcal{F}'_t$-martingales satisfying

$$\langle Z^i(\varphi_i), Z^j(\varphi_j)\rangle_t = \delta_{ij} \int_0^t X_s^i(\varphi_i^2)ds,$$

A_t^i $(i = 1,2)$ are non-decreasing, continuous, $\mathcal{F}'_t$-adapted $M_F(\mathbb{R}^d)$-valued processes, each starting at 0.

Let $\mathcal{M} = \mathcal{M}(m^1, m^2)$ denote the set of all continuous adapted processes (X^1, X^2) satisfying (M_{m^1,m^2}) for a given pair (m^1, m^2) in $M_F(\mathbb{R}^d)^2$. The Tanaka formula (Theorem 5.9) exhibits $L_t(X^1, X^2)(\varphi)$ in the semimartingale decomposition of $(X_t^1 \times X_t^2)(\psi)$ for appropriate ψ. Many of the terms involve singular stochastic integrals and the results of Section 4 are used to control these expressions. The Tanaka formula establishes the continuity of $L_t(X^1, X^2)$ in $(X^1, X^2) \in \mathcal{M}$ (as well as t). A key result in this direction is the fact that in $d \leq 3$ the rate of convergence of $L_t^\epsilon(X^1, X^2)$ to $L_t(X^1, X^2)$ is uniform in $(X^1, X^2) \in \mathcal{M}$ (and t) (Theorem 5.10). This will allow us to easily prove existence theorems for pointwise interacting superprocesses in a forthcoming work. More specifically, in this forthcoming work we will prove existence and uniqueness theorems for M_{m^1,m^2} when $A_t^i = L_t(X^1, X^2)$ for $i = 1,2$. For now, however, the reader should treat $\mathcal{M}_{m^1,m^2}$ as a working hypothesis that could potentially include other types of interaction. The case $A^1 = A^2 = 0$ (independent super-Brownian motions), which will be a setting for a good part of this work, shows that we are not working in a vacuum.

Our results in Section 5 were motivated by a Tanaka formula for the ordinary local time of the super α-stable process ($d < 2\alpha$) in Adler-Lewin

(1989). In Section 6 we show how the bounds of Section 4 lead to a simple proof of a slightly more general formula in the Brownian setting.

Section 7 contains a technical estimate needed to control the martingale terms in the Tanaka formula of Section 5.

The system of approximating branching Brownian motions is introduced in Section 2 along with an associated nonstandard model. These are used in Sections 3 and 4. Those unfamiliar with nonstandard analysis should be able to still follow these arguments using the appropriate weak convergence techniques. The historical process (Dawson-Perkins (1990), Dynkin (1990a), Le Gall (1989)) would give another approach here but the nonstandard setting allows us to refer more easily to parallel arguments in D.I.P (1989).

The process Z_t arising in $(M_{s,m,\mu})$ is an orthogonal martingale measure. As in Walsh (1986, Ch.2) we can extend the stochastic integral $Z_t(\varphi) = \int_0^t \int \varphi(x) dZ(s,x)$ to $\int_0^t \int \varphi(s,w,x) dZ(s,x)$ where $\varphi(s,w,x)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable ($\mathcal{P}$ is the predictable σ-field on $[0,\infty) \times \Omega$) and

$$\mathbb{P}\Big(\int_0^t \int \varphi(s,w,x)^2 X_s(dx)ds\Big) < \infty \quad \text{for all} \quad t \geq 0.$$

The resulting stochastic integral (still denoted $Z_t(\varphi)$) is a continuous L^2-martingale satisfying

$$\langle Z(\varphi)\rangle_t = \int_0^t X_s(\varphi_s^2)ds.$$

The same extension holds for processes Z_t^i in (M_{m^1,m^2}). These extensions will be used without further comment. c_1, $c_2 \ldots$ will denote positive constants arising in the course of an argument. Positive constants introduced in Section i which arise in subsequent arguments are denoted $c_{i.1}, c_{i,2}, \ldots$.

2 Branching Particle systems and the Nonstandard Model

We will introduce a system of branching Brownian motions which converge weakly to a super-Brownian motion Y. Section 2 of Perkins (1988) contains some additional properties of the labelling scheme we now introduce.

Let $I = \cup_{n=0}^{\infty} \mathbb{Z}_+ \times \{0,1\}^n$ and if $\beta = (\beta_0, \ldots, \beta_i) \in I$ let $|\beta| = i$ and $\beta|j = (\beta_0, \ldots, \beta_j)$ for $j \leq i$. Let $\{B^\beta : \beta \in I\}$ be a collection of i.i.d.

Brownian motions and let $\{e^\beta : \beta \in I\}$ be a collection of i.i.d. fair coin-tossing random variables taking on the values 0 and 2. These two collections are independent and are defined on a common $(\Omega, \mathcal{A}, \hat{\mathbb{Q}})$. Given $\mu \in \mathbb{N}$ and $\{x_1, \ldots, x_K\} \subset \mathbb{R}^d$ define a system of critical branching Brownian particles $\{N^\beta : \beta \in J\}$ as follows. N^β starts at x_{β_0} if $\beta_0 \le K$ and N^β is identically Δ (the point at infinity in $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\Delta\}$) if $\beta_0 > K$. If $j \le |\beta|$ and $N^\beta_{j/\mu} \ne \Delta$, then $N^\beta_t - N^\beta_{j/\mu} = B^{\beta|j}_t - B^{\beta/j}_{j|\mu}$ for $t \in [j/\mu, (j+1)/\mu)$ and

$$N^\beta_{(j+1)/\mu} - N^\beta_{j/\mu} = \begin{cases} B^{\beta|j}_{(j+1)/\mu} - B^{\beta|j}_{j/\mu} & \text{if } e^{\beta|j} = 2 \text{ and } j < |\beta| \\ \Delta & \text{if } e^{\beta|j} = 0 \text{ or } j = |\beta| \end{cases}.$$

Once N^β hits Δ it stays there.

Hence $\{N^\beta : \beta \in I\}$ describes a system of particles which follow independent Brownian motions on each $[j/\mu, (j+1)/\mu)$ and independently die or split into two with equal probability at each time $(j+1)/\mu$.

Write $\beta \sim t$ if $\beta \in I$ satisfies $|\beta|/\mu \le t < (|\beta|+1)/\mu$ and define a measure-valued process by

$$N^{(\mu)}_t(A) = N_t(A) = \mu^{-1} \sum_{\beta \sim t} 1_A(N^\beta_t) \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Fix an initial measure $m \in M_F(\mathbb{R}^d)$. If $N^{(\mu)}_0 \to m$ in $M_F(\mathbb{R}^d)$ (allow $(x_i)_{i \le K}$ to depend on μ) then S. Watanabe showed that $N^{(\mu)}_\cdot$ converges weakly to $\mathbb{Q}^m(Y \in \cdot)$, the law of super-Brownian motion, on $D([0,\infty), M_F(\mathbb{R}^d))$ (see Perkins (1990, Theorem 2.2)). It will be convenient to use the nonstandard formulation of this result.

In the next two sections we will work on w_1-saturated enlargement of a super-structure containing $\mathbb{R}$ and $(\Omega, \mathcal{A}, \hat{\mathbb{Q}})$. Fix $\eta \in {}^*\mathbb{N} - \mathbb{N}$ and let $\mu = 2^\eta$. Choose $K \in {}^*\mathbb{N} - \mathbb{N}$ and $(x_i : i \le K)$ an internal sequence in ${}^*\mathbb{R}^d$ such that $st_d(N^{(\mu)}_0) = m$, where st_d is the standard part map on ${}^*M_F(\mathbb{R}^d)$ and $N^{(\mu)}_\cdot$ is the internal ${}^*M_F(\mathbb{R}^d)$-valued process defined as above on $({}^*\Omega, {}^*\mathcal{A}, {}^*\hat{\mathbb{Q}})$. If $\hat{\mathbb{Q}}^m = L({}^*\hat{\mathbb{Q}})$, the Loeb measure on the measurable space $({}^*\Omega, L({}^*\mathcal{A}))$, then (DIP (1989, Thm. 2.3)) $N^{(\mu)}_\cdot$ is $\hat{\mathbb{Q}}^m$-a.s. S-continuous and $Y(t) = st(N^{(\mu)})(t)$ (st is the standard part map on ${}^*D([0,\infty), M_F(\mathbb{R})))$ is a super-Brownian motion starting at m under $\hat{\mathbb{Q}}^m$. (Define $Y \equiv 0$ on the $\hat{\mathbb{Q}}^m$-null set on which $N^{(\mu)}$ is not S-continuous.)

The following Lévy modulus was proved in DIP (1989, Thm. 4.5) as a standard result. The nonstandard formulation we now give is then immediate. We let $h(v) = ((v \wedge e^{-1})(\log 1/(v \wedge e^{-1})))^{1/2}$. The function h is non-decreasing.

Theorem 2.1 *If $c > 2$ there are positive constants $c_{2.1}(c)$, $c_{2.2}(c)$ and $c_{2.3}(c)$ and a random variable $\delta(w, c)$ on $({}^*\Omega, L({}^*\mathcal{A}), \hat{Q}^m)$ such that*

$$\hat{Q}^m(\delta \leq \rho) \leq c_{2.1} m(\mathbb{R}^d)\rho^{c_{2.2}} \quad \text{for} \quad 0 \leq \rho \leq c_3. \tag{2.1}$$

(2.2) *If* $0 < t - s \leq \delta(w, c)$, $s, t \in {}^*[0, \infty)$, $\beta \sim t$ *and* $N_t^\beta \neq \Delta$, *then*

$$|N_t^\beta - N_s^\beta| < ch(t - s).$$

Notation. $T = \{j/\mu : j \in {}^*\mathbb{N}_0\}$, $\mathcal{T}$ is the set of internal subsets of T, and $\lambda = \lambda^\mu$ is the internal measure on $(T, \mathcal{T})$ which assigns mass μ^{-1} to each point.

3 Non-existence of Collisions for $d \geq 6$

In showing that two independent super-Brownian motions do not collide, and hence can only have a trivial collision local time, if $d = 6$, we use techniques similar to those of Sections 3 and 5 of DIP (1989). In particular we work in the nonstandard setting introduced at the end of the previous section.

Definition. V denotes the random (finite) measure on $\mathcal{B}([0, \infty) \times \mathbb{R}^d)$ defined by

$$V(A) = \int_0^\infty \int_{\mathbb{R}^d} 1_A(s, x) Y_s(dx) ds.$$

$U = U^\mu$ denotes the random internal measure on $T \times {}^*\mathcal{B}(\mathbb{R}^d)$ defined by

$$U(A) = \int_T \int_{{}^*\mathbb{R}^d} 1_A(s, x) N_s(dx) d\lambda(s).$$

It is clear that $st_M(U) = V$ as where st_M denotes the standard part map on ${}^*M_F([0, \infty) \times \mathbb{R}^d)$ and we consider U also as an internal measure on ${}^*([0, \infty) \times \mathbb{R}^d)$. We want to show V distributes its mass over the graph of Y,

$G(Y)$, in a uniform manner. Theorems of this type were proved in Perkins (1989) and DIP (1989) for Y_t and $\int_0^T Y_t dt$, respectively, where these measures were shown to be equivalent with the appropriate Hausdorff measures on their supports. The scaling properties of Brownian motion suggest a modified Hausdorff measure used by Taylor and Watson (1985)-see also the earlier work of Hawkes (1978).

Let $C(x,r)$ $(\subset \mathbb{R}^d)$ denote the closed cube of side-length r and "lower left-hand corner" x, and let

$$C(t,x,r) = [t,t+r^2] \times C(x,r) \subset [0,\infty) \times \mathbb{R}^d \quad (t \geq 0,\ x \in \mathbb{R}^d)$$

and

$$C_0(t,x,r) = (t,t+r^2] \times \prod_{i=1}^{d}(x_i, x_i + r] \subset (0,\infty) \times \mathbb{R}^d.$$

If $C \subset [0,\infty) \times \mathbb{R}^d$, $B \subset \mathbb{R}^d$ and $r > 0$, then

$$C^r = \{(t,x) \in [0,\infty) \times \mathbb{R}^d : |t-t'| \leq r^2 \text{ and } |x-x'| \leq r \text{ for some } (t',x') \in C\}$$

$$B^r = \{x \in \mathbb{R}^d : |x - x'| \leq r \text{ for some } x' \in B\}.$$

Definition. If $f : [0,\epsilon) \to [0,\infty)$ is non-decreasing near 0 and $f(0+) = 0$, let

$$q^f(A) = \lim_{\delta \downarrow 0} \inf\{\sum_{i \geq 1}^{\infty} f(r_i) : A \subset \cup_{i=1}^{\infty} C(t_i, y_i, r_i),\ r_i < \delta\}, \quad A \subset [0,\infty\} \times \mathbb{R}^d.$$

q^f is a measure on $\mathcal{B}([0,\infty) \times \mathbb{R}^d)$ and if f varies regularly at 0 (as will be the case for the only f we will consider) q^f is a constant multiple of the restricted Hausdorff measure $P - f - m$ introduced by Taylor and Watson (1985). Let us set $f(r) = r^4 \log\log(1/r)$.

Theorem 3.1 *If* $d \geq 5$ *there are constants* $0 < c_{3.1} \leq c_{3.2} < \infty$ *such that for any* $m \in M_F(\mathbb{R}^d)$

$$c_{3.1} q^f(A \cap \bar{G}(Y)) \leq V(A) \leq c_{3.2} q^f(A \cap G(Y)) \text{ for all } A \in \mathcal{B}((0,\infty) \times \mathbb{R}^d)\ \mathbb{Q}^m\text{–}a.s.$$

This result will be proved by making minor changes in the proof of Theorem 14. of DIP (1989) which gives a similar connection between $\int_0^t Y_s ds$ and the ordinary Hausdorff f-measure of the range of Y (both are measures on $\mathbb{R}^d$). We will need some notation and terminology from DIP (1989).

Notation. $a_n = 2^{-n/2}$, $j_n = 2^{n+1}$, $b_n = 2^{-j_n}$,

$$\Lambda_n = \{C(t,x,2^{-n/2}) : t = (n_0+e_0)2^{-n}, x_i = (n_i+e_i)2^{-n/2}\ (1 \le i \le d), n_0 \in \mathbb{N}_0,$$

$$n_i \in \mathbf{Z}\ \ (i \ge 1), e_i = 0 \ \text{ or }\ 1/2\},$$

$$\Lambda_{n,L} = \{C \in \Lambda_n : C \subset [L^{-1}, L] \times [-L,L]^d\},$$

$$c_{3.3} = P^o(|B_2| \le 1/2)/24,$$

$$I(t,\epsilon) = \{\gamma \in I : \gamma \sim t-\epsilon \ \text{ there is a }\ \beta \sim t\ \text{ such that }\ \beta|[\mu(t-\epsilon)] = \gamma\ \text{ and }$$

$$N_t^\beta \ne \Delta\},\ \ 0 \le \epsilon \le t\ ([x]\ \text{ is the integer part of } x).$$

Definition. $C \in \Lambda_{j_n}$ *is bad* iff $U({}^*C) > 0$ and $U({}^*C^{7a_K}) \le c_{3.3} f(a_K)$ for $K = 2^n, \ldots, 2^{n+1} - n$.

Lemma 3.2 *There are constants $c_{3.4}(L)$ $(L \in \mathbb{N})$ and $c_{3.5}$ such that for $L \in \mathbb{N}$ and n sufficiently large $(n \ge n_0(L))$,*

$$\text{(3.1)}\qquad \hat{Q}^m(C\ \ \textit{bad and}\ \ 2b_n < \delta(w,c))$$

$$\le c_{3.4}(L)m(1)2^{-2^n(d-2)}2^{nd/2}\exp(-c_{3.5}2^{n/2})\quad \textit{for all}\ \ C \in \Lambda_{j_n,L}.$$

Proof. Let $C = [t, t+b_n] \times C_1 \in \Lambda_{j_n}$ and assume C is bad. Assume in addition that $7b_n \le L^{-1}$ and $2b_n < \delta(w,3)$. Since $V({}^*C) > 0$ we may choose $\beta \sim t'$ such that $t' \in {}^*[t, t+b_n]$ and $N_{t'}^\beta \in {}^*C_1$. Clearly $\gamma = \beta|[\mu(t-b_n)] \in I(t, b_n)$. Since $|t - b_n - t'| \le 2b_n < \delta(w,3)$ we have

$$|N_{t-b_n}^\gamma - N_{t'}^\beta| \le 3h(2b_n) \le 6a_k\ \text{ whenever }\ k \le 2^{n+1} - n,$$

and therefore

$$\text{(3.2)}\qquad N_{t-b_n}^\gamma \in C_1^{3h(2b_n)},$$

and

$$[t - 7a_k^2, t] \times \bar{B}(N_{t-b_n}^\gamma, a_k) \subset [t - 7a_k^2, t) \times \bar{B}(N_{t'}^B, 7a_k) \subset C^{7a_k}$$

$$\text{for all } \; k = 2^n, \ldots, 2^{n+1} - n.$$

As C is bad this latter inclusion means that

$$(3.3) \quad U([t - 7a_k^2, t] \times \bar{B}(N_{t-b_n}^{\gamma} a_k)) \le c_{3.3} f(a_k) \;\text{ for }\; k = 2^n, \ldots, 2^{n+1} - n.$$

We have shown that if $7b_n \le L^{-1}$ then w.p.l. C bad and $2b_n < \delta(w,3)$ implies there is a $\gamma \in I(t, k_n)$ satisfying (3.2) and (3.3), which in turn implies the conclusion of Lemma 5.1 of DIP (1989). Now argue just as in the derivation of Proposition 5.6 of DIP (1989) to complete the proof. The only difference is that since t is fixed there is no need to sum over the time grid $\{(j+1)b_n\} \cap (L^{-1}, L + b_n]$ and this results in the factor of $2^{-2^n(d-2)}$ instead of the $2^{-2^n(d-4)}$ obtained in DIP (1989, Prop. 5.6). □

Proof of Theorem 3.1. Let $L \in \mathbb{N}$. If $n \ge n_0(L)$ then since $\mathrm{card}(\Lambda_{j_n,L}) \le 2 \cdot 4^d L^{d+1} 2^{j_n(1+d/2)}$, Lemma 3.2 implies

$$\hat{Q}^m (1(2b_n < \delta(w,3)) \sum_{C \in \Lambda_n, L} 1(C \;\text{ bad }) f(2^{(-j_n/2)}))$$

$$\le c_1(L) m(1) n 2^{nd/2} \exp\{-c_{3.5} 2^{n/2}\} \equiv \epsilon_n(L)^2.$$

Since $\sum_n \epsilon_n(L) < \infty$, the above together with (2.1) allows us to conclude there is an $N_0(L, w) < \infty$ a.s. such that

$$(3.4) \qquad \sum_{C \in \Lambda_{j_n,L}} 1(C \;\text{ bad }) f(2^{-j_n/2}) \le \epsilon_n(L) \;\text{ for all }\; n \ge N_0(L; w).$$

Fix w outside a null set so that $N_0(L, w) < \infty$ for all $L \in \mathbb{N}$ and $V = st_M(U)$. Introduce

$$\Lambda_{j_n,L}^b = \{C \in \Lambda_{j_n,L} : C \;\text{ bad }\}$$

$$\Lambda_{j_n,L}^{g,1} = \{C \in \Lambda_{j_n,L} : U({}^*C) > 0,\; C \;\text{ good (i.e. not bad)}\}.$$

If $C \in \Lambda_{j_n,L}^{g,1}$ choose $k \in \{2^n, \ldots, 2^{n+1} - n\}$ such that $V({}^*C^{7a_k}) > c_{3.3} f(a_k)$. $C^{7a_k} \subset [t, t+r] \times C_1$, where C_1 is a closed cube of side-length

$$2^{-2^n} + 14a_k \le 15a_k < a_{k'}/2$$

where $k' = k - 10$ or $k - 11$ whichever is even (hence $k' \in [2^n - 11, 2^{n+1} - n - 10]$), and $r = 2^{-2^{n+1}} + 2(7a_k)^2 < (a_{k'}/2)^2$. Therefore there is a C' in $\Lambda_{k'}$

such that $C^{7a_k} \subset C'_0$. Let $\Lambda^{g,2}_{j_n,L}$ be the set of cubes obtained by choosing one such C' for each C in $\Lambda^{g,1}_{j_n,L}$. Note that

$$U(\,{}^*C'_0) \geq U(\,{}^*C^{7a_k}) > c_{3.3} f(a_k) \geq c_{3.3} 2^{-22} f(a_{k'}) \tag{3.5}$$

$$\text{for all } C' \text{ in } \Lambda^{g,2}_{j_n,L}.$$

As in the proof of Lemma 3 of Taylor and Watson (1985) there is a $\Lambda^{g}_{j_n,L} \subset \Lambda^{g,2}_{j_n,L}$ such that

$$\bigcup_{\Lambda^g_{j_n,L}} C = \bigcup_{\Lambda^{g,2}_{j_n,L}} C \supset \bigcup_{C \in \Lambda^{g,1}_{j_n,L}} C \tag{3.6}$$

and

$$\text{no point in } (0,\infty) \times \mathbb{R}^d \text{ is covered by more than} \tag{3.7}$$

$$2^{d+1} \text{ cubes in } \{C_0 : C \in \Lambda^g_{j_n,L}\}$$

(the latter property is the reason we chose k' even and must sometimes work with the semi-closed cubes C_0).

Let $A \subset [L^{-1}, L] \times [-L, L]^d$ be compact. Since $st_M(U) = V$ and each C in $\Lambda_{j_n,L}$ is closed, we have (writing G and $\bar{G}$ for $G(Y)$ and $\bar{G}(Y)$, respectively)

$$A \cap \bar{G} \subset (\cup_{\Lambda^b_{j_n,L}} C) \cup (\cup'_{\Lambda^g_{j_n,L}} C) \tag{3.8}$$

where $\cup'$ means the union is taken over those C which intersect A (note that if $x \in A \cap \bar{G}$ then $x \in$ Int (C) for some $C \in \Lambda_{j_n,L}$ satisfying $V(C) > 0$). If $n \geq N_0(L, w)$ then (3.4) and (3.5) show that

$$\sum_{C \in \Lambda^b_{j_n,L}} f(2^{-j_n/2}) + \sum_{C \in \Lambda^g_{j_n,L}} 1(C \cap A \neq \emptyset) f(2^{-j_n/2})$$

$$\begin{aligned}
&\leq \epsilon_n(L) + \sum_{C \in A^g_{j_n,L}} 1(C \cap A \neq \emptyset)\, c_2^o\, U(\,{}^*C_0) \\
&\leq \epsilon_n(L) + c_2 2^{d+1}\, {}^oU(\,{}^*A^{a_{2n-11}}) \quad (\text{ by } (3.7)) \\
&\to c_2 2^{d+1} V(A) \text{ as } n \to \infty,
\end{aligned}$$

where we have used the compactness of A in the last. In light of (3.8), this shows

$$q^f(A \cap \bar{G}) \leq c_2 2^{d+1} V(A) \tag{3.9}$$

for all compact A (let $L \to \infty$) and hence all Borel sets $A \subset (0,\infty) \times \mathbb{R}^d$ by the inner regularity of the finite measures $q^f(\cdot \cap \bar{G})$ and $V(\cdot)$.

Consider now the upper bound on V. By Theorem 1.4 of DIP (1989) we may fix w outside a null set such that

$$\int_r^s Y_u(B)du \leq c_3 f-m(R(r,s] \cap B) \text{ for all } B \in \mathcal{B}(\mathbb{R}^d), \quad 0 < r \leq s \leq \infty,$$

where $R(r,s] = \cup_{r<u\leq s} S(Y_u)$ and $f-m$ denotes the ordinary f-Hausdorff measure (see also Proposition 4.7 of Perkins (1990)). Let $0 < r \leq s$, $B \in \mathcal{B}(\mathbb{R}^d)$ and $A = (r,s] \times B$. Choose a sequence of covers $\{C(t_i^n, x_i^n, r_i^n) : i \in \mathbb{N}\}$ $(n \in \mathbb{N})$ of $A \cap G$ such that

$$\lim_{n\to\infty} \sum_i f(r_i^n) = q^f(A \cap G), \quad \lim_{n\to\infty} \sup_i r_i^n = 0. \tag{3.10}$$

Then $\{C(x_i^n, r_i^n) : i \in \mathbb{N}\}$ covers $B \cap R(r,s]$ and therefore

$$\begin{aligned} V(A) = \int_r^s Y_u(B)du &\leq c_3 f - m(R(r,s] \cap B) \\ &\leq c_3 \lim_{n\to\infty} \sum_i f(c_4 r_i^n) \\ &\leq c_5 q^f(A \cap G) \quad \text{(by (3.10))}. \end{aligned}$$

For a fixed $M \in \mathbb{N}$ this gives the required upper bound first for A in the field of subsets of $(M^{-1}, M] \times \mathbb{R}^d$ of the form

$$\bigcup_{i=1}^n (r_i, s_i] \times B_i, \quad M^{-1} \leq r_1 \leq s_1 \leq r_2 \ldots \leq s_n \leq M, \ B_i \in \mathcal{B}(\mathbb{R}^d),$$

and then for all A in $\mathcal{B}((M^{-1}, M] \times \mathbb{R}^d)$ (note that $q^f(\cdot \cap G)$ is a.s. a finite measure by (3.9)). Let $M \to \infty$ to complete the proof. □

In DIP (1989, Theorem 3.1(a)) it was shown that for $d \geq 3$

$$\mathbb{Q}^m(Y_t(B(x,\epsilon)) > 0) \leq c_{3.6} t^{-d/2} m(1) \epsilon^{d-2} \text{ for all } t \geq \epsilon^2. \tag{3.11}$$

In fact exact asymptotics for the left-hand side as $\epsilon \downarrow 0$ were found. We now use the same techniques to prove a slightly stronger result.

Proposition 3.3 *Assume* $d \geq 3$.

$$Q^m\left(\int_t^{t+\epsilon^2} Y_s(B(x,\epsilon))ds > 0\right) \leq c_{3.7} t^{-d/2} m(1) \epsilon^{d-2} \quad \text{for all} \quad t \geq 4\epsilon^2.$$

Proof. If $\epsilon > 0$ and $x \in \mathbb{R}^d$ let

$$m(\epsilon,x)(A) = \epsilon^{-2} \int 1_A((y-x)/\epsilon) dm(y).$$

Then

$$\begin{aligned} q(m,t,\epsilon) &\equiv Q^m\left(\int_t^{t+\epsilon^2} Y_s(B(x,\epsilon))ds > 0\right) \\ &= 1 - \lim_{\theta\to\infty} Q^m\left(\exp\left\{-\theta \int_t^{t+\epsilon^2} Y_s(B(x,\epsilon))ds\right\}\right) \\ &= 1 - \lim_{\theta\to\infty} Q^{m(\epsilon,x)}\left(\exp\left\{-\theta\epsilon^4 \int_{t/\epsilon^2}^{t/\epsilon^2+1} Y_s(B(0,1))ds\right\}\right) \\ &= 1 - \lim_{\theta\to\infty} Q^{m(\epsilon,x)}\left(Q^{Y_{t/\epsilon^2}}\left(\exp\left\{-\theta\epsilon^4 \int_0^1 Y_s(B(0,1))ds\right\}\right)\right) \end{aligned}$$

by scaling, spatial homogeneity and the Markov property. Use Iscoe (1986, Theorem 3.1) and Lemma 3.5 of DIP (1989) to see there is a non-negative function $u^\infty(x)$ ($u^\infty(1,x)$ in the notation of Lemma 3.5 of DIP (1989)) such that

$$q(m,t,\epsilon) = 1 - Q^{m(\epsilon,x)}(\exp\{-Y_{t/\epsilon^2}(u^\infty)\})$$

and, if $T_r = \inf\{t : |B_t| \leq r\}$, then

$$\begin{aligned} u^\infty(x) &\leq c_1 P^x(T_{3/2} \leq 1) \quad \text{for all} \quad |x| \geq 2 \\ &\leq c_1 P^o\left(\sup_{s\leq 1} |B_s| > |x|/4\right) \\ &\leq c_2 \exp(-|x|^2/34) \quad \text{(reflection principle)} \\ &\equiv g(x). \end{aligned}$$

From Theorem 1.1 (with $\mu \equiv 0$) we see that for $t \geq 4\epsilon^2$

$$\begin{aligned} q(m,t,\epsilon) &\leq 1 - Q^{m(\epsilon,x)}(\exp(-Y_{t/\epsilon^2}(g))) + Q^{m(\epsilon,x)}(Y_{t/\epsilon^2}(B(0,2)) > 0) \\ &= 1 - \exp\{-m(\epsilon,x)(U_{t/\epsilon^2}g)\} + c_{3.6}\epsilon^d t^{-d/2} \epsilon^{-2} m(1) 2^{d-2} \quad \text{(by (3.11))}. \end{aligned}$$

An elementary comparison theorem (e.g. Lemma 3.0 of DIP (1989)) allows us to dominate $U_{t/\epsilon^2}g(y)$ by $P_{t/\epsilon^2}g(y)$ (P_t is the Brownian semigroup) and the latter equals $c_3 p_{t/\epsilon^2+17}(y)$ ($p_t(y)$ is the Brownian transition density). Therefore for $t \geq 4\epsilon^2$,

$$\begin{aligned} q(m,t,\epsilon) &\leq 1-\exp\{-\epsilon^{-2}c_3\int p_{t/\epsilon^2+17}((y-x)/\epsilon)dm(y)\} \\ &\quad +c_{3.6}t^{-d/2}2^{d-2}m(1)\epsilon^{d-2} \\ &\leq c_{3.7}t^{-d/2}m(1)\epsilon^{d-2}. \quad \square \end{aligned}$$

Remark 3.4. By using the more precise estimate given in Theorem 3.1(b) of DIP (1989) and being slightly more careful in the above argument one can show that

$$\mathbb{Q}^m\Big(\int_t^{t+\epsilon^2} Y_s(B(x,\epsilon))ds > 0\Big) \leq c_{3.8}\epsilon^{d-2}\Big(\int p_t(y-x)dm(y)+c(\epsilon,\delta,m)\Big) \text{ for all } t \geq \delta$$

where $\lim_{\epsilon\downarrow 0} c(\epsilon,\delta,m) = 0$ for each $\delta > 0$, $m \in M_F(\mathbb{R}^d)$. Theorem 3.1(b) of DIP (1989) shows this bound is best possible up to the value of $c_{3.8}$.

The following elementary corollary is also a consequence of the more precise results of Dynkin (1990c) and the general results of Taylor-Watson (1985).

Write $q^\alpha(A)$ for $q^g(A)$ when $g(r) = r^\alpha$.

Corollary 3.5. *Assume $d \geq 3$. If $A \subset (0,\infty)\times\mathbb{R}^d$ satisfies $q^{d-2}(A) = 0$, then $A \cap \bar{G}(Y) = \emptyset$ $\mathbb{Q}^m$-a.s.*

Proof. We may assume $A \subset [\delta,\infty)\times\mathbb{R}^d$ for some $\delta > 0$. If $q^{d-2}(A) = 0$ there is a sequence $\{C(t_i^n,x_i^n,r_i^n) : i \in \mathbb{N}\}$ ($n \in \mathbb{N}$) such that $A \subset \cup_{i=1}^\infty \text{Int}\,(C(t_i^n,x_i^n,r_i^n))$ and $\lim_{n\to\infty}\sum_i (r_i^n)^{d-2} = 0$.

$$\begin{aligned} \mathbb{Q}^m(A\cap\bar{G}(Y)\neq\emptyset) &\leq \sum_{i=1}^\infty \mathbb{Q}^m\Big(\int_{t_i^n}^{t_i^n+(r_i^n)^2} Y_s(C(x_i^n,r_i^n))ds > 0\Big) \\ &\leq c_1\delta^{-d/2}m(1)\sum_{i=1}^\infty (r_i^n)^{d-2} \quad \text{Proposition 3.3)} \\ &\to 0 \text{ as } n\to\infty. \quad \square \end{aligned}$$

Theorem 3.6. *Assume $d \geq 6$. If Y^1, Y^2 are independent super-Brownian motions starting at m^1 and m^2, respectively, then $\bar{G}(Y^1) \cap \bar{G}(Y^2) = \emptyset$ and hence $S(Y_t^1) \cap S(Y_t^2) = \emptyset$ for all $t > 0$ $\mathbb{Q}^{m^1} \times \mathbb{Q}^{m^2}$-a.s.*

Proof. Theorem 3.1 implies $q^{d-2}(\bar{G}(Y^1)) = 0$ $\mathbb{Q}^{m^1}$-a.s. if $d \geq 6$. Hence Corollary 3.5 and a Fubini argument shows that $\bar{G}(Y^1) \cap \bar{G}(Y^2) = \emptyset$ a.s. if $d \geq 6$. □

Remark 3.7 (i) It is easy to use Theorem 3.1 and Proposition 3.3 to see that $q^g(\bar{G}(Y^1) \cap \bar{G}(Y^2) \cap ([\delta, \infty) \times \mathbb{R}^d)) < \infty$ for all $\delta > 0$ $\mathbb{Q}^{m^1} \times \mathbb{Q}^{m^2}$-a.s. where $g(r) = r^{6-d} \log\log 1/r$ $(d \geq 3)$. As above one can then show that for $d \geq 4$, if Y^i $(i = 1, 2, 3)$ are independent super-Brownian motions then $\cap_{i=1}^3 S(Y_t^i) = \emptyset$ for all $t > 0$ a.s. Similarly there are no quintuple collisions in three or more dimensions.

(2) The non-existence of collisions between independent super-Brownian motions for $d \geq 7$ is a simple consequence of Proposition 3.3 alone. Theorem 3.1 was needed to treat the critical six-dimensional case.

4 Uniform Regularity Results for Super-Brownian Motion

In this section we derive uniform (in (t, x)) bounds on $Y_t(B(x,r))$ as $r \downarrow 0$ for the super-Brownian motion Y. Estimates on $Y_t(B(x,r))$ for a typical x in $S(Y_t)$ were obtained in Perkins (1988) to find an exact Hausdorff measure function associated with Y_t but additional work seems necessary to get uniform bounds. We continue to work with the nonstandard model $N^{(\mu)}$ $(\mu = 2^\eta \in {}^*\mathbb{N} - \mathbb{N})$ introduced in Section 2. This richer nonstandard model allows us to define the following processes, originally introduced in Perkins (1990, Sec. 4). If $r \in {}^*[0, \infty)$, let $\underline{r} = [r\mu]/\mu$ where $[x]$ denotes the integer part of x. If $B \in \mathcal{B}(\mathbb{R}^d)$ and $r \geq 0$, let

$$N_t^{r,B}(A) = \mu^{-1} \sum_{\gamma \sim \underline{r}+t} 1(N_{\underline{r}}^\gamma \in {}^*B, N_{t+\underline{r}}^\gamma \in A), \quad t \in {}^*[0, \infty), \quad A \in {}^*\mathcal{B}(\mathbb{R}^d).$$

Hence $N^{r,B}$ is an internal ${}^*M_F(\mathbb{R}^d)$-valued process which at time t records the contribution to $N_{t+\underline{r}}$ from descendants of particles which were in ${}^*B$ at time $\underline{r}$.

Notation. $\{\mathcal{A}_t : t \in {}^*[0,\infty)\}$ is the internal filtration on $({}^*\Omega, {}^*\mathcal{A})$ defined by

$$\mathcal{A}_t = {}^*\sigma(B^\beta, e^\beta : |\beta| < [t\mu]) \vee (\cap_{u>t} {}^*\sigma(B^\beta_s : |\beta| = [t\mu], s \le u))$$

(here ${}^*\sigma(X)$ is the internal $* - \sigma$-algebra generated by the internal X). Let $\mathcal{F}^\circ_t = \sigma(\mathcal{A}_t)$ for $t \in [0,\infty)$ and let $\mathcal{F}_t = \mathcal{F}^\circ_t \vee \{\hat{\mathbb{Q}}^m\text{-null sets}\}$.

$\{\mathcal{F}_t : t \ge 0\}$ is a standard filtration (but is not right-continuous) and Y is $(\mathcal{F}_t)$-adapted. It follows from the nonstandard construction of Y that for each fixed $r \in [0,\infty)$ and $B \in \mathcal{B}(\mathbb{R}^d)$, $N^{r,B}$ is $\hat{\mathbb{Q}}^m$-a.s. S-continuous and $Y^{r,B}(t) = st(N^{r,B})(t)$ satisfies

$$\text{(4.1)} \quad \hat{\mathbb{Q}}^m(Y^{r,B} \in C|\mathcal{F}_r) = \mathbb{Q}^{Y_r|B}(C) \;\; \hat{\mathbb{Q}}^m \text{ -a.s. for all Borel measurable}$$
$$C \text{ in } C([0,\infty), M_F(\mathbb{R}^d)) \text{ and all Borel } B \text{ such}$$
$$\text{that } \partial B \text{ is Lebesgue - and m-null}.$$

(See (4.7) and (4.8) of Perkins (1990) where this is shown if B is a ball. The same proof works for B as above.)

Notation. (a) If $\nu \in M_F(\mathbb{R}^d)$ let $\nu P_t(A) = P^\nu(B_t \in A)$, and let

$$D(\nu, r) = \sup\{\nu(B(y,r)) : y \in \mathbb{R}^d\}.$$

(b) If $x, y \in \mathbb{R}^d$, write $x \sim y$ if $x - y \in \mathbb{Z}^d$ and let $\pi(x)$ denote the unique point in $[0,1)^d$ such that $x \sim \pi(x)$. If $A \subset \mathbb{R}^d$ let $\tilde{A} = \cup_{k \in \mathbb{Z}^d}(k + A) = \{x : x \sim a \;\exists a \in A\}$.

(c) $\alpha(d) = \begin{cases} 2 & \text{if } d = 2 \\ 1 & \text{if } d > 2 \end{cases}$

$$\varphi(r) = r^2(\log^+ 1/r + 1)^{\alpha(d)}, \; \psi(r) = r^2(\log^+ 1/r + 1)^{\alpha(d)-1}.$$

Let $c_{4.1} \in [\sqrt{d}+4, \sqrt{d}+5)$ and let $\mathcal{B}_n$ denote the set of open balls, $B_n(y)$, of radius $r_n = c_{4.1}h(2^{-n})$ and centered at a point y in $(2^{-n}\mathbb{Z})^d$. (Here $a\mathbb{Z} = \{an : n \in \mathbb{Z}\}$.) We may, and shall, increase $c_{4.1}$ slightly so that $m(\partial B) = 0$ for all B in $\cup_n \mathcal{B}_n$. In what follows we will always assume n is large enough to ensure $r_n < \frac{1}{2}$. For each such n and each $j \in \mathbb{Z}_+$ we construct a class $\mathcal{C}_n(j)$ of subsets of $\mathbb{R}^d$ with the following properties:

(4.2) $\forall B \in \mathcal{B}_n$, $\exists C \in \mathcal{C}_n(j)$ such that $B \subset C$

(4.3) $\forall C \in \mathcal{C}_n(j)$, $\exists B \in \mathcal{B}_n$ such that $C \subset \tilde{B}$

(4.4) $\forall C \in \mathcal{C}_n(j)$, $P^m(B(j2^{-n}) \in C) \leq D(mP_{j2^n}, r_n) + \varphi(h(2^{-n}))$

(4.5) card $\mathcal{C}_n(j) \leq (m(1)+1)2^{c_{4.2}n}$ ($c_{4.2}$ depends only on d).

For each $B_n(y) \in \mathcal{B}_n$ with $y \in [0,1)^d$, let C be a (disjoint) union of balls of the form $k + B_n(y)$ ($k \in \mathbf{Z}^d$) where we keep adding on translates of $B_n(y)$ until the mP_{j2^n} measure of the union exceeds $\varphi(h(2^{-n}))$. Hence C will satisfy (4.4). Continue in this manner using new translates of $B_n(y)$ until every $k + B_n(y)$ is contained in one (and only one C). Each such C will be a disjoint union of sets in $\widetilde{B_n(y)}$ and hence satisfy (4.3). The class of C's constructed in this way (for $B_n(y)$) are mutually disjoint and all but one will satisfy $mP_{j2^n}(C) > \varphi(h(2^{-n}))$. Hence by (4.3)

$$\text{(4.6)} \qquad \text{no. of } C's \text{ constructed from one } B_n(y)$$
$$\leq m(\widetilde{B_n(y)})\varphi(h(2^{-n}))^{-1} + 1.$$

Now repeat this procedure for each of the 2^{nd} choices of y in $[0,1)^d \cap (2^{-n}\mathbf{Z})^d$. $\mathcal{C}_n(j)$ is the set of all C's constructed in this manner. (4.5) is then clear from (4.6). (4.2) is immediate from this construction.

Lemma 4.1 *For $\hat{\mathbf{Q}}^m$ – a.a.w, if $2^{-n} < \delta(w,3)$ ($\delta(w,3)$ as in Theorem 2.1) then*

$$\sup\{Y_t(B(x,h(2^{-n}))) : x \in \mathbb{R}^d,\ j2^{-n} \leq t < (j+1)2^{-n}\}$$
$$\leq \sup\{Y_t^{j2^{-n},C}(\mathbb{R}^d) : C \in \mathcal{C}_n(j),\ t \in [0,2^{-n})\}\ \forall j \in \mathbf{Z}_+.$$

Proof. Fix w outside a $\hat{\mathbf{Q}}^m$-null set such that $Y = st(N)$ and $Y^{u,C} = st(N^{u,C})$ for all non-negative rational u and $C \in \cup_{j,n}\mathcal{C}_n(j)$. Assume $2^{-n} < \delta(w,3)$. Let $x \in \mathbb{R}^d$ and choose $y \in 2^{-n}\mathbf{Z}^d$ such that $|x-y| \leq \sqrt{d}2^{-n} \leq \sqrt{d}h(2^{-n})$. Fix $j \in \mathbf{Z}_+$. Then

$$B(x, 4h(2^{-n})) \subset B(y, (\sqrt{d}+4)h(2^{-n})) \subset B_n(y) \subset C \text{ for some } C \in \mathcal{C}_n(j)$$

(the latter inclusion by (4.2)). If $t \in [j2^{-n}, (j+1)2^{-n})$, $\beta \sim t$ and $N_t^\beta \neq \Delta$, then $|N_t^\beta - N_{j2^{-n}}^\beta| \leq 3h(2^{-n})$ (recall $\delta(w,3) \geq 2^{-n}$) and hence

$$N_t^\beta \in {}^*B(x, h(2^{-n})) \Rightarrow N_{j2^{-n}}^\beta \in {}^*B(x, 4h(2^{-n})) \subset {}^*C. \tag{4.7}$$

Therefore

$$\begin{aligned} Y_t(B(x,h(2^{-n}))) &\leq {}^oN_t({}^*B(x,h(2^{-n}))) \\ &\leq {}^o\frac{1}{\mu}\sum_{\beta\sim t} 1(N_{j2^{-n}}^\beta \in {}^*C,\ N_t^\beta \neq \Delta) \quad \text{(by (4.7))} \\ &= {}^oN_t^{j2^{-n},C}({}^*\mathbb{R}^d) = Y_t^{j2^{-n},C}(\mathbb{R}^d). \end{aligned}$$

The result follows. □

Notation. If $f : \mathbb{R}^d \to [0,\infty]$ is Borel, let

$$G(f,t) = \int_0^t \sup_y P^y(f(B_s))ds.$$

The next result is readily obtained by taking limits in Proposition 2.6(c) of Perkins (1988).

Proposition 4.2 *If $f : \mathbb{R}^d \to [0,\infty]$ is Borel and $4P^m(f(B_t)) \leq \lambda$, then*

$$\mathbb{Q}^m(Y_t(f) \geq \lambda) \leq \exp\{-\lambda(4G(f,t))^{-1}\}.$$

Let $q_s(y) = \sum_{k\in\mathbb{Z}^d} P_s(y+k)$ denote the transition density (on $[0,1)^d$) of $\pi(B_s)$ (and set $q_s(y) = 0$ if $y \notin [0,1)^d$).

Lemma 4.3 *Let $r \in [0, \frac{1}{2})$.*

(a) $\sup_y P^y(B_s \in \tilde{B}(0,r)) \leq c_{4.3} r^d (s^{-d/2} + 1)$

(b) If B is a ball of radius r then

$$G(1_{\tilde{B}}, t) \leq c_{4.4}\psi(r)$$

providing $t \leq r^{2-d}$, if $d > 2$, or $t \leq \log \frac{1}{r}$, if $d = 2$.

Proof. (a) If $y \in [0,1)^d$ then

$$\begin{aligned} q_s(y) &= (2\pi s)^{-d/2} \sum_{k \in \mathbb{Z}^d} \exp\{-|y+k|^2/2s\} \\ &\leq \prod_{i=1}^{d} \Big(\sum_{k_i=-\infty}^{\infty} \exp\{-(y_i+k_i)^2/2s\}(2\pi s)^{-1/2} \Big) \\ &\leq \Big(2(2\pi s)^{-1/2} + 2\sum_{k=1}^{\infty} \exp\{-k^2/2s\}(2\pi s)^{-1/2}\Big)^d \\ &\leq \Big(2(2\pi s)^{-1/2} + 2\int_0^{\infty} \exp\{-x^2/2s\}(2\pi s)^{-1/2} dx\Big)^d \\ &\leq (s^{-1/2}+1)^d. \end{aligned}$$

Therefore

$$\begin{aligned} P^y(B_s \in \tilde{B}(0,r)) &= P^o(\pi(B_s) \in \pi(B(-y,r))) \\ &= \int 1(z \in \pi(B(-y,r)))q_s(z)dz \\ &\leq c_d r^d (s^{-1/2}+1)^d \end{aligned}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$ (recall $r \leq 1/2$). This gives (a).
(b) If $d > 2$, B is a ball of radius $r \leq 1/2$ and t is as in the statement of (b), then

$$\begin{aligned} G(1_{\tilde{B}},t) &= \textstyle\int_0^t \sup_y P^y(B_s \in \tilde{B}(0,r))ds \\ &\leq r^2 + \textstyle\int_{r^2}^{t\vee r^2} c_{4.3} r^d (s^{-d/2}+1)ds \\ &\leq r^2 + c_{4.3}tr^d + c_{4.3}r^d r^{2(1-d/2)}(d/2-1)^{-1} \\ &\leq r^2 + c_{4.3}r^2 + c_{4.3}(d/2-1)^{-1}r^2 \equiv c_{4.3}r^2. \end{aligned} \tag{4.8}$$

If $d = 2$, then (4.8) still holds and we get

$$\begin{aligned} G(1_{\tilde{B}},t) &\leq r^2 + c_{4.3}tr^2 + c_{4.3}r^2 \log^+ t/r^2 \\ &\leq r^2 + c_{4.3}r^2 \log 1/r + c_{4.3}r^2[\log\log\frac{1}{r} + 2\log\frac{1}{r}]. \quad \square \end{aligned}$$

Notation. Let y_t denote the diffusion on $[0,\infty)$ with generator $(y/2)d^2/dy^2$ and let $\{P_0^{y'} : y' \geq 0\}$ denote the laws of y on path space. The process is absorbed at 0.

Lemma 4.4 $P_o^{y'}(\sup_{t\le T} y(t) \ge \lambda) \le \exp\{-2(\sqrt{\lambda}-\sqrt{y'})^2/T\}\ \forall \lambda \ge y'$.

Proof. The transition density of y is given on p.100 of Knight (1981). This readily gives $P_0^{y'}(\exp\theta y(t)) = \exp\{2\theta y'(2-\theta t)^{-1}\}$ for $\theta < 2/t$. The result now follows from the weak L^1 inequality by the usual application of Markov's inequality (the optimal θ is $t^{-1}(2-2(y'/\lambda)^{1/2})$. □

We now combine the three previous estimates to derive the required bound on $Y^{j2^{-n},C}$. Recall $r_n = c_{4.1}h(2^{-n})$.

Lemma 4.5 *There is a* $c_{4.5}$ *such that*

$$\hat{\mathbb{Q}}^m(\sup_{t\le 2^{-n}} Y_t^{j2^{-n},C}(\mathbb{R}^d) \ge \lambda) \le 2\exp\{-c_{4.5}\lambda 2^n n^{-\alpha(d)}\}$$

for all $j2^{-n}$ *and* $C \in \mathcal{C}_n(j)$ *such that*

$$8(D(mP_{j2^{-n}}, r_n) + \varphi(h(2^{-n}))) \le \lambda, \tag{4.9}$$

$r_n < \frac{1}{2}$, *and*

$$j2^{-n} \le \begin{cases} r_n^{2-d} & \text{if } d, \ge 3, \\ \log 1/r_n, & \text{if } d = 2. \end{cases}$$

Proof. Use (4.1) to bound the required probability by

$$\mathbb{Q}^m(Y_{j2^{-n}}(C) \ge \lambda/2) + \mathbb{Q}^m(1(Y_{j2^{-n}}(C) < \lambda/2)\mathbb{Q}^{Y_{j2^{-n}}|_C}(\sup_{t\le 2^{-n}} Y_t(\mathbb{R}^d) \ge \lambda))$$

$$\le \exp\{-\lambda(8G(1_C, j2^{-n}))^{-1}\} + P_o^{\lambda/2}(\sup_{t\le 2^{-n}} y(t) \ge \lambda)$$

((4.4) and (4.9) allow us to apply Proposition 4.2)

$$\le \exp\{-\lambda(c_{4.4}8\psi(r_n))^{-1}\} + \exp\{-2^n(\sqrt{2}-1)^2\lambda\}, \tag{4.10}$$

where we have used (4.3) and Lemma 4.3 to bound the first term and Lemma 4.4 to bound the second term. An easy calculation now shows the first term in (4.10) is the larger and gives the required bound. □

Lemma 4.6 *(a) $t \mapsto D(mP_t, r)$ is non-increasing on $[0, \infty)$ for each $m \in M_F(\mathbb{R}^d)$, $r > 0$.*
(b) There is a $c_{4.6}$ such that

$$D(mP_t, r) \geq c_{4.6} D(m, r) \quad \forall\, 0 < t \leq r^2, \quad m \in M_F(\mathbb{R}^d).$$

Proof. (a) By the Markov property it suffices to show $D(mP_t, r) \leq D(m, r)$. If $y \in \mathbb{R}^d$ then

$$\begin{aligned} mP_t(B(y,r)) &= \int\int 1(|z-y| < r) p_t(z-x) dm(x) dz \\ &= \int\int 1(|w + x - y| < r) dm(x)\; p_t(w) dw \\ &\leq D(m,r) \int p_t(w) dw = D(m,r). \end{aligned}$$

(b) Choose y such that $m(B(y,r)) \geq D(m,r)/2$. Then

$$\begin{aligned} mP_t(B(y,r)) &\geq \int 1(|x-y| < r) P^x(B_t \in B(y,r)) dm(x) \\ &= \int 1(|x-y| < r) P^o(B_1 \in B((y-x)t^{-1/2}, rt^{-1/2})) dm(x) \\ &\geq c_1 m(B(y,r)) \geq (c_1/2) D(m,r) \end{aligned}$$

where $c_1 = \inf\{P^o(B_1 \in B)$: B a ball of radius 1 containing 0 $\}$. $\square$

Here finally is the main result of this section. Recall that

$$D(m,r) = \sup\{m(B(y,r)) : y \in \mathbb{R}^d\} \text{ and } \varphi(r) = \begin{cases} r^2(1 + \log^+ \frac{1}{r}) & d > 2 \\ r^2(1 + \log^+ \frac{1}{r})^2 & d = 2 \end{cases}.$$

Theorem 4.7 *Assume $d \geq 2$. There exists $c_{4.7}$, $c_{4.8}$ such that for any m in $M_F(\mathbb{R}^d)$ and $\mathbb{Q}^m$-a.s. $\exists r_o(w) > 0$ such that*

$$D(Y_t, r) \leq c_{4.7} \max(D(mP_t, c_{4.8} r), \varphi(r)) \text{ for all } t \geq 0,\ 0 < r < r_0.$$

Proof. Let

$$A_n(j) = \{ \sup_{j2^{-n} \leq t < (j+1)2^{-n}} D(Y_t, h(2^{-n})) \geq c_1(D(mP_{j2^{-n}}, r_n) + \varphi(h(2^{-n})))\},$$

where $c_1 \geq 8$ will be chosen below. Let

$$T_n = \begin{cases} r_n^{2-d} & \text{if } d \geq 3 \\ \log(1/r_n) & \text{if } d = 2 \end{cases}.$$

Then (2.1) and Lemmas 4.1 and 4.5 imply that for $r_n < \frac{1}{2}$,

$$\begin{aligned}
\mathbb{Q}^m(\cup_{j \leq 2^n T_n} A_n(j)) &\leq \sum_{j \leq 2^n T_n} \hat{\mathbb{Q}}^m(A_n(j), \delta(w,3) > 2^{-n}) + c_{2.1} m(\mathbb{R}^d) 2^{-nc_{2.2}} \\
&\leq \sum_{j \leq 2^n T_n} \sum_{C \in \mathcal{C}_n(j)} 2 \exp\{-c_{4.5} c_1 (D(mP_{j2^{-n}}, r_n) + \varphi(h(2^{-n}))2^n n^{-\alpha(d)}\} \\
&+ c_{2.1} m(\mathbb{R}^d) 2^{-nc_{2.2}} 2^{c_{4.2} n} \\
&\leq 2^n T_n (m(1)+1) 2 \exp\{-c_{4.5} c_1 \varphi(h(2^{-n})) 2^n n^{-\alpha(d)}\} \\
&+ c_{2.1} m(\mathbb{R}^d) 2^{-nc_{2.2}} \quad \text{(by (4.5))} \\
&\leq 2^{n(1+c_{4.2})} T_n (m(1)+1) 2 \exp\{-c_1 c_2 n\} + c_{2.1} m(\mathbb{R}^d) 2^{-nc_{2.2}},
\end{aligned}$$

where c_2 depends only on d. Now fix c_1 large enough so that the last expression is summable (note that $T_n \leq 2^{nc_3}$). By Borel-Cantelli we have $\mathbb{Q}^m$-a.s. for large enough n

$$\sup_{j2^{-n} \leq t < (j+1)2^{-n}} D(Y_t, h(2^{-n})) < c_1(D(mP_{j2^{-n}}, c_{4.1} h(2^{-n})) + \varphi(h(2^{-n})))$$

$$\forall\, 0 \leq j2^{-n} \leq T_n,$$

and therefore by Lemma 4.6(b) (with $mP_{j2^{-n}}$ in place of m in that result)

$$D(Y_t, h(2^{-n})) \leq c_3(D(mP_t, c_{4.1} h(2^{-n})) + \varphi(h(2^{-n}))) \ \forall\, 0 \leq t \leq T_n. \tag{4.11}$$

Take n larger still (if necessary) so that T_n exceeds the lifetime of Y to get (4.11) for all $0 \leq t < \infty$. The required result follows by a standard argument which allows us to replace $h(2^{-n})$ by a sufficiently small continuous parameter r at the cost of increasing c_3 and $c_{4.1}$. □

Corollary 4.8 *If $d \geq 2$ then for any m in $M_F(\mathbb{R}^d)$*

$$\limsup_{r \downarrow 0} \sup_{t \geq \delta} D(Y_t, r) \varphi(r)^{-1} \leq c_{4.7} \ \text{ for all } \ \delta > 0 \ \mathbb{Q}^m - a.s.$$

Proof. Lemma 4.6 implies

$$\begin{aligned}\sup_{t\geq\delta} D(mP_t, c_{4.8}r) &= D(mP_\delta, c_{4.8}r)\\ &\leq m(1)\delta^{-d/2}(2c_{4.8}r)^d.\end{aligned}$$

The result is now clear from Theorem 4.7. □

In the next section we will also need a uniform (in t) upper bound on $\int\int 1(|y_1-y_2|\leq r)dY_t^1dY_t^2$ as $r\downarrow 0$ when Y^1 and Y^2 are independent super-Brownian motions starting at m^1 and m^2, respectively. This in turn requires an elementary covering argument implicit in Perkins (1988) and DIP (1989). This argument (Proposition 4.10) is of independent interest since it gives an elementary proof of the fact that $\dim S(Y_t)\leq 2$ for all $t>0$ a.s. (the reader is invited to derive this from the proof of Proposition 4.10).

Notation. If $0\leq\epsilon\leq t$, let $Z(t,\epsilon)$ denote the cardinality of $I(t,\epsilon)$ (the latter was introduced just prior to Lemma 3.3).

Lemma 4.9 *Conditional on $\mathcal{F}_{t-\epsilon}$, $Z(t,\epsilon)$ is a Poisson random variable with mean $2Y_{t-\epsilon}(1)\epsilon^{-1}$.*

Proof. The internal conditional distribution of $Z(t,\epsilon)$ given $\mathcal{A}_{t-\epsilon}$ (or equivalently $\mathcal{A}_{t-\epsilon}$) is binomial with mean $\approx 2Y_{t-\epsilon}(1)\epsilon^{-1}$ and number of trials $\mu N_{t-\epsilon}(1)$ (see DIP (1989, Lemma 4.1, (4.2))). The lemma is now a simple consequence of the well-known weak convergence of the binomial distribution to the Poisson. □

Proposition 4.10 *Let $Y^*(1)=\sup_{t\geq 0}Y_t(1)$. For $\hat{Q}^m$-a.a.w for n sufficiently large ($n\geq N(w)$) for any $j\in\mathbb{N}$ $S(Y_{j2^{-n}})$ is contained in a union of $2^{n+2}(Y^*(1)\vee 1)$ balls of radius $3h(2^{-n})$.*

Proof. The previous lemma and an elementary calculation shows that

$$\hat{Q}^m(Z(j2^{-n},2^{-n})>2^{n+2}(Y_{(j-1)2^{-n}}(1)\vee 1)|\mathcal{F}_{(j-1)2^{-n}})\leq\exp\{-2^{n+1}(2\log 2-1)\}$$

(bound the conditional distribution of $Z(j2^{-n},2^{-n})$ by a Poisson distribution with mean $2^{n+1}(Y_{(j-1)2^{-n}}(1)\vee 1)$ and apply an elementary large deviations estimate). Therefore

$$\hat{Q}^m(\max_{1\le j\le 2^{2n}} Z(j2^{-n},2^{-n}) > 2^{n+2}(Y^*(1)\vee 1)) \le 2^{2n}\exp\{-2^{n+1}(2\log 2-1)\}$$

is summable over n. By Borel-Cantelli we can fix w outside a $\hat{Q}^m$ null set such that there is on $N_1(w)$ so that

$$\max_{1\le j\le 2^{2n}} Z(j2^{-n},2^{-n}) \le 2^{n+2}(Y^*(1)\vee 1) \ \text{ for all } \ n\ge N_1 \tag{4.12}$$

and $\delta(w,3)>0$ ($\delta(w,c)$ as in Theorem 2.1). Choose $n\ge N_1$ so that $2^{-n}<\delta(w,3)$ and $Y_{2^n+t}\equiv 0$. Then for any $j\in \mathbb{N}$

$$S(Y_{j2^{-n}}) \subset \bigcup_{x\in I(j2^{-n},2^{-n})} B(x,3h(2^{-n})), \tag{4.13}$$

and (4.12) together with $I(j2^{-n},2^{-n})=\emptyset$ for $j>2^{2n}$ shows that (4.13) covers $S(Y_{j2^{-n}})$ by a union of at most $2^{n+2}(Y^*(1)\vee 1)$ open balls of radius $3h(2^{-n})$. $\square$

Theorem 4.11 *Assume $d\ge 2$. There is a $c_{4.9}$ such that for any m^1, m^2 in $M_F(\mathbb{R}^d)$, $\beta>0$, and $\mathbb{Q}^{m^1}\times\mathbb{Q}^{m^2}$-a.a. w there is an $r_1(\beta,w)>0$ such that*

$$\sup_{t\ge\beta}\int\int 1(|y_1-y_2|\le r)dY_t^1(y_1)dY_t^2(y_2) \tag{4.14}$$

$$\le c_{4.9}(\sup_t Y_t^1(1)\vee 1)r^{4-4/d}(\log 1/r)^{2/d+2} \ \text{ for } \ r<r_1(\beta,w).$$

Proof. If $d=2$ this is a simple consequence of Corollary 4.8 (in fact one can reduce the power of the logarithm) so assume $d>2$. We work on the obvious product of Loeb spaces so that $w=(w^1,w^2)$. Fix $\beta>0$.

Condition on w^1 satisfying $\delta(w^1,3)>0$ ($\delta(w,c)$ as in Theorem 2.1), $r_0(w^1)>0$ (r_0 as in Theorem 4.7), and the conclusion of Proposition 4.10 (for $n\ge N(w^1)$ say). We now argue conditionally on w^1 and hence will work with respect to $\hat{Q}^{m^2}$. Assume n is taken sufficiently large so that $n\ge N(w^1)$, $h(2^{-n})<r_0(w^1)$, $2^{-n}<\delta(w^1,3)$, and $D(mP_t,c_{4.8}r)\le\varphi(r)$ for all $t\ge\beta$ and

$r \leq h(2^{-n})$ (see the proof of Corollary 4.8). Then on $\{\delta(w^2,3) > 2^{-n}\}$ we have for $\beta \leq j2^{-n} \leq t < (j+1)2^{-n}$)

$$\int\int 1(|y_1-y_2| \leq h(2^{-n}))dY_t^1 dY_t^2 \leq c_{4.7}\varphi(h(2^{-n}))\int 1(d(y_2, S(Y_t^1)) \leq h(2^{-n}))dY_t^2$$

(by Theorem 4.7 since $h(2^{-n}) < r_0(w^1)$)

(4.15)

$$\leq c_{4.7}\varphi(h(2^{-n})\int 1(d(y_2, S(Y^1_{j2^{-n}})) \leq 4h(2^{-n}))dY_t^2$$

(since $2^{-n} < \delta(w^1,3)$)

$$\leq c_{4.7}\varphi(h(2^{-n}))Y^{j,n},$$

where

$$S_{j,n} = \{y : d(y, S(Y^1_{j2^{-n}})) \leq 7h(2^{-n})\}$$

$$Y^{j,n} = \sup_{t\leq 2^{-n}} Y_t^{2,S_{j,n},j2^{-n}}.$$

and we have used $\delta(w^2,3) > 2^{-n}$ to conclude that each particle in $S(Y_t^2)$ within a distance $4h(2^{-n})$ of $S(Y^1_{j2^{-n}})$, is the descendent of a particle in $S(Y^2_{j2^{-n}})$ within a distance $7h(2^{-n})$ of $S(Y^1_{j2^{-n}})$. Use the nonstandard construction of Y^2 to make this rigorous. By the choice of w^1 and Proposition 4.10

$$\text{(4.16)} \quad \begin{aligned} P^{m^2}(B_s \in S_{j,n}) &\leq 2^{n+2}((Y^1)^*(1) \vee 1)m^2(1)s^{-d/2}c_1 h(2^{-n})^d \\ &= c_2 c_3(w^1)m^2(1)s^{-d/2}2^{-n(d/2-1)}n^{d/2} \end{aligned}$$

where $c_3(w^1) = (Y^1)^*(1) \vee 1$. Therefore

$$\begin{aligned} G(1_{S_{j,n}}, t) &\leq \int_0^t (c_2 c_3(w^1)s^{-d/2}2^{-n(d/2-1)}n^{d/2}) \wedge 1 ds \\ &\leq c_4 c_3(w^1)^{2/d} n 2^{-n(1-2/d)}. \end{aligned}$$

By (4.16) if $j2^{-n} \geq \beta$ and

$$\text{(4.17)} \quad \lambda \geq 4c_2 c_3(w^1)m^2(1)\beta^{-d/2}n^{d/2}2^{-n(d/2-1)}$$

we may apply Proposition 4.2 to conclude

$$\hat{Q}^{m^2}(Y^2_{j2^{-n}}(S_{j,n}) \geq \lambda) \leq \exp\{-\lambda(4c_4 c_3(w^1)^{2/d})^{-1}n^{-1}2^{n(1-2/d)}\}.$$

Let $\lambda_n = (4c_4c_3(w^1)^{2/d})17n^2 2^{-n(1-2/d)}$. Then (4.17) holds for $n \geq n_0(w^1, \beta, m^1(1))$ and so

$$\hat{Q}^{m^2}(Y^2_{j2^{-n}}(S_{j,n}) \geq \lambda_n) \leq e^{-17n}.$$

Argue as in Lemma 4.5 to obtain for $n \geq n_1(w^1, \beta, m^1(1))$

$$\hat{Q}^{m^2}(Y^{j,n} \geq 2\lambda_n) \leq 2e^{-17n}.$$

By Borel Cantelli for $\hat{Q}^{m_2}$-a.s. w^2 there is an $N_1(w^2)$ such that

$$\begin{aligned} \sup_{\beta \leq j2^{-n} \leq 2^n} Y^{j,n} < 2\lambda_n \text{ for all } n &\geq N_1(w^2) \vee n_1(w^1, \beta, m^1(1)) \\ &\equiv N_2(w). \end{aligned}$$

Returning to (4.15) we see that for a.a. w there is an $N_3(w) \in \mathbb{N}$ such that for $n \geq N_3(w)$

$$\begin{aligned} \sup_{2\beta \leq t} \int\int 1(|y_1 - y_2| \leq h(2^{-n}))dY^1_t dY^2_t &\leq c_{4.7}\varphi(h(2^{-n}))2\lambda_n \\ &\leq c_5c_3(w^1)^{2/d}h(2^{-n})^{4-4/d}(\log 1/h(2^{-n}))^{2+2/d} \end{aligned}$$

(4.14) is a trivial consequence of the above. □

5 A Tanaka Formula for Collision Local Time

Let $m^1, m^2 \in M_F(\mathbb{R}^d)$ and assume (X^1, X^2) are continuous adapted $M_F(\mathbb{R}^d)$-valued processes on $(\Omega', \mathcal{F}', \mathcal{F}'_t, P')$ which satisfy (M_{m^1,m^2}) (in the Introduction). We first show how to enlarge the probability space to construct a pair of independent super-Brownian motions (Y^1, Y^2) so that $Y^i \geq X^i$ $i = 1, 2$. Recall the notation $(\Omega^o, \mathcal{F}^o, \mathcal{F}^o[s, t+], \mathbb{Q}_{s,m,\mu})$ from Theorem 1.1. Let $\mathcal{F}^o_t = \mathcal{F}^o[0, t+]$.

Define

$$\Omega = \Omega' \times \Omega^o \times \Omega^o,\ \mathcal{F} = \mathcal{F}' \times \mathcal{F}^o \times \mathcal{F}^o,\ \mathcal{F}_t = \mathcal{F}'_t \times \mathcal{F}^o_t \times \mathcal{F}^o_t,$$

and if $\mathbb{Q}_\mu = \mathbb{Q}_{0,0,\mu}$, define $\mathbb{P}$ on $(\Omega, \mathcal{F})$ by (see Theorem 1.1(d))

$$\mathbb{P}(B \times C_1 \times C_2) = \int_{\Omega'} 1_B(w')\mathbb{Q}_{A^1(w')}(C_1)\mathbb{Q}_{A^2(w')}(C_2)d\mathbb{P}'(w').$$

Here $A^i(w')$ is the random measure on $[0,\infty) \times \mathbb{R}^d$ defined by

$$A^i(w')([0,t] \times B) = A^i_t(w')(B).$$

Let $\pi : \Omega \to \Omega'$ denote the projection mapping and denote points in Ω by $w = (w', \tilde{Y}^1, \tilde{Y}^2)$. Let $Y^i_t(w) = X^i_t(w') + \tilde{Y}^i_t \geq X^i_t(w')$. Roughly speaking, when an individual of population X^i is killed by A^i we think of the particle living on in an afterlife and use $\tilde{Y}^i$ to record the subsequent evolution of the descendents of these dead particles. There is no further interaction between these deceased particles and the two living populations and hence, conditioned on w', $\tilde{Y}^i$, $i = 1, 2$, should be independent.

Theorem 5.1 *(a) If $W \in b\mathcal{F}'$, then*

$$\mathbb{P}(W \circ \pi | \mathcal{F}_t) = \mathbb{P}'(W|\mathcal{F}'_t) \circ \pi \quad \mathbb{P} - a.s. \tag{5.1}$$

(b) (Y^1, Y^2) are independent $(\mathcal{F}_t)$-super Brownian motions starting at m^1 and m^2, respectively (under $\mathbb{P}$).

(c) If $Z^{Y^i}_t(\varphi)$ is the martingale part of $Y^i_t(\varphi)$ $(\varphi \in \mathcal{D}(A))$ then

$$\langle Z^{Y^i}(\varphi_i), Z^j(\varphi_j) \circ \pi \rangle_t = \delta_{ij} \langle Z^i(\varphi_i), Z^j(\varphi_j) \rangle_t \circ \pi = \delta_{ij} \int_0^t X^i_s \circ \pi(\varphi^2_i) ds.$$

Proof. (a) Let $B \in \mathcal{F}'_t$ and C_1, $C_2 \in \mathcal{F}^o_t$. Then if $W \in b\mathcal{F}'$,

$$\begin{aligned}
\int 1_{B \times C_1 \times C_2} W \circ \pi d\mathbb{P} &= \int 1_B(w') W(w') \mathbb{Q}_{A^1(w')}(C_1) \mathbb{Q}_{A^2(w')}(C_2) d\mathbb{P}'(w') \\
&= \int 1_B(w') \mathbb{P}'(W|\mathcal{F}'_t)|(w') \mathbb{Q}_{A^1(w')}(C_1) \mathbb{Q}_{A^2(w')}(C_2) d\mathbb{P}'(w')
\end{aligned}$$

since Theorem 1.1 (d) shows that $\mathbb{Q}_{A^j(w')}(C_i)$ is $\mathcal{F}'_t$-measurable. Therefore

$$\int 1_{B \times C_1 \times C_2} W \circ \pi d\mathbb{P} = \int 1_{B \times C_1 \times C_2} \mathbb{P}'(W|\mathcal{F}'_t) \circ \pi d\mathbb{P}$$

and (5.1) follows.

(b) Since $Z^i_t(\varphi)$ is on $(\mathcal{F}'_t)$-martingale (a) shows that $Z^i_t(\varphi) \circ \pi$ is an $(\mathcal{F}_t)$-martingale under $\mathbb{P}$ and

$$\langle Z^i(\varphi_i) \circ \pi, Z^j(\varphi_j) \circ \pi \rangle_t = \delta_{ij} \int_0^t X^i_s \circ \pi(\varphi^2_i) ds \tag{5.2}$$

because $Z_t^i(\varphi_j)\circ Z_t^j(\varphi_j)\circ\pi-\delta_{ij}\int_0^t X_s^i\circ\pi(\varphi_i^2)ds$ is an $(\mathcal{F}_t)$-martingale by the same reasoning. Let $\tilde{\mathcal{F}}_t=\mathcal{F}'\times\mathcal{F}_t^o\times\mathcal{F}_t^o\supset\mathcal{F}_t$ and

$$\tilde{Z}_t^i(\varphi)=\tilde{Y}_t^i(\varphi)-\int_0^t\tilde{Y}_r^i(A\varphi)dr-A_t^i(\varphi)\circ\pi,\ \varphi\in\mathcal{D}(A).$$

If $s<t$, $B\in\mathcal{F}'$ and $C_1,C_2\in\mathcal{F}_s^o$, then

$$\begin{aligned}
&\mathbb{P}\left(1_{B\times C_1\times C_2}(\tilde{Z}_t^i(\varphi)-\tilde{Z}_s^i(\varphi))\right)\\
&=\int_B \mathbb{Q}_{A^1(w')}\times\mathbb{Q}_{A^2(w')}(1_{C_1\times C_2}(\tilde{Z}_t^i(\varphi)(w',\cdot)-\tilde{Z}_s^i(\varphi)(w',\cdot)))d\mathbb{P}'(w')\\
&=0
\end{aligned}$$

because $\tilde{Z}_t^i(\varphi)(w',\cdot)$ is an $(\mathcal{F}_t^o\times\mathcal{F}_t^o)$-martingale under $\mathbb{Q}_{A^1(w')}\times Q_{A^2(w')}$ by $(M_{0,0,A^i(w')})$. Therefore $\tilde{Z}_t^i(\varphi)$ is an $(\tilde{\mathcal{F}}_t)$, and therefore an $(\mathcal{F}_t)$, martingale. The same reasoning shows $\tilde{Z}_t^i(\varphi_i)\tilde{Z}_t^j(\varphi_j)-\delta_{ij}\int_0^t\tilde{Y}_s^i(\varphi_i^2)ds$ is an $(\tilde{\mathcal{F}}_t)$, and hence also on $(\mathcal{F}_t)$, martingale and so

$$\langle\tilde{Z}^i(\varphi_i)\tilde{Z}^j(\varphi_j)\rangle_t=\delta_{ij}\int_0^t\tilde{Y}_s^i(\varphi_i^2)ds. \tag{5.3}$$

If $\varphi\in\mathcal{D}(A)$ then $Z_t^{Y^i}(\varphi)=Z_t^i(\varphi)\circ\pi+\tilde{Z}_t^i(\varphi)$ is a $(\mathcal{F}_t)$-martingale by the above and

$$Y_t^i(\varphi)=m^i(\varphi)+Z_t^{Y^i}(\varphi)+\int_0^t Y_r^i(A\varphi)dr\quad i=1,2,\quad\varphi\in\mathcal{D}(A).$$

If $s<t$ then

$$\begin{aligned}
\mathbb{P}(\tilde{Z}_t^i(\varphi_i)Z_t^j(\varphi_j)\circ\pi|\mathcal{F}_s)&=\mathbb{P}(Z_t^j(\varphi_j)\circ\pi\,\mathbb{P}(\tilde{Z}_t^i(\varphi_i)|\tilde{\mathcal{F}}_s)|\mathcal{F}_s)\\
&=\mathbb{P}(Z_t^j(\varphi_j)\circ\pi\,\tilde{Z}_s^i(\varphi_i)|\mathcal{F}_s)\\
&=Z_s^j(\varphi_j)\circ\pi\,\tilde{Z}_s^i(\varphi_i)
\end{aligned}$$

and so

$$\langle\tilde{Z}^i(\varphi_i),Z^j(\varphi_j)\circ\pi\rangle=0. \tag{5.4}$$

(5.2), (5.3) and (5.4) imply

$$\langle Z^{Y^i}(\varphi_i),Z^{Y^j}(\varphi_j)\rangle_t=\delta_{ij}\int_0^t Y_s^i(\varphi_i^2)ds.$$

Theorem 1.2 now implies (b).

(c) is immediate from (5.2) and (5.4). □

The above result extends easily to more general super-processes such as super-Feller processes.

(a) implies that $(X, A) \equiv (X^1, X^2, A^1, A^2)$ (on $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbb{P}')$ and $(X, A) \circ \pi$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ have the same law and, more significantly have the same adapted distribution in the sense of Hoover-Keisler (1984). This means that all random variables obtained from (X, A), respectively $(X, A) \circ \pi$, by the operations of composition with bounded continuous functions and taking conditional expectation with respect to $(\mathcal{F}'_t)$, respectively $(\mathcal{F}_t)$, have the same laws. It implies, for example, that if we identify $(X^1, X^2, A^1, A^2, Z^1, Z^2)$ with $(X^1, X^2, A^1, A^2, Z^1, Z^2) \circ \pi$, then (M_{m^1,m^2}) holds on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ (this is established in the previous proof). Therefore in studying properties of (X, A) we may as well work with $(X, A) \circ \pi$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and hence we may, and shall, assume there are independent $(\mathcal{F}_t)$-super-Brownian motions (Y^1, Y^2) starting at m^1 and m^2, respectively, and such that $Y^i_t \geq X^i_t$ for all $t \geq 0$, $i = 1, 2$. In the future when working with systems such as (M_{m^1,m^2}) we will simply assert the existence of dominating super-Brownian motions (Y^1, Y^2) "without loss of generality by enlarging the probability space if necessary".

Let $\mathcal{A}$ and R_α denote the generator and α-resolvent of $2d$-dimensional Brownian motion, respectively. It will be understood that $\alpha > 0$ if $d \leq 2$.

Lemma 5.2 *For any φ in $\mathcal{D}(\mathcal{A})$,*

$$\begin{aligned} X^1_t \times X^2_t(\varphi) &= m^1 \times m^2(\varphi) \\ &+ \int_0^t \int \int \varphi(x_1, x_2)[X^1_s(dx_1) Z^2(ds, dx_2) + X^2_s(dx_2) Z^1(ds, dx_1)] \\ &- \int_0^t \int \int \varphi(x_1, x_2)[X^1_s(dx_1) A^2(ds, dx_2) + X^2_s(dx_2) A^1(ds, dx_1)] \\ &+ \int_0^t \int \int \mathcal{A}\varphi(x_1, x_2) X^1_s(dx_1) X^2_s(dx_2) ds. \end{aligned}$$

Proof. If $\varphi(x_1, x_2)$ is a linear combination of functions of the form $\varphi_1(x_1)\varphi_2(x_2)$ where $\varphi_i \in \mathcal{D}(A)$ (call this class of φ's $\mathcal{L}$) this is immediate from (M_{m^1,m^2}) and Itô's lemma. A theorem of S. Watanabe (see Ethier-Kurtz (1986, p.17)) implies $\mathcal{L}$ is a core for $\mathcal{A}$ and the result follows for all φ in $\mathcal{D}(\mathcal{A})$ by taking limits. $\square$

To obtain a Tanaka formula for collision local time we want to set $\phi(x_1, x_2) = -\delta_0(x_1 - x_2)\psi((x_1 + x_2)/2)$ in the above. If $\psi \in b\mathcal{B}(\mathbb{R}^d)$ we introduce some approximate identities and set $\psi_\epsilon(x_1, x_2) = p_\epsilon(x_1 - x_2)\psi((x_1 + x_2)/2)$ and (B_1, B_2 are independent Brownian motions in $\mathbb{R}^d$)

$$\begin{aligned} G_{\alpha,\epsilon}\psi(x_1, x_2) = R_\alpha\psi_\epsilon(x_1, x_2) &= P^{x_1} \times P^{x_2}(\int_0^\infty e^{-\alpha S} p_\epsilon(B_s^1 - B_s^2)\psi((B_s^1 + B_s^2)/2)ds \\ &= k_d \int_0^\infty e^{-2\alpha s} p_{s+\epsilon/4}((x_1 - x_2)/2)P_s\psi((x_1 + x_2)/2)ds, \end{aligned}$$

where $k_d = 2(4^{-d/2})$ and we have used the independence of $B^1 + B^2$ and $B^1 - B^2$. Let

$$G_\alpha\psi(x_1, x_2) = k_d \int_0^\infty e^{-2\alpha s} p_s((x_1 - x_2)/2)P_s\psi((x_1 + x_2)/2)ds \equiv G_{\alpha,0}\psi(x_1, x_2).$$

If $\psi \in C_0(\mathbb{R}^d)$ then $\psi_\epsilon(x_1, x_2) \in C_0(\mathbb{R}^d)$ and $G_{\alpha,\epsilon}\psi \in \mathcal{D}(\mathcal{A})$ solves $\mathcal{A}G_{\alpha,\epsilon}\psi = \alpha G_{\alpha,\epsilon}\psi - \psi_\epsilon$. (If $\alpha = 0$ and $d \geq 3$ it is easy to check $G_{\alpha,\epsilon}\psi \overset{\|\,\|_\infty}{\Longrightarrow} G_{0,\epsilon}\psi$ and $\mathcal{A}G_{\alpha,\epsilon}\psi \overset{\|\,\|_\infty}{\Longrightarrow} -\psi_\epsilon$ as $\alpha \downarrow 0$.) Hence the previous lemma gives us, first for $\psi \in C_0(\mathbb{R}^d)$ and then for any $\psi \in b\mathcal{B}(\mathbb{R}^d)$ by taking bounded pointwise limits,

$$\begin{aligned} (T_\epsilon) \qquad X_t^1 \times X_t^2(G_{\alpha,\epsilon}\psi) &= m^1 \times m^2(G_{\alpha,\epsilon}\psi) \\ &+ \textstyle\int_0^t \int\int G_{\alpha,\epsilon}\psi(x_1, x_2)[X_s^1(dx_1)Z^2(ds, dx_2) + X_s^2(dx_2) + Z^1(ds, dx_1)] \\ &- \textstyle\int_0^t \int\int G_{\alpha,\epsilon}\psi(x_1, x_2)[X_s^1(dx_1)A^2(ds, dx_2) + X_s^2(dx_2)A^1(ds, dx_1)] \\ &+ \alpha \textstyle\int_0^t \int\int G_{\alpha,\epsilon}\psi(x_1, x_2)X_s^1(dx_1)X_s^2(dx_2)ds \\ &- \textstyle\int_0^t \int\int p_\epsilon(x_1 - x_2)\psi((x_1 + x_2)/2)X_s^1(dx_1)X_s^2(dx_2)ds, \\ &\qquad \text{for all } \psi \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

We now show that each term in (T_ϵ) converges as $\epsilon \downarrow 0$.

Notation. $g_\alpha(x) = \int_0^\infty e^{-\alpha t} p_t(x)dt$ if $d \geq 3$ or $\alpha > 0$. Here $g_0(x) = c_{5.1}|x|^{2-d}$

if $d \geq 3$. Define

$$g_0(x) = \begin{cases} 1 + \log^+(1/|x|) & d = 2 \\ 1 & d = 1 \end{cases}.$$

A bit of integration shows that

$$2G_{2\alpha}1(x_1, x_2) = g_\alpha(x_1 - x_2) \quad d \geq 3 \text{ or } \alpha > 0, \tag{5.5}$$

$$c_{5.2}(\alpha)\log^+(1/|x|) \leq g_\alpha(x) \leq c_{5.3}(\alpha) g_0(x) \quad d = 2, \tag{5.6}$$

$$g_\alpha(x) \leq c_{5.4}(\alpha) g_0(x) \quad d = 1. \tag{5.7}$$

Lemma 5.3 *Assume $\psi \in b\mathcal{B}(\mathbb{R}^d)$ and $\alpha \geq 0$ ($\alpha > 0$ if $d \leq 2$).*

$$(a) \quad |G_{\alpha,\epsilon}\psi(x_1, x_2) - G_\alpha\psi(x_1, x_2)| \leq \begin{cases} c_{5.5}\|\psi\|_\infty \epsilon |x_1 - x_2|^{-d} & \text{if } d \geq 2 \\ c_{5.5}\|\psi\|_\infty \epsilon^{1/2} & \text{if } d = 1 \end{cases}$$

If $\psi \geq 0$, $\lim_{\epsilon \downarrow 0} G_{\alpha,\epsilon}\psi(x_1, x_2) = G_\alpha\psi(x_1, x_2)$ for all (x_1, x_2).

$$(b) \quad |G_{\alpha,\epsilon}\psi(x_1, x_2)| \leq e^{\alpha\epsilon/2}\|\psi\|_\infty g_{\alpha/2}(x_1 - x_2)$$

Proof. (a)

$$\begin{aligned} &|G_{\alpha,\epsilon}\psi(x_1, x_2) - G_\alpha\psi(x_1, x_2)| \leq \\ &k_d\|\psi\|_\infty \int_0^\infty e^{-2\alpha s}|p_{s+\epsilon/4}((x_1 - x_2)/2) - p_s((x_1 - x_2)/2)|ds \\ &\leq K_d\|\psi\|_\infty \int_0^\infty e^{-2\alpha s} \int_s^{s+\epsilon/4} |(\partial p_u/\partial u)((x_1 - x_2)/2)|duds. \end{aligned} \tag{5.8}$$

Use $|(\partial p_u/\partial u)(z)| \leq p_u(z)(2u)^{-1}(d + |z|^2/u)$ and an elementary calculation to obtain the inequality in (a) for $d \geq 2$. If $d = 1$ the above bound on $|\partial p_u/\partial u|$ implies $|(\partial p_u/\partial u)(z)| \leq c_1 u^{-3/2}$. As we also have $|p_{s+\epsilon/4}(x) - p_s(x)| \leq s^{-1/2}(2\pi)^{-1/2}$ (5.8) shows that

$$\begin{aligned} |G_{\alpha,\epsilon}\psi - G_\alpha\psi| &\leq k_1\|\psi\|_\infty[\int_0^\epsilon (2\pi s)^{-1/2}ds + \int_\epsilon^\infty e^{-2\alpha s}c_1 s^{-3/2}(\epsilon/4)ds] \\ &\leq k_1\|\psi\|_\infty[\epsilon^{1/2} + c_1\epsilon^{1/2}/2]. \end{aligned}$$

For $d \geq 2$, pointwise convergence off the diagonal is clear from the above. On the diagonal, the pointwise convergence of $G_{\alpha,\epsilon}\psi(x_1, x_1)$ to $G_\alpha\psi(x_1, x_1)$ if $\psi \geq 0$ is a simple consequence of Monotone Convergence.

$$(b) \qquad \begin{aligned} |G_{\alpha,\epsilon}\psi(x_1, x_2)| &\leq k_d\|\psi\|_\infty \textstyle\int_0^\infty e^{-2\alpha s} p_{s+\epsilon/4}((x_1 - x_2)/2)ds \\ &= k_d\|\psi\|_\infty \textstyle\int_{\epsilon/4}^\infty e^{\alpha\epsilon/2} e^{-2\alpha u} p_u((x_1 - x_2)/2)du \\ &\leq e^{\alpha\epsilon/2}\|\psi\|_\infty g_{\alpha/2}(x_1 - x_2). \qquad \square \end{aligned}$$

In order to control the martingale term on the right-hand side of (T_ϵ) as $\epsilon \downarrow 0$ we need the following estimate which is proved in Section 7.

Lemma 5.4 *Let Y^1 and Y^2 be independent $(\mathcal{F}_t)$-super-Brownian motions starting at m^1 and m^2, respectively.*

(a) If $3 \leq d \leq 5$ there are constants $c_{5.6}(t)$ $(t > 0)$ such that

$$I\!P\Big(\int_0^L \int \Big(\int g_\alpha(y_1 - y_2) Y_s^1(dy_1)\Big)^2 Y_s^2(dy_2)ds\Big)$$

$$\leq c_{5.6}(t)(m^1(1) + 1)\begin{cases} \int\int(\log^+(1/|x_1 - x_2|) + 1)m^1(dx_1)m^2(dx_2), & \textit{if } d = 3, \\ \int\int(|x_1 - x_2|^{6-2d} + 1)m^1(dx_1)m^2(dx_2), & \textit{if } d = 4, 5, \end{cases}$$

$$\textit{for all } \alpha \geq 0, t > 0.$$

(b) If $d \leq 2$ there are constants $c_{5.7}(\alpha, t)$ $(\alpha, t > 0)$ such that

$$I\!P\Big(\int_0^t \int \Big(\int g_\alpha(y_1 - y_2) Y_s^1(dy_1)\Big)^2 Y_s^2(dy_2)ds\Big)$$

$$\leq c_{5.7}(\alpha, t)m^1(1)(m^1(1) + 1)m^2(1).$$

Recall that $\mathcal{M} = \mathcal{M}(m^1, m^2)$ is the set of all continuous adapted processes (X^1, X^2) satisfying (M_{m^1,m^2}).

Corollary 5.5 *Assume $d \leq 5$ and*

$$(5.9) \qquad \int\int \log^+(1/|x_1 - x_2|)m^1(dx_1)m^2(dx_2) < \infty \quad \textit{if } d = 3$$

$$(5.10) \qquad \int\int |x_1 - x_2|^{6-2d} m^1(dx_1)m^2(dx_2) < \infty \quad \textit{if } d = 4, 5.$$

Then for any $\psi \in \mathcal{B}(\mathbb{R}^d)$, $T > 0$ and $\alpha \geq 0$ ($\alpha > 0$ if $d \leq 2$),

$$\lim_{\epsilon\to 0+} \sup_{(X^1,X^2)\in\mathcal{M}} \| \sup_{t\leq T} \int_0^t \int\int G_{\alpha,\epsilon}\psi(x_1,x_2)X_s^1(dx_1)Z^2(dx_1,dx_2)$$

$$- \int_0^t \int\int G_\alpha\psi(x_1,x_2)X_s^1(dx_1)Z^2(dx_1 dx_2)\|_2.$$

For any $(X^1,X^2) \in \mathcal{M}$, $\int_0^t \int\int G_\alpha\psi(x_1,x_2)X_s^1(dx_1)Z^2(ds_1,dx_2)$ is a continuous L^2-martingale.

Proof. For $\epsilon > 0$ and $(X^1,X^2) \in \mathcal{M}$, Doob's inequality and $X^i \leq Y^i$ shows the above L^2-norm is bounded by

$$c_1 \mathbb{P}\Big(\int_0^T \int\Big(\int |G_{\alpha,\epsilon}\psi(y_1,y_2) - G_\alpha\psi(y_1,y_2)|Y_s^1(dy_1)\Big)^2 Y_s^2(dy_2)ds\Big)$$

$$= c_1 \mathbb{P}\Big(\int_0^T \int\int\int |G_{\alpha,\epsilon}\psi(y_1,y_2) - G_\alpha\psi(y_1,y_2)|$$

$$\times|G_{\alpha,\epsilon}(y_1',y_2) - G_\alpha\psi(y_1',y_2)|Y_s^1(dy_1)Y_s^1(dy_1')Y_s^2(dy_2)ds\Big). \tag{5.11}$$

By Lemma 5.3(b), if $\epsilon \leq 1$ the integrand in (5.11) is bounded by $4e^\alpha\|\psi\|_\infty^2$ $g_{\alpha/2}(y_1 - y_2)g_{\alpha/2}(y_1' - y_2)$ which is integrable with respect to the measure $(Y_s^1(dy_1)Y_s^1(dy_1')Y_s^2(dy_2)dsd\mathbb{P})$ by Lemma 5.4. The integrand approaches 0 as $\epsilon \downarrow 0$ on $\{(y_1,y_1',y_2)\colon y_1 \neq y_2 \text{ and } y_1' \neq y_2\}$ and this set is not charged by the above measure because Y^2 is independent of Y^1 and does not charge points. The Dominated Convergence Theorem implies (5.11) approaches 0 as $\epsilon \to 0+$ and the corollary follows. □

Lemma 5.6 *(a) Assume $d \leq 5$, and (5.9) and (5.10) hold. Then for any $\psi \in b\mathcal{B}(\mathbb{R}^d)$, $T > 0$ and $\alpha \geq 0$ ($\alpha > 0$ if $d \leq 2$)*

$$\lim_{\epsilon\to 0+} \sup_{(X^1,X^2)\in\mathcal{M}} \| \sup_{t\leq T} \Big| \int_0^t \int\int G_{\alpha,\epsilon}\psi(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds$$

$$- \int_0^t \int\int G_\alpha\psi(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds\Big|\|_1$$

$$= 0. \tag{5.12}$$

For any $(X^1, X^2) \in \mathcal{M}$, $\int_0^t \int\int G_\alpha\psi(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds$ is a continuous process with $I\!P$ integrable total variation over compact time intervals.

(b) Assume $d \le 5$ and

$$(5.13) \qquad \int\int g_0(x_1,x_2)dm^1(x_1)dm^2(x_2) < \infty \quad \text{if} \quad d \le 4,$$

$$(5.14) \qquad \int\int |x_1 - x_2|^{-4} dm^1(x_1)dm^2(x_2) < \infty \quad \text{if} \quad d = 5.$$

Then for any $\psi \in b\mathcal{B}(I\!R^d)$, $T > 0$ and $\alpha \ge 0$ ($\alpha > 0$ if $d \le 2$)

$$(5.15) \qquad \lim_{\epsilon \to 0+} \sup_{(X^1,X^2)\in\mathcal{M}} \| \sup_{t \le T} | \int\int G_{\alpha,\epsilon}\psi(x_1,x_2)X_t^1(dx_1)X_t^2(dx_2)$$

$$- \int\int G_\alpha\psi(x_1,x_2)X_t^1(dx_1)X_t^2(dx_2)| \|_1 = 0.$$

For any $(X^1, X^2) \in \mathcal{M}$, $X_t^1 \times X_t^2(G_\alpha\psi)$ is a continuous process such that $\sup_{t\le T} |\int\int G_\alpha\psi d(X_t^1 \times X_t^2)| \in L^1$ for any $T > 0$.

Proof. (a) The supremum inside the L^1-norm in (5.12) is bounded by

$$(5.16) \qquad \int_0^t \int\int |G_{\alpha,\epsilon}\psi(y_1,y_2) - G_\alpha\psi(y_1,y_2)| Y_s^1(dy_1)Y_s^2(dy_2)ds.$$

If $d = 1$ the above integrand converges uniformly to 0 and hence (5.16) approaches 0 in L^1. Assume now $d > 1$. By Lemma 5.3 the integrand in (5.16) converges to 0 as $\epsilon \downarrow 0$ for all $y_1 \ne y_2$, and is bounded by $2e^{\alpha/2}\|\psi\|_\infty g_{\alpha/2}(y_1 - y_2)$ for $\epsilon \le 1$. Lemma 5.4 and Cauchy-Schwarz imply $g_{\alpha/2}(y_1 - y_2)$ belongs to $L^1(1(s \le T)Y_s^1(dy_1)Y_s^2(dy_2)ds dI\!P)$, and since this measure does not charge the diagonal $\{y_1 = y_2\}$, Dominated Convergence implies (5.16) converges to 0 in $L^1(I\!P)$ as $\epsilon \downarrow 0$. This gives (5.12). Since $G_{\alpha,\epsilon}\psi$ is bounded and continuous, $\int_0^t \int\int G_{\alpha,\epsilon}\psi(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds$ is a continuous process with integrable total variation over compact intervals. The convergence established above now gives the same conclusion with $\epsilon = 0$.

(b) If $d = 1$ the uniform convergence of $|G_{\alpha,\epsilon}\psi - G_\alpha\psi|$ to 0 (Lemma 5.3) and the bound $X_t^i \le Y_t^i$ make (5.15) obvious. Assume therefore $d \ge 2$ and for a fixed $\psi \in \mathcal{B}(I\!R^d)$ define

$$\eta(\alpha, \epsilon, \delta) = \sup\{|G_\alpha\psi(x_1,x_2) - G_{\alpha,\epsilon}\psi(x_1,x_2)| : |x_1 - x_2| \ge \delta\}.$$

Recall that $\lim_{\epsilon\to 0+}\eta(\alpha,\epsilon,\delta)=0$ for all $\delta>0$ by Lemma 5.3 (a). Let $(X^1,X^2)\in\mathcal{M}$ and $T>0$. Then for $\epsilon\le 1$,

$$\sup_{t\le T}|X_t^1\times X_t^2(G_{\alpha,\epsilon}\psi)-X_t^1\times X_t^2(G_\alpha\psi)|\le\eta(\alpha,\epsilon,\delta)\sup_{t\le T}X_t^1(1)X_t^2(1)$$

$$+c_1(\alpha,\psi)\sup_{t\le T}\int\int 1(|x_1-x_2|\le\delta)g_{\alpha/2}(x_1-x_2)X_t^1(dx_1)X_t^2(dx_2)\quad\text{(Lemma 5.3 (b))}$$

$$\le\eta(\alpha,\epsilon,\delta)\sup_{t\le T}Y_t^1(1)Y_t^2(1) \tag{5.17}$$

$$+c_1(\alpha,\psi)\sup_{t\le T}\int\int 1(|y_1-y_2|\le\delta)g_{\alpha/2}(y_1-y_2)Y_t^1(dy_1)Y_t^2(dy_2).$$

Let R_t denote the image of $Y_t^1\times Y_t^2$ under the mapping $(y_1,y_2)\to|y_1-y_2|$. Then if $\beta>0$, $r_1(\omega,\beta)\ge\delta$ (r_1 as in Theorem 4.11) and we write $g_0(|y|)=g_0(y)$,

$$\begin{aligned}
&\sup_{\beta\le t\le T}\int\int 1(|y_1-y_2|\le\delta)g_0(y_1-y_2)Y_t^1(dy_1)Y_t^2(dy_2)\\
&\le\sup_{\beta\le t\le T}R_t([0,\delta])g_0(\delta)-\int 1(r\le\delta)g_0'(r)R_t([0,r])dr\\
&\le c_{4.4}(\sup_t Y_t^1(1)\vee 1)[\delta^{4-4/d}(\log 1/\delta)^{2/d+2}g_0(\delta)-\int 1(r\le\delta)g_0'(r)r^{4-4/d}(\log 1/r)^{2/d+2}dr\\
&\qquad\text{(Theorem 4.11)}\\
&\to 0\quad\text{as}\quad\delta\downarrow 0\quad(\text{recall}\quad d\le 5).
\end{aligned}$$

We have shown

(5.18)

$$\lim_{\delta\downarrow 0}\sup_{\beta\le t\le T}\int\int 1(|y_1-y_2)\le\delta)g_0(y_1-y_2)Y_t^1(dy_1)Y_t^2(dy_2)=0\quad\forall\beta>0\quad\text{a.s.}$$

Let $\psi=1$ in (T_ϵ) with $(X^1,X^2)=(Y^1,Y^2)$ to see

$$\sup_{t\le T}\int\int G_{\alpha,\epsilon}1(y_1,y_2)Y_t^1(dy_1)Y_t^2(dy_2) \tag{5.19}$$

$$\le m^1\times m^2(G_{\alpha,\epsilon}1)+\sup_{t\le T}\int_0^t\int\int G_{\alpha,\epsilon}1[Y_s^1(dy_1)Z^2(ds,dy_2)+Y_s^2(dy_2)Z^1(ds,dy_1)]$$

$$+\alpha\int_0^T\int\int G_{\alpha,\epsilon}1Y_s^1(dy_1)Y_s^2(dy_2)ds.$$

The right side converges in L^1 as $\epsilon\downarrow 0$ to

(5.20)

$$m^1\times m^2(G_\alpha 1)+\sup_{t\le T}\int_0^t\int\int G_\alpha 1[Y_s^1(dy_1)Z^2(ds,dy_2)+Y_s^2(dy_2)Z^1(ds,dy_1)]$$

$$+\alpha\int_0^T\int\int G_\alpha 1Y_s^1(dy_1)Y_s^2(dy_2)ds\equiv m^1\times m^2(G_\alpha 1)+\Gamma_T$$

by Corollary 5.5, (a) and the hypotheses on m^1, m^2 (see (5.5)). Apply Fatou's lemma in (5.19) (use Lemma 5.3 and (5.5)) to conclude that

$$\begin{aligned}\sup_{t\le T}\Big(&\int\int g_{\alpha/2}(y_1-y_2)Y_t^1(dy_1)Y_t^2(dy_2)\\ &-\int\int g_{\alpha/2}(y_1-y_2)m^1(dy_1)m^2(dy_2)\Big)\\ &\le 2\Gamma_T\end{aligned}\tag{5.21}$$

and in particular

$$E(\sup_{t\le T}\int\int g_\alpha(y_1-y_2)Y_t^1(dy_1)Y_t^2(dy_2))<\infty.\tag{5.22}$$

Let $\delta_n\downarrow 0$ satisfy $m^1\times m^2(|y_1-y_2|=\delta_n)=0$. Then

$$\sup_{t\le\beta}\int\int g_{\alpha/2}(y_1-y_2)1(|y_1-y_2|\le\delta_n)Y_t^1(dy_1)Y_t^2(dy_2)$$

$$\begin{aligned}&\le\sup_{t\le\beta}\Big(\int\int g_{\alpha/2}(y_1-y_2)Y_t^1(dy_1)Y_t^2(dy_2)-\int\int g_{\alpha/2}(y_1-y_2)m^1(dy_1)m^2(dy_2)\Big)\\ &\quad+\sup_{t\le\beta}\Big|\int\int g_{\alpha/2}(y_1-y_2)1(|y_1-y_2|>\delta_n)Y_t^1(dy_1)Y_t^2(dy_2)\\ &\quad-\int\int g_{\alpha/2}(y_1-y_2)1(|y_1-y_2|>\delta_n)m^1(dy_1)m^2(dy_2)\Big|\\ &\quad+\int\int g_{\alpha/2}(y_1-y_2)1(|y_1-y_2|\le\delta_n)m^1(dy_1)m^2(dy_2)\end{aligned}$$

$$=I(\beta)+II(\delta_n,\beta)+III(\delta_n).\tag{5.23}$$

Fix ω outside of a null set such that $\Gamma_\beta(\omega) \to 0$ as $\beta \downarrow 0$, (5.18) holds and $II(\delta_n, \beta) \to 0$ as $\beta \downarrow 0$ for all n. The latter is a simple consequence of the weak continuity of $Y_t^1 \times Y_t^2$ and the choice of $\{\delta_n\}$. If $\epsilon_0 > 0$ choose N_1 so that $III(\delta_n) < \epsilon_0$ for $n \geq N_1$. Next use (5.21) and the choice of ω to find $\beta > 0$ such that $I(\beta) + II(\delta_{N_1}, \beta) < \epsilon_0$. (5.23) implies that for $n \geq N_1$,

$$\sup_{t \leq \beta} \int\int g_{\alpha/2}(y_1 - y_2) 1(|y_1 - y_2| \leq \delta_n) Y_t^1(dy_1) Y_t^2(dy_2) < 2\epsilon_0.$$

Now use (5.18) to find another N_2 so that for $n \geq N_2$

$$\sup_{\beta \leq t \leq T} \int\int g_{\alpha/2}(y_1 - y_2) 1(|y_1 - y_2| \leq \delta_n) Y_t^1(dy_1) Y_t^2(dy_2) < 2\epsilon_0.$$

We have proved

$$\lim_{\delta \downarrow 0} \sup_{0 \leq t \leq T} \int\int g_{\alpha/2}(y_1 - y_2) 1(|y_1 - y_2| \leq \delta) Y_t^1(dy_1) Y_t^2(dy_2) = 0 \quad \text{a.s.}$$

and hence also in L^1 by (5.22). Returning to (5.17) we may first choose δ sufficiently small and then ϵ so that the L^1 norm of (5.17) is less than our given ϵ_0. As the estimate is uniform in $(X^1, X^2) \in \mathcal{M}$, the proof of (5.15) is complete. The last statement is immediate from (5.15) and the weak continuity of $X_t^1 \times X_t^2$. $\square$

Remark. It is clear from the above arguments that if one drops the sup over (X^1, X^2) and deals with a single $(X^1, X^2) \in \mathcal{M}$ then a.s. convergence as well as L^1-convergence holds in both (a) and (b).

Lemma 5.7 *Assume $(X^1, X^2) \in \mathcal{M}$, $d \leq 5$, and (5.9) and (5.10) hold. Then for any $\psi \in b\mathcal{B}(\mathbb{R}^d)$, $T > 0$ and $\alpha \geq 0$ ($\alpha > 0$ if $d \leq 2$),*

$$\int_0^T \int\int |G_{\alpha,\epsilon}\psi(x_1, x_2) - G_\alpha \psi(x_1, x_2)| X_s^1(dx_1) A^2(ds, dx_2)$$

$$\text{(5.24)} \qquad \to 0 \quad \textit{a.s. and in } L^1,$$

and $\int_0^t \int\int G_\alpha \psi(x_1, x_2) X_s^1(dx_1) A^2(ds, dx_2)$ is a continuous process with integrable total variation over compact time intervals.

Proof. We may assume $\psi \geq 0$. From (T_ϵ) we have

$$\int_0^t \int \int G_{\alpha,\epsilon}\psi(x_1,x_2)X_s^1(dx_1)A^2(ds,dx_2)$$

(5.25)

$$\leq m^1 \times m^2(G_{\alpha,\epsilon}\psi) + \int_0^t \int \int G_{\alpha,\epsilon}\psi(x_1,x_2)[X_s^1(dx_1)Z^2(ds,dx_2) + X_s^2(dx_2)Z^1(ds,dx_1)]$$

$$+\alpha \int_0^t \int \int G_{\alpha,\epsilon}\psi(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds.$$

The hypotheses on m^i, Corollary 5.5 and Lemma 5.6(a) imply the right side of (5.25) converges in L^1 uniformly in $t \leq T$ to

$$\Lambda_t = m^1 \times m^2(G_\alpha\psi) \quad + \quad \int_0^t \int \int G_\alpha\psi(x_1,x_2)[X_s^1(dx_1)Z^2(ds,dx_2) + X_s^2(dx_2)Z^1(ds,dx_1)]$$
$$+ \quad \alpha \int_0^t \int \int G_\alpha\psi(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds.$$

Apply Lemma 5.3 and Fatou's lemma to conclude (take $\psi = 1$ and use (5.5))

$$(5.26) \quad \int_0^t \int \int g_{\alpha/2}(x_1 - x_2)X_s^1(dx_1)A^2(ds,dx_2) \leq 2\Lambda_t \quad \text{for all } \ t \geq 0 \ \text{ a.s.}$$

If $d \geq 2$ (5.26) shows that $X_s^1(dx_1)A^2(ds,dx_2)$ does not charge $\{x_1 = x_2\}$ a.s. Since $\sup_{t \leq T} \Lambda_t \in L^1$, (5.26) and Lemma 5.3(b) allow us to apply Dominated Convergence to derive (5.24). (If $d = 1$ $\|G_{\alpha,\epsilon}\psi - G_\alpha\psi\|_\infty \to 0$ and there is no need to worry about the diagonal.) The last statement of the lemma is then obvious from the convergence in (5.24) and (5.25) (the latter for the integrability of the total variation on compacts.) □

Lemma 5.8 *Assume $d \leq 3$ and*

$$(5.27) \qquad \int_0^1 r^{1-d}D(m^1,r)dr < \infty.$$

Then for any $\psi \in b\mathcal{B}(\mathbb{R}^d)$, $T > 0$ and $\alpha \geq 0$ ($\alpha > 0$ if $d \leq 2$)

$$\lim_{\epsilon \to 0+} \sup_{(X^1,X^2)\in\mathcal{M}} \|\min(\int_0^T \int \int |G_{\alpha,\epsilon}\psi(x_1,x_2) - G_\alpha\psi(x_1,x_2)|X_s^1(dx_1)A^2(ds,dx_2), 1)\|_1 = 0.$$

Proof. If $\eta(\alpha,\epsilon,\delta)$ is defined as in the proof of Lemma 5.6(b) then Lemma 5.3(b) gives us

$$\int |G_{\alpha,\epsilon}\psi(x_1,x_2) - G_\alpha\psi(x_1,x_2)|X_s^1(dx_1)$$

$$\begin{aligned}
&\le \eta(\alpha,\epsilon,\delta)X_s^1(1) \\
&+c_1(\alpha,\psi)\textstyle\int 1(|x_1-x_2|\le\delta)g_{\alpha/2}(x_1-x_2)X_s^1(dx_1) \\
&\le \eta(\alpha,\epsilon,\delta)Y_s^1(1) \\
&+c_2(\alpha,\psi)\textstyle\int 1(|x_1-x_2|\le\delta)g_0(x_1-x_2)Y_s^1(dx_1) \quad \text{(by (5.6, 5.7))}\ .
\end{aligned} \tag{5.28}$$

If $R_s^{x_2}([0,r]) = Y_s^1(B(x_2,r))$ then, integrating by parts, we have for $d=2$ or 3, and $\delta < r_0(\omega)$ ($r_0(w)$ as in Theorem 4.7)

$$\int 1(|x_1-x_2|\le\delta)g_0(x_1-x_2)Y_s^1(dx_1) = \int 1(r\le\delta)g_0(r)dR_s^{x_2}(r) \tag{5.29}$$

$$\begin{aligned}
&\le\ R_s^{x_2}([0,\delta])g_0(\delta) - \int_0^\delta g_0'(r)R_s^{x_2}[0,r]dr \\
&\le\ c_{4.7}(D(m^1,c_{4.8}\delta)+\varphi(\delta))g_0(\delta) \\
&-\ \int_0^\delta g_0'(r)c_{4.7}(D(m^1,c_{4.8}r)+\varphi(r))dr \\
&\qquad \text{(Lemma 4.6(a) and Theorem 4.7)} \\
&\equiv\ f(\delta).
\end{aligned}$$

Note that if $\delta_0 > \delta$

$$\begin{aligned}
\int_\delta^{\delta_0} -g_0'(r)D(m^1,c_{4.8}r)dr &= c_1\int_{c_{4.8}\delta}^{c_{4.8}\delta_0} -g_0'(s)D(m^1,s)ds \\
&\ge c_1D(m^1,c_{4.8}\delta)(g_0(c_{4.8}\delta)-g_0(c_{4.8}\delta_0)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\limsup_{\delta\downarrow 0}D(m^1,c_{4.8}\delta)g_0(\delta) &\le c_2(\limsup_{\delta\downarrow 0}D(m^1,c_{4.8}\delta)g_0(c_{4.8}\delta_0) \\
&+ \int_0^{\delta_0} -g_0'(r)D(m^1,c_{4.8}r)dr) \\
&= c_3\int_0^{c_{4.8}\delta_0} -g_0'(s)D(m^1,s)ds.
\end{aligned}$$

Taking $\delta_0 \downarrow 0$ we see that the left side of the above is zero and hence by (5.27) the same is true for $f(\delta)$. Returning to (5.28), we have proved (for $d = 2$ or 3)

$$\min(\int_0^T \int\int |G_{\alpha,\epsilon}\psi(x_1, x_2) - G_\alpha\psi(x_1, x_2)| X_s^1(dx_1) A^2(ds, dx_2), 1)$$

(5.30)
$$\le 1(r_0(w) \le \delta) + \eta(\alpha, \epsilon, \delta) \sup_{s \le T} Y_s^1(1) A_T^2(1) + c_2(\alpha, \psi) f(\delta) A_T^2(1)$$
$$\le 1(r_0(w) \le \delta) + \eta(\alpha, \epsilon, \delta) \sup_{s \le T} Y_s^2(1)(m^2(1) + Z_T^2(1))$$
$$+ c_2(\alpha, \psi) f(\delta)(m^2(1) + Z_T^2(1)) \text{ (by } (M_{m^1,m^2})).$$

Recall that

$$E(Z_T^2(1)^2) = E\int_0^T X_s^2(1) ds \le E\int_0^T Y_s^2(1) ds = Tm^2(1), \tag{5.31}$$

$\lim_{\epsilon \downarrow 0} \eta(\alpha, \epsilon, \delta) = 0$, and $\lim_{\delta \downarrow 0} P(r_0 \le \delta) = 0$ (Theorem 4.7). It is therefore clear that by first choosing δ and then ϵ sufficiently small we can make the mean value of (5.30) uniformly small over all (X^1, X^2) in $\mathcal{M}$. This gives the required result for $d = 2$ or 3. If $d = 1$ the result is immediate from the uniform convergence of $|G_{\alpha,\epsilon}\psi - G_\alpha\psi|$ to 0 (see Lemma 5.3) and the fact that

$$\begin{aligned} E(\int_0^T X_s^1(1) A_s^2(1)) &\le E(\sup_{s \le T} X_s^1(1)^2)^{1/2} E(A_T^2(1)^2)^{1/2} \\ &\le E(\sup_{s \le T} Y_s^1(1)^2)^{1/2} [m^2(1) + E(Z_T^2(1)^2)^{1/2}] \text{ (by } (M_{m^1,m^2})) \end{aligned}$$

is uniformly bounded (over $(X^1, X^2) \in \mathcal{M}$) by (5.31). $\square$

Theorem 5.9 *Assume* $d \le 5$,

$$\int\int g_0(x_1 - x_2) dm^1(x_1) dm^2(x_2) < \infty \quad \textit{if } d \le 4, \tag{5.32}$$

$$\int\int |x_1 - x_2|^{-4} dm^1(x_1) dm^2(x_2) < \infty \quad \textit{if } d = 5, \tag{5.33}$$

and (X^1, X^2) *satisfies* (M_{m^1,m^2}). *Then* (X^1, X^2) *has a continuous (in* t*) collision local time* $L_t(X^1, X^2)$. *For any* $\psi \in b\mathcal{B}(\mathbb{R}^d)$, $L_t(X^1, X^2)(\psi)$ *is a.s.-continuous in* t *and satisfies*

$$(5.34) \quad \lim_{\epsilon \downarrow 0} \| \sup_{t \leq T} |L_t^\epsilon(X^1, X^2)(\psi) - L_t(X^1, X^2)(\psi)| \|_1 = 0, \quad \textit{for all } T > 0,$$

and, for all $\alpha \geq 0$ *(*$\alpha > 0$ *if* $d \leq 2$*)*

$$X_t^1 \times X_t^2(G_\alpha \psi)$$

$$= m^1 \times m^2(G_\alpha \psi) +$$

$$(T) \qquad \int_0^L \int \int G_\alpha(x_1, x_2)[X_s^1(dx_1) Z^2(ds, dx_2) + X_s^2(dx_2) Z^1(ds, dx_1)]$$

$$- \int_0^t \int \int G_\alpha \psi(x_1, x_2)[X_s^1(dx_1) A^2(ds, dx_2) + X_s^2(dx_2) A^1(ds, dx_1)]$$

$$+\alpha \int_0^t \int \int G_\alpha \psi(x_1, x_2) X_s^1(dx_1) X_s^2(dx_2) ds - L_t(X^1, X^2)(\psi) \quad \textit{for all } t \geq 0 \quad \textit{a.s.}$$

Each process in (T) is a.s. continuous in t. *The first process on the right-hand side of (T) is an* L^2*-martingale and each of the other processes on the right-hand side has integrable total variation on compact time intervals.*

Proof. Let $\psi \in b\mathcal{B}(\mathbb{R}^d)$. The hypotheses on (m^1, m^2) and Lemma 5.3 allow us to apply Dominated Convergence to conclude

$$(5.35) \qquad \lim_{\epsilon \downarrow 0} m^1 \times m^2(G_{\alpha,\epsilon} \psi) = m^1 \times m^2(G_\alpha \psi).$$

(Note that if $d = 1$ this is trivial by the uniform convergence of $G_{\alpha,\epsilon}\psi$ and for $d > 1$, $m^1 \times m^2$ does not charge the diagonal by (5.32) and (5.33).) Corollary 5.5 and Lemmas 5.6 and 5.7 show that as $\epsilon \downarrow 0$ each of the remaining terms in (T_ϵ), except possibly for the last term on the right-hand side, converge in L^1 uniformly in $t \leq T$ for any $T > 0$. Hence there exists an adapted continuous process $\bar{L}_t(X^1, X^2)(\psi)$ such that

$$(5.36) \quad \lim_{\epsilon \downarrow 0} \| \sup_{t \leq T} |L_t^\epsilon(X^1, X^2)(\psi) - \bar{L}_t(X^1, X^2)(\psi)| \|_1 = 0 \ \text{ for all } \ T > 0.$$

We may now obtain $(\bar{T})$ (this is (T) but with $\bar{L}$ in place of L) by letting $\epsilon \downarrow 0$ in (T_ϵ) and applying Corollary 5.5 and Lemmas 5.6 and 5.7.

We now show there is a continuous, non-decreasing $M_F(\mathbb{R}^d)$-valued process $L_t(X^1, X^2)$ such that $L_t(X^1,X^2)(\psi) = \bar{L}_t(X^1,X^2)(\psi)$ for all $t \geq 0$ a.s. and all $\psi \in b\mathcal{B}(\mathbb{R}^d)$. Let $\mathcal{D}$ be a countable dense set in $C_\ell(\mathbb{R}^d)$ containing 1. (5.36) and a diagonalization argument show there is a sequence $\epsilon_n \downarrow 0$ such that

$$\text{(5.37)} \qquad \lim_{n\to 0} \sup_{t \leq T} |L_t^{\epsilon_n}(X^1,X^2)(\psi) - \bar{L}_t(X^1,X^2)(\psi)| = 0$$

$$\text{for all } \ T > 0 \ \text{ and } \ \psi \in \mathcal{D} \ \text{ a.s.}$$

The bound $X^i \leq Y^i$ and Theorems 1.1 and 1.2 of DIP (1989) imply w.p.1 for any $\delta > 0$ there is a compact set K which supports $L_\infty^{\epsilon_n}(X^1,X^2) - L_\delta^{\epsilon_n}(X^1,X^2)$ for all $n \in \mathbb{N}$. Fix w outside a $\mathbb{P}$-null set so that this conclusion and (5.37) both hold and let $\eta > 0$. Since $\bar{L}_t(X^1,X^2)(1)$ is continuous there is a $\delta > 0$ such that $L_\delta^{\epsilon_n}(X^1,X^2)(1) < \eta$ for all $n \in \mathbb{N}$. Choose a compact K as above. Then

$$L_\infty^{\epsilon_n}(X^1,X^2)(K^c) < \eta \ \text{ for all } \ n \in \mathbb{N}$$

and hence $\{L_t^{\epsilon_n}(X^1,X^2)\colon n \in \mathbb{N}\}$ is tight. (5.34) and Prohorov's theorem imply $L_t^{\epsilon_n}(X^1,X^2) \xrightarrow{w} L_t(X^1,X^2)$ where the limiting $M_F(\mathbb{R}^d)$-valued process satisifies

$$\text{(5.38)} \ \ L_t(X^1,X^2)(\psi) = \bar{L}_t(X^1,X^2)(\psi) \ \text{ for all } \ t \geq 0 \ \text{ and all } \ \psi \in \mathcal{D} \ \text{ a.s. .}$$

This implies $L_t(X^1,X^2)$ is weakly continuous. It is also clearly increasing in t since $L_t^{\epsilon_n}(X^1,X^2)$ is. Let $\psi_n \xrightarrow{bp} \psi$ ($\psi \in b\mathcal{B}(\mathbb{R}^d)$ and $\xrightarrow{bp}$ indicates bounded pointwise convergence). Then $G_\alpha\psi_n(x_1,x_2) \to G_\alpha\psi(x_1,x_2)$ for all $x_1 \neq x_2$ (and for all (x_1,x_2) if $d = 1$) and $|G_\alpha\psi_n(x_1,x_2)| \leq c_1 g_{\alpha/2}(x_1 - x_2)$ (this is clear from the estimates in Lemma 5.3.) Set $\psi = \psi_n$ in $(\bar{T})$ and let $n \to \infty$. The martingale term will converge uniformly in $t \leq T$ in L^2 to the same expression with ψ in place of ψ_n (by Lemma 5.4). Each of the remaining terms in $(\bar{T})$ except possibly the very last will converge for all $t \geq 0$ a.s. by Dominated Convergence to the corresponding term with ψ in place of ψ_n. Therefore for an appropriate subsequence we have

$$\bar{L}_t(X^1,X^2)(\psi_{n_k}) \to \bar{L}_t(X^1,X^2)(\psi) \ \text{ for all } \ t \geq 0 \ \text{ a.s. .}$$

As this is trivial for L, it follows from (5.38) that $L_t(X^1,X^2)(\psi) = \bar{L}_t(X^1,X^2)(\psi)$ for all $t \geq 0$ a.s. for all $\psi \in b\mathcal{B}(\mathbb{R}^d)$. Therefore (5.36) implies (5.34) and shows $L_t(X^1,X^2)$ is the collision local time of (X^1,X^2). This also gives the a.s.-continuity of $L_t(X^1,X^2)(\psi)$ for each $\psi \in b\mathcal{B}(\mathbb{R}^d)$, and shows that $(\bar{T})$ implies (T). The required properties of the other processes in (T) (continuity, martingale property and integrable variation over compacts) are immediate from Lemmas 5.6, 5.7 and Corollary 5.5. □

Theorem 5.10 *Assume $d \leq 3$ and*

$$\int_0^1 r^{1-d} D(m^i, r)dr < \infty \quad i = 1,2. \tag{5.39}$$

Then for any $\psi \in b\mathcal{B}(\mathbb{R}^d)$ and $T > 0$

$$\lim_{\epsilon \downarrow 0} \sup_{(X^1,X^2)\in\mathcal{M}} \|\min(\sup_{t\leq T} |L_t^\epsilon(X^1,X^2)(\psi) - L_t(X^1,X^2)(\psi)|, 1)\|_1 = 0.$$

Proof. An integration by parts shows that (5.39) implies the finite energy conditions ((5.32), (5.33)) in Theorem 5.9 (see (5.29)). Comparing (T) and (T_ϵ) we see that the result is a consequence of Corollary 5.5, Lemmas 5.6, 5.8 and (5.35). □

Proposition 5.11 *Suppose that (Y^1,Y^2) are as in Theorem 1.2, and are therefore independent super-Brownian motions starting at m_1 and m_2, respectively. Assume $d \leq 5$, and $m^i \neq 0$ $(i = 1,2)$.*

(a) $\mathbb{P}(G(Y^1) \cap G(Y^2) \neq \emptyset) > 0$.

(b) If (m^1,m^2) satisfy (5.32) and (5.33) then $\mathbb{P}(L_t(Y^1,Y^2) > 0) > 0$ for all $t > 0$.

Proof. (a) An easy first moment argument shows that for any $\delta > 0$, (X_δ^1, X_δ^2) a.s. satisfies the hypotheses (5.32) and (5.33) on (m^1,m^2). An application of Theorem 5.9 and the Markov property shows that if

$$L_{\delta,t}^\epsilon(Y^1,Y^2) = L_t^\epsilon(Y^1,Y^2) - L_\delta^\epsilon(Y^1,Y^2), \quad t \geq \delta$$

then there is a continuous measure-valued process $L_{\delta,t}(X^1,X^2)$ such that

$$\lim_{\epsilon\downarrow 0} \mathbb{P}\big(\sup_{\delta\leq t\leq T} |L_{\delta,t}^\epsilon(Y^1,Y^2)(\psi) - L_{\delta,t}(Y^1,Y^2)(\psi)||F_\delta\big) = 0$$

for all $\psi \in b\mathcal{B}(\mathbb{R}^d)$, and $T \geq \delta$. Note therefore that (take $\psi = 1$) for $t > \delta$

$$\mathbb{P}(L_{\delta,t}(Y^1,Y^2)(1) \mid \mathcal{F}_\delta) = \lim_{\epsilon\downarrow 0} \mathbb{P}(L^\epsilon_{\delta,t}(Y^1,Y^2)(1)|\mathcal{F}_\delta)$$

$$= \lim_{\epsilon\downarrow 0} \int_0^{t-\delta} \int\int p_\epsilon(x_1 - x_2)(Y^1_\delta P_s)(dx_1)(Y^2_\delta P_s)(dx_2)ds$$

$$= \int_0^{t-\delta} \int\int p_{2s}(x_1-x_2)Y^1_\delta(dx_1)Y^2_\delta(dx_2)ds \quad \text{((5.5), (5.6) and Dominated Convergence)}$$

$$> 0 \quad \text{a.s. on } \{Y^1_\delta \neq 0, Y^2_\delta \neq 0\}. \tag{5.40}$$

Therefore (1.1) shows that $\mathbb{P}(\bar{G}(Y^1) \cap \bar{G}(Y^2) \neq \emptyset) > 0$. The results of Section 4 of Perkins (1990) (see especially Proposition 4.7) show that $\bar{G}(Y^1) - G(Y^1)$ is countable. As $t \mapsto L_{\delta,t}(Y^1,Y^2)(1)$ is continuous we can in fact infer from (1.1) that $\mathbb{P}(G(Y^1) \cap G(Y^2) \neq \emptyset) > 0$.

(b) This is immediate from (5.40) because in this case $L_{\delta,t}(Y^1,Y^2) = L_t(Y^1,Y^2) - L_\delta(Y^1,Y^2)$. □

Remarks 5.12. (1) If $d \geq 6$, Theorem 3.6 implies that

$$\lim_{\epsilon\downarrow 0} \int_\delta^\infty \int\int p_\epsilon(y_1 - y_2)Y^1_t(dy_1)Y^2_t(dy_2)dt = 0 \quad \text{for all } \delta > 0 \text{ a.s.} \tag{5.41}$$

since $\bar{G}^i \cap ([\delta,\infty) \times \mathbb{R}^d)$ $(i = 1,2)$ ($\bar{G}^i$ is the closed graph of Y^i) are a positive distance apart. Since $X^i \leq Y^i$, (5.41) must also hold for X^1 and X^2. Hence the only possible collision local time for (X^1, X^2) is 0 (recall $L_t(X^1,X^2)$ is right-continuous at $t = 0$ by definition).

(2) Theorem 5.10 is false for $d = 4$ or 5. As was mentioned in the Introduction it will be used in a future work to construct pairs of superprocesses which may interact when particles collide. We will show there that these interacting processes can only exist if $d \leq 3$ and the failure of Theorem 5.10 for $d > 3$ will then follow.

(3) The hypothesis on (m^1, m^2) in Theorem 5.9 are clearly necessary for (T) to make sense if $d \leq 4$. If $d = 5$ we do not know if the power in (5.33) may be increased to -3.

(4) The results of Section 4 of Evans-Perkins (1989) show that given a pair of independent super-Brownian motions (Y^1, Y^2), as in Theorem 1.2, if

$d \leq 3$ there is a progressively measurable measure-valued process $K_s(Y^1, Y^2)$ such that

$$\int\int p_\epsilon(y_1-y_2)\psi((y_1+y_2)/2)Y_s^1(dy_1)Y_s^2(dy_2) \to K_s(Y^1,Y^2)(\psi) \text{ a.s. and in } L^p$$

for each $p < 2$ and $s > 0$ (the L^p-convergence is implicit in the Fourier analytic approach of Evans-Perkins (1989)). Assuming (m^1, m^2) satisfy the hypotheses of Theorem 5.9, it is then easy to see that

$$L_t(Y^1,Y^2)(\psi) = \int_0^t K_s(Y^1,Y^2)(\psi)ds \text{ for all } \psi \in b\mathcal{B}(\mathbb{R}^d) \text{ and } t \geq 0 \text{ a.s.}$$

($\geq$ is clear by Fatous lemma and equality follows by checking the means of the total mass are equal.) A Radon-Nikodym argument, using the fact that $X^i \leq Y^i$, then shows that if $d \leq 3$ and the hypotheses of Theorem 5.9 are satisfied there is a progressively measurable measure-valued $K_s(X^1, X^2)$ such that

$$L_t(X^1,X^2)(\psi) = \int_0^t K_s(X^1,X^2)(\psi)ds \tag{5.42}$$

$$\text{for all } \psi \in b\mathcal{B}(\mathbb{R}^d) \text{ and } t \geq 0 \text{ a.s.}$$

6 A Tanaka Formula for a Class of Measure-Valued Processes

Let us briefly show how the methods of the previous section also apply to the Tanaka formula of Adler-Lewin (1989) for super-Brownian motion in three or fewer dimensions (see Tribe (1989) for an interesting treatment of the one-dimensional case). The estimates required are considerably simpler than those in Section 5.

Assume X_t is a cadlag, adapted $M_F(\mathbb{R}^d)$-valued process on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that

$$X_t(\varphi) = m(\varphi) + Z_t(\varphi) + \int_0^t X_s(A\varphi)ds - A_t(\varphi), t \geq 0, \varphi \in \mathcal{D}(A), \tag{6.1}$$

$Z_t(\varphi)$ is a continuous $\mathcal{F}_t$-local martingale such that $\langle Z(\varphi)\rangle_t = \int_0^t X_s(\varphi^2)ds$, $A_t(\varphi)$ is a cadlag, non-decreasing, adapted $M_F(\mathbb{R}^d)$-valued process, starting at 0. Here m is a fixed measure in $M_F(\mathbb{R}^d)$.

Recall that the resolvent densities g_α were introduced at the beginning of the previous section.

Theorem 6.1 *Assume $d \leq 3$ and $\int g_0(x)dm(x) < \infty$. There is a continuous non-decreasing, adapted process L_t^0 (the local time of X at 0) such that*

$$\sup_{t\leq T} |L_t^0 - \int_0^t X_s(p_\epsilon)ds| \xrightarrow{L^1} 0 \quad \textit{for any} \quad T > 0 \tag{6.2}$$

and for all $\alpha \geq 0$ ($\alpha > 0$ if $d \leq 2$)

$$\begin{aligned} X_t(g_\alpha) \ &= m(g_\alpha) + Z_t(g_\alpha) - A_t(g_\alpha) \\ &\quad + \alpha \textstyle\int_0^t X_s(g_\alpha)ds - L_t^0 \quad \textit{for all} \quad t \geq 0 \quad \textit{a.s.} \end{aligned} \tag{6.3}$$

Each term in (6.3) is cadlag, all but $X_t(g_\alpha)$ and $A_t(g_\alpha)$ are a.s. continuous, and if A_t is continuous then each term in (6.3) is continuous. $Z_t(g_\alpha)$ is on L^2-martingale and the other terms on the right-hand side of (6.3) have integrable variation on compacts.

Remark. Assume $A = 0$ and m has a bounded density with respect to Lebesgue measure. Theorem 6.1 is then due to Adler-Lewin (1989) (see Tribe (1989) for a stronger result if $d = 1$). Under these hypotheses Sugitani (1987) proved the existence of a jointly continuous process L_t^x such that

$$\int_0^t X_s(B)ds = \int_B L_t^x dx \quad \text{for all} \quad B \in \mathcal{B}(\mathbb{R}^d), t \geq 0.$$

(6.2) and the continuity of L_t^x shows our notations are consistent. By replacing g_α with $g_{\alpha,x}(y) = g_\alpha(y - x)$ in (6.3) one obtains a Tanaka formula for L_t^x. (The latter observation holds in the general context of Theorem 6.1 where the obvious change in the hypothesis on m is required.)

Proof of Theorem 6.1. Let $g_{\alpha,\epsilon}(x) = \int_0^\infty e^{-\alpha s} p_{s+\epsilon}(x)ds$. Then $Ag_{\alpha,\epsilon} = \alpha g_{\alpha,\epsilon} - p_\epsilon$ ($\alpha > 0$ if $d \leq 2$) and so

$$X_t(g_{\alpha,\epsilon}) = m(g_{\alpha,\epsilon}) + Z_t(g_{\alpha,\epsilon}) - A_t(g_{\alpha,\epsilon}) + \alpha \int_0^t X_s(g_{\alpha,\epsilon})ds - \int_0^t X_s(p_\epsilon)ds. \tag{6.4}$$

As in Theorem 5.9 we will obtain (6.2) and (6.3) by letting $\epsilon \downarrow 0$ in (6.4) and showing that each term in (6.4), except for the last, converges uniformly in $t \le T$ in L^1 to the corresponding term in (6.3). As the argument required for each term is similar to but much simpler than the argument used to handle the corresponding term in Theorem 5.9 we only give the details for $X_t(g_{\alpha,\epsilon})$, i.e., we will show

$$\sup_{t\le T} |X_t(g_{\alpha,\epsilon}) - X_t(g_\alpha)| \xrightarrow{L^1} 0 \quad \text{as} \quad \epsilon \downarrow 0 \tag{6.5}$$

It is easy to see that $g_{\alpha,\epsilon}(x) \to g_\alpha(x)$ as $\epsilon \downarrow 0$, the convergence is uniform on $|x| \ge \delta$ for any $\delta > 0$ (and uniform if $d = 1$), and

$$g_{\alpha,\epsilon}(x) \le c_1(\alpha) g_0(x) \quad \text{for all} \quad 0 \le \epsilon \le 1 \ (\alpha > 0 \ \text{ if } \ d \le 2).$$

(6.5) is obvious if $d = 1$ by the above uniform convergence so we will assume $d = 2$ on 3. If $Y_t \ge X_t$ is a super-Brownian motion (see Theorem 5.1 and the ensuing discussion) and

$$\eta(\epsilon, \delta) = \sup_{|x|\ge\delta} |g_{\alpha,\epsilon}(x) - g_\alpha(x)|,$$

then

$$\sup_{t\le T} X_t(|g_{\alpha,\epsilon} - g_\alpha|) \le \eta(\epsilon,\delta) \sup_{t\le T} Y_t(1) + c_1 2 \sup_{t\le T} \int 1(|x| < \delta) g_\alpha(x) Y_t(dx).$$

If $T > \beta > 0$ and $R_t(A) = Y_t(|y| \in A)$ then for $\delta < r_0(w)$ (r_0 as in Theorem 4.7)

$$\sup_{\beta\le t\le T} \int 1(|x| < \delta) g_0(x) Y_t(dx) \tag{6.6}$$

$$= \sup_{\beta\le t\le T} \int_0^\delta g_0(r) R_t(dr)$$

$$\le \sup_{\beta\le t\le T} c_{4.7}[g_0(\delta)(D(mP_t, c_{4.8}\delta) \vee \varphi(\delta)) + \int_0^\delta (D(mP_t, c_{4.8}r) \vee \varphi(r) g_0'(r) dr \quad \text{(Theorem 4.7)}$$

$$\to 0 \quad \text{as} \quad \delta \downarrow 0$$

because $D(mP_t, c_{4.8}r) \le m(1)\beta^{-\delta/2}c_2 r^d$ for all $t \ge \beta$. Just as in the proof of Lemma 5.6(b), (6.4) with Y in place of X and Fatou's lemma show that

$$\begin{aligned} \sup_{t\le T} Y_t(g_\alpha) - m(g_\alpha) &\le (\sup_{t\le T} Z_t(g_\alpha)) + \alpha \textstyle\int_0^T Y_s(g_\alpha)ds \\ &\in L^1 \end{aligned} \tag{6.7}$$

(the last by the arguments used to handle the second and fourth terms on the right side of (6.3) – the martingale term is now much easier to handle than in Theorem 5.9). (6.6) and (6.7) imply that

$$\sup_{\beta\le t\le T} \int 1(|x| < \delta) g_\alpha(x) Y_t(dx) \xrightarrow{L^1} 0 \quad \text{for all} \quad 0 < \beta < T < \infty.$$

To handle the supremum over $t \in [0,\beta]$ we use (6.7) and argue exactly as in the proof of Lemma 5.6(b). This completes the proof of (6.5) and hence (6.2) and (6.3). The remaining properties of the processes in (6.3) are clear from the uniform (in $t \le T$) L^1 convergence of each of the terms in (6.4). □

7 Proof of Lemma 5.4

Lemma 7.1 *(a)* $P^x(g_\alpha(B_t - y)) = e^{\alpha t}\int_t^\infty e^{-\alpha v} p_v(x-y)dv$

(b) If

$$h(\alpha, v) = \begin{cases} \alpha^{-1/2}e^{-\alpha v} & d = 1 \\ (\log^+(1/v) + \alpha^{-1})e^{-\alpha v} & d = 2 \\ v^{1-d/2}e^{-\alpha v} & d \ge 3 \end{cases}$$

then

$$P^x([P^{B_u}(g_\alpha(B_t - y))]^2) \le c_{7.1}e^{2\alpha t}\int_t^\infty h(\alpha, v)p_{v+u}(x-y)dv.$$

Proof. (a) is trivial.

(b) $P^x(P^{B_u}(g_\alpha(B_t - y))^2)$

$$\begin{aligned} &= 2e^{2\alpha t}\int\int_t^\infty\int_{v_1}^\infty e^{-\alpha(v_1+v_2)}p_{v_1}(w-y)p_{v_2}(w-y)p_u(w-x)dv_2 dv_1 dw \quad \text{(by (a))} \\ &\le c_1 e^{2\alpha t}\int\int_t^\infty\int_{v_1}^\infty e^{-\alpha v_2}v_2^{-d/2}dv_2 e^{-\alpha v_1}p_{v_1}(w-y)p_u(w-x)dv_1 dw \end{aligned}$$

If $d > 2$ the above is bounded by

$$c_{7.1}e^{2\alpha t}\int_t^\infty v_1^{1-d/2}e^{-\alpha v_1}p_{v_1+u}(x-y)dv_1.$$

The results for $d \le 2$ are similar. $\square$

Proof of Lemma 5.4. We may assume $t \ge 1$.

$$\mathbb{P}\Big(\int_0^t\int\Big(\int g_\alpha(y_1-y_2)Y_s^1(dy_1)\Big)^2Y_s^2(dy_2)ds\Big)$$

$$\begin{aligned}&= \int\int\int_0^t \mathbb{P}\Big(\Big(\int g_\alpha(y_1-y_2)Y_s^1(dy_1)\Big)^2\Big)p_s(y_2-x_2)dsdy_2m^2(dx_2)\\ &= \int\int\int_0^t [P^{m^1}(g_\alpha(B_s-y_2))]^2p_s(y_2-x_2)dsdy_2m^2(dx_2)\end{aligned}$$

$$(7.1)\quad +\int\int\int_0^t P^{m^1}\Big(\int_0^s[P^{B_u}(g_\alpha(B_{s-u}-y_2))]^2\Big)dup_s(y_2-x_2)dsdy_2m^2(dx_2)$$

(see for example Fitzsimmons (1988, (2.7)))

$$\equiv I + II.$$

Consider I first. By Lemma 7.1(a)

$$I = \int\int\int_0^t e^{2\alpha s}\Big[\int\int_s^\infty e^{-\alpha v}p_v(x_1-x_2)dvm^1(dx_1)\Big]^2p_s(y_2-x_2)dsdy_2m^2(dx_2)$$

$$= 2\int\int\int_0^t e^{2\alpha s}\Big[\int\int\int_s^\infty\int_{v_1}^\infty e^{-\alpha v_2}p_{v_2}(x_1'-y_2)e^{-\alpha v_1}p_{v_1}(x_1-y_2)dv_2dv_1m^1(dx_1')m^1(dx_1)$$

$$p_s(y_2-x_2)dsdy_2m^2(dx_2)$$

(7.2)

$$\le c_1\int\int\int_0^t e^{2\alpha s}\int_s^\infty\Big(\int_{v_1}^\infty e^{-\alpha v_2}v_2^{-d/2}dv_2\Big)e^{-\alpha v_1}p_{v_1+s}(x_1-x_2)dv_1dsm^1(dx_1)m^2(dx_2)m^1(\mathbb{R}^d).$$

Assume now $d > 2$. Then

$$\begin{aligned}I &\le c_2m^1(\mathbb{R}^d)\int\int\int_0^t\int_s^\infty v_1^{1-d/2}p_{v_1+s}(x_1-x_2)dv_1dsm^1(dx_1)m^2(dx_2)\\ &\le c_3m^1(\mathbb{R}^d)\int\int\int_0^t\int_s^\infty v_1^{1-d/2}p_{2v_1}(x_1-x_2)dv_1dsm^1(dx_1)m^2(dx_2)\\ &\le c_3m^1(\mathbb{R}^d)\int\int\Big[\int_0^t v_1^{2-d/2}p_{2v_1}(x_1-x_2)dv_1 + t\int_t^\infty v_1^{1-d}dv_1\Big]m^1(dx_1)m^2(dx_2)\end{aligned}$$

(7.3)

$$\leq \begin{cases} c_4 m^1(\mathbb{R}^d)[\int\int(|x_1-x_2|^{6-2d}+1)m^1(dx_1)m^2(dx_2) & \text{if } d>3 \\ c_4 m^1(\mathbb{R}^d)[\int\int(\log^+(t/|x_1-x_2|^2)+1)m^1(dx_1)m^2(dx_2) & \text{if } d=3 \end{cases}$$

by an elementary calculation where c_4 depends only on d (recall $t \geq 1$). By making minor modifications to the above arguments we find

$$I \leq c_4(\alpha,t)m^1(\mathbb{R}^d)^2 m^2(\mathbb{R}^d) \ \text{ if } \ d \leq 2,\ \alpha > 0. \tag{7.4}$$

Now consider II in (7.1). Lemma 7.1 implies that

$$II \leq$$

$$c_5 \int\int\int\int_0^t \int_0^s e^{2\alpha(s-u)} \int_{s-u}^{\infty} h(\alpha,v_1)p_{v_1+u}(x_1-y_2)dv_1 du p_s(y_2-x_2)ds dy_2 m^2(dx_2)m^1(dx_1)$$

$$= c_5 \int\int\int_0^t \int_0^s \int_{s-u}^{\infty} e^{2\alpha(s-u)}h(\alpha,v_1)p_{v_1+u+s}(x_1-x_2)dv_1 du ds m^2(dx_2)m^1(dx_1)$$

$$\leq c_6 \int\int\int_0^t \int_0^s \int_{s-u}^{\infty} e^{2\alpha(s-u)}h(\alpha,v_1)p_{2(u+v_1)}(x_1-x_2)dv_1 du ds m^2(dx_2)m^1(dx_1)$$

because $p_{v_1+u+s}(x_1-x_2) \leq c p_{2(u+v_1)}(x_1-x_2)$ when $u \leq s \leq u+v_1$. Therefore, setting $w = u+v_1$, we get

$$\begin{aligned} II &\leq c_6 e^{2\alpha t}\int\int\int_0^{\infty}\int_0^t \min(t-u,v_1)h(\alpha,v_1)p_{2(u+v_1)}(x_1-x_2)du dv_1 m^2(dx_2)m^1(dx_1) \\ &\leq c_6 e^{2\alpha t}\int\int\int_0^{\infty}\int_{v_1}^{v_1+t} \min(t-w+v_1,v_1)h(\alpha,v_1)p_{2w}(x_1-x_2)dw dv_1 m^2(dx_2)m^1(dx_1) \\ &\leq c_6 e^{2\alpha t}\int\int[\int_0^t\int_0^w v_1 h(\alpha,v_1)dv_1 p_{2w}(y_1-y_2)dw \\ &+ \int_0^{\infty}\int_{v_1}^{v_1+t} t1(w \geq t)h(\alpha,v_1)p_{2w}(x_1-x_2)dw dv_1)m^2(dx_2)m^1(dx_1) \end{aligned}$$

(7.5)

$$\leq c_6 e^{2\alpha t}\int\int[\int_0^t\int_0^w v_1 h(\alpha,v_1)dv_1 p_{2w}(x_1-x_2)dw + t\int_0^t h(\alpha,v_1)t^{-d/2}v_1 dv_1$$

$$+t\int_t^{\infty} h(\alpha,v_1)\int_{v_1}^{v_1+t} w^{-d/2}dw dv_1]m^2(dx_2)m^1(dx_1).$$

If $3 \le d \le 5$ and $\alpha = 0$ this gives

$$\begin{aligned} II &\le c_6 \int\int [\int_0^t \int_0^w v_1^{2-d/2} dv_1 p_{2w}(x_1 - x_2) dw \\ &+ t^{1-d/2} \int_0^t v_1^{2-d/2} dv_1 + t^2 \int_t^\infty v_1^{1-d} dv_1] m^2(dx_2) m^1(dx_1) \\ &\le c_7 \int\int [\int_0^t w^{3-d/2} p_{2w}(x_1 - x_2) dw + t^{4-d}] m^2(dx_2) m^1(dx_1) \end{aligned}$$

$$(7.6) \qquad \le c_8(t) \begin{cases} m^2(\mathbb{R}^d) m^1(\mathbb{R}^d) \ \text{ if } \ d = 3 \\ \int\int (1 + \log^+(1/|x_1 - x_2|)) m^2(dx_2) m^1(dx_1) \ \text{ if } \ d = 4 \\ \int\int (1 + |x_1 - x_2|^{-2}) m^2(dx_2) m^1(dx_1) \ \text{ if } \ d = 5 \end{cases}$$

If $d = 2$ and $\alpha > 0$, (7.5) implies

$$\begin{aligned} II \le c_6 e^{2\alpha t} \int\int [\int_0^t \int_0^w & v_1((\log^+ 1/v_1) + \alpha^{-1}) e^{-\alpha v_1} dv_1 p_{2w}(x_1 - x_2) dw \\ &+ \int_0^t (\log^+(1/v_1) + \alpha^{-1}) e^{-\alpha v_1} v_1 dv_1 \\ +t \int_t^\infty & (\log^+(1/v_1) + \alpha^{-1}) \log(1 + t/v_1) e^{-\alpha v_1} dv_1] m^2(dx_2) m^1(dx_1) \end{aligned}$$

$$(7.7) \qquad \le c_9(\alpha, t) m^1(\mathbb{R}^d) m^2(\mathbb{R}^d)$$

and a simpler argument gives the same upper bound if $d = 1$ and $\alpha > 0$.

The result now follows from (7.3) and (7.6) if $d \ge 3$ (it suffices to consider $\alpha = 0$) and (7.4) and (7.7) if $d \le 2$. □

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