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Abstract: We explicitly identify the possible probability entrance laws for a class of measure-valued processes that are constructed by taking a particular measure-valued Markov branching process and conditioning it to stay away from the zero measure trap. The set of extreme points of the entrance space is larger than the state space of the conditioned process, and contains elements which correspond to "starting" the conditioned process at the zero measure.

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The *continuous state branching process* was discussed in [Feller, 1951] as a diffusion approximation to a suitably scaled critical Galton-Watson branching process when the initial number of individuals becomes large. This process, known also as the $BES^2(0)$ process, is a diffusion on $[0,\infty[$ with generator that extends $2xd^2/dx^2$. The state 0 is a trap.

Suppose we start a $BES^2(0)$ process away from 0 and "condition on never hitting 0". That is, we consider the limit as $T \to \infty$ for the process conditioned to have not hit 0 up to time T. It is a part of the folk lore that this limit exists, that the resulting process correspond to performing to a Doob h-transform on the $BES^2(0)$ process using the function $h(x) = x$, and that the resulting process is a $BES^2(4)$ process, i.e. a diffusion on $]0,\infty[$ with generator that extends $2xd^2/dx^2 + 4d/dx$ (see, for example, Example 3.5 of [Pitman and Yor, 1982]). The state 0 now becomes an entrance but not an exit boundary.

Our aim is to investigate the analogue of this result for a class of measure-valued branching Markov processes that generalise the continuous state branching process. The class of processes is a particular case of a construction given in [Watanabe, 1968] which we now review.

Suppose that E is a locally compact, separable space. If E is non-compact, let Δ denote the point at infinity and put $E^\Delta = E \cup \{\Delta\}$. For uniformity of notation, set $E^\Delta = E$ when E is compact. Write $\overline{C}(E)$ for the Banach space of continuous functions on E with continuous extension to E^Δ (equipped, of course, with the supremum norm). Let $(P_t)_{t\geq 0}$ be the semigroup of a conservative Markov process on E. Assume that $(P_t)_{t\geq 0}$ is Feller; that is $(P_t)_{t\geq 0}$ maps $\overline{C}(E)$ to $\overline{C}(E)$ and is strongly continuous on $\overline{C}(E)$. We can extend $(P_t)_{t\geq 0}$ to a Feller semigroup for E^Δ by setting $P_t f(\Delta) = f(\Delta)$.

Let $M(E)$ (respectively, $M(E^\Delta)$) denote the space of finite Borel measures on E (respectively, E^Δ) with the topology of weak convergence, so that $M(E^\Delta)$ is a locally compact, separable space.

In [Watanabe, 1968] it is shown that for each $f \in C(E^\Delta)$ with $f \geq 0$ the integral equation

$$(1) \qquad u_t(x) = P_t f(x) - \int_0^t P_s(x, u_{t-s}^2)\, ds$$

has a unique solution $u_t = U_t f$; and there exists a unique Feller semigroup, $(Q_t)_{t\geq 0}$ on $M(E^\Delta)$ for which

$$(2) \qquad \int Q_t(\mu, d\nu)\, e^{-\nu(f)} = \exp(-\mu\, U_t f)$$

for all such f. Let $X = (W, G, G_t, \Theta_t, X_t, \mathbb{P}^\mu)$ be a Feller process with the semigroup $(Q_t)_{t\geq 0}$. It is shown in [El Karoui and Roelly-Coppoletta, 1987] and [Fitzsimmons, 1988] that if X_0 belongs to $M(E)$ almost surely then X almost surely has continuous, $M(E)$-valued paths. We refer the reader to the Introduction of [Fitzsimmons, 1988] for an up-to-date selection of the considerable amount of work that has been done on this and related classes of measure-valued processes. Chapter 9 of [Ethier and Kurtz, 1986] provides a discussion of how such processes arise as high density approximations for the configuration of a branching Markov process.

It is easy to show that the total mass process $\{X_t(1) : t \geq 0\}$ is a $BES^2(0)$ process (one can use the Laplace transform calculations on p.100 of [Knight, 1981], for instance). The zero measure is a trap. As above, we can start the process X off at $\mu \in M(E) \setminus \{0\}$, condition the process to be "alive" at time T (that is, to be away from 0) and then let $T \to \infty$. It is shown in [Evans and Perkins, 1990] and [Roelly-Coppoletta and Rouault, 1989] that the result of this procedure is a right Markov process $(\tilde{W}, \tilde{G}, \tilde{G}_t, \tilde{\Theta}_t, \tilde{X}_t, \tilde{\mathbf{P}}^\mu)$ with state space $M(E) \setminus \{0\}$ and semigroup $(\tilde{Q}_t)_{t \geq 0}$, that is the Doob h-transform of X using the function $h(\nu) = \nu(1)$. That is,

$$
\begin{aligned}
(3) \qquad \tilde{Q}_t F(\mu) &= \tilde{\mathbf{P}}^\mu F(\tilde{X}_t) \\
&= \mu(1)^{-1} \mathbf{P}^\mu [F(X_t) X_t(1)].
\end{aligned}
$$

Moreover, it is shown in [Roelly-Coppoletta and Rouault, 1989] that if $f \in \overline{C}(E)$ with $f \geq 0$ then

$$
(4) \qquad \int \tilde{Q}_t(\mu, d\nu) e^{-\nu(f)} = \left[\frac{\mu V_t f}{\mu(1)} \right] \exp(-\mu U_t f),
$$

where U_t is as above and $V_t f = v_t$ is the solution of

$$
(5) \qquad v_t = 1 + 2 \int_0^t P_s(v_{t-s}(U_{t-s} f))\, ds
$$

(the results in [Roelly-Coppoletta and Rouault, 1989] are for the case $E = \mathbb{R}^d$, but they carry over to this setting).

From the introductory remarks above, it is clear that the total mass process $\{\tilde{X}_t(1) : t \geq 0\}$ is a $BES^2(4)$ process (this can also be seen, of course, from an explicit computation of Laplace transforms). By analogy, we might hope that we can, in some sense, "start $\tilde{X}$ off" at the zero measure and treat the zero measure as some kind of "entrance boundary". This turns out to be the case, but now there will be many ways to start from 0 — one for each "direction" in which $\tilde{X}$ can make its initial infinitesimal move away from 0. To make this claim precise, we recall the following concept.

DEFINITION. A family $(N_t)_{t>0}$ of probability measures on $M(E) \setminus \{0\}$ is a *probability entrance law* for the semigroup $(\tilde{Q}_t)_{t\geq 0}$ if for all $s, t > 0$ we have $N_s \tilde{Q}_t = N_{s+t}$.

We can now state our result classifying the possible probability entrance laws for $(\tilde{Q}_t)_{t\geq 0}$. We let $M_1(E)$ to denote the space of probability measures on E. Part of our proof is along the lines set out in [Dynkin, 1988b] and [Fitzsimmons, 1988] for determining the entrance space of a class of measure-valued Markov branching processes.

THEOREM. There is a one-to-one correspondence between the class of probability entrance laws for $(\tilde{Q}_t)_{t\geq 0}$ and the class of probability measures on $M_1(E) \times [0,\infty[$. A probability entrance law $(N_t)_{t>0}$ corresponds to a probability measure Γ on $M_1(E) \times [0,\infty[$ by the relationship

$$\int N_t(d\mu) \exp(-\mu(f)) = \int \Gamma(d\nu, dx)(\nu V_t f) \exp(-x\nu U_t f) \tag{6}$$

for all $f \in \overline{C}(E)$ with $f \geq 0$. Moreover, if we set $p(\mu) = \mu(\cdot)/\mu(1)$ and $m(\mu) = \mu(1)$ for $\mu \in M(E) \setminus \{0\}$ then Γ is the weak limit as $t \downarrow 0$ of $N_t \circ (p, m)^{-1}$.

PROOF. Suppose firstly that $(N_t)_{t>0}$ is a probability entrance law for $(\tilde{Q}_t)_{t\geq 0}$. Let W^+ denote the space of continuous paths from $]0,\infty[$ to $M(E)$ and write $(X_t^+)_{t>0}$ for the coordinate process on W^+. There is a unique probability measure $\mathbb{P}$ on W^+ under which (X_t^+) is Markovian with semigroup $(\tilde{Q}_t)$ and 1-dimensional distributions (N_t) (cf §40 of [Sharpe, 1988]). Let $(G_t^+)_{t\geq 0}$ denote the $\mathbb{P}$-augmentation of the natural filtration of (X_t^+) made right-continuous.

Recalling remarks made above, we see that $(m(X_t^+))_{t>0}$ is Markovian under $\mathbb{P}$ with the $BES^2(4)$ semigroup and that $(N_t \circ m^{-1})_{t>0}$ is a probability entrance law for the $BES^2(4)$ semigroup. From the theory of 1-dimensional diffusions we conclude that there is a G_0^+-measurable, $[0,\infty[$-valued random variable x_{0+} such that $\lim_{t\downarrow 0} m(X_t^+) = x_{0+}$ $\mathbb{P}$-a.s. and $N_t \circ m^{-1} = (\mathbb{P} \circ x_{0+}^{-1}) S_t$, where we write S_t for the semigroup of $BES^2(4)$ considered as a diffusion on $[0,\infty[$.

Suppose that $f \in \overline{C}(E)$ with $f \geq 0$. For $\alpha > 0$ set $U^\alpha f = \int_0^\infty e^{-\alpha t} P_t f dt$. From the first moment calculations in Proposition 2.7 of [Fitzsimmons, 1988] or Theorem 1.1 of [Dynkin, 1988a] and (3) we see that the process $\exp(-\alpha t) X_t^+(U^\alpha f)/X_t^+(1)$ is a bounded supermartingale, and so $\lim_{t\downarrow 0} X_t^+(U^\alpha f)/X_t^+(1)$ exists $\mathbb{P}$-a.s. As the set of functions $\{U^\alpha f : f \in \overline{C}(E), f \geq 0\}$ is dense in $\overline{C}(E) \cap \{f : f \geq 0\}$, we can conclude that there exists a G_0^+-measurable, $M_1(E)$-valued random variable Y_{0+} such that $\lim_{t\downarrow 0} X_t^+(\cdot)/X_t^+(1) = Y_{0+}$.

Set $\Gamma = \mathbb{P} \circ (y_{0+}, x_{0+})^{-1}$. Then, by part (ii) of the following lemma and (4) we have for each $f \in \bar{C}(E)$ with $f \geq 0$ that

$$\begin{aligned}
\int N_t(d\mu) \exp(-\mu(f)) &= \mathbb{P}[\exp(-X_t^+(f))] \\
&= \lim_{s \downarrow 0} \mathbb{P}[\exp(-X_{t+s}^+(f))] \\
&= \lim_{s \downarrow 0} \mathbb{P}[\int \tilde{Q}_t(X_s^+, d\mu) \exp(-\mu(f))] \\
&= \lim_{s \downarrow 0} \mathbb{P}\left[\left(\frac{X_s^+ V_t f}{X_s^+(1)}\right) \exp(-X_s U_t f)\right] \\
&= \mathbb{P}[(Y_{0+} V_t f) \exp(-x_{0+} Y_{0+} U_t f)] \\
&= \int \Gamma(d\nu, dx)(\nu V_t f) \exp(-x \nu U_t f).
\end{aligned}$$

To show the reverse correspondence, it suffices to consider the case when Γ is the unit point mass at some pair (ν, x). When $x \neq 0$ the family of measures $(\tilde{Q}_t(x\nu, \cdot))_{t>0}$ is clearly an entrance probability law for $(\tilde{Q}_t)_{t\geq 0}$ which satisfies (6). Suppose that $x = 0$. From (4) we see that

$$\lim_{\varepsilon \downarrow 0} \int \tilde{Q}_t(\varepsilon\nu, d\mu) \exp(-\mu(f)) = \nu V_t f$$

for all $f \in \bar{C}(E)$ with $f \geq 0$. As $\nu V_t 0 = 1$, it follows that for each $t > 0$ there exists a probability measure N_t on $M(E) \setminus \{0\}$ such that $\lim_{\varepsilon \downarrow 0} \tilde{Q}_t(\varepsilon\nu, \cdot) \Rightarrow N_t$ and $\int N_t(d\mu) \exp(-\mu(f)) = \nu V_t f$ (cf. Lemma 5.1 of [Kallenberg, 1983]). In order to show that $(N_t)_{t>0}$ is a probability entrance law for $(\tilde{Q}_t)_{t\geq 0}$ and hence complete the proof, it will suffice to show that $N_s \tilde{Q}_t F = N_{s+t} F$ for all $s, t > 0$ and all bounded, continuous functions F on $M(E) \setminus \{0\}$. This, however, is clear from the above, part (i) of the following lemma and the fact that $\tilde{Q}_s(\tilde{Q}_t F)(\varepsilon\nu) = \tilde{Q}_{s+t} F(\varepsilon\nu)$.

We required the following lemma in the course of the preceeding proof.

LEMMA (i). For each $t \geq 0$ the map $\mu \mapsto \tilde{Q}_t(\mu, \cdot)$, $\mu \in M(E) \setminus \{0\}$ is continuous.

(ii). For each $t \geq 0$ and each $f \in \bar{C}(E)$ with $f \geq 0$ the maps $x \mapsto U_t f(x)$ and $x \mapsto V_t f(x)$ are continuous.

PROOF (i). By the Feller property, $\mu \mapsto Q_t(\mu, \cdot)$ is continuous. From Theorem 1.1 of [Dynkin, 1988a] or Proposition 2.7 of [Fitzsimmons, 1988] we know that $\mu \mapsto \mathbb{P}^\mu[X_t(1)^2]$ is locally bounded. The result now follows from (3) and a standard uniform integrability argument.

(ii). The claim regarding $U_t f$ is a consequence of (2) and the Feller property of $(Q_t)_{t\geq 0}$. The claim regarding $V_t f$, now follows from (4) and part (i).

REMARK. In [Roelly-Coppoletta and Rouault, 1989] the process $\tilde{X}$ is identified as the solution to a martingale problem that resembles the martingale problem for X except for the addition of an extra drift term, which the authors describe as representing an interactive immigration effect. For probability entrance laws that correspond to pairs of the form $(\nu, 0) \in M_1(E) \times [0, \infty[$, the measure ν can be thought of in these terms as giving the disposition of an initial immigration that puts mass into the system and pushes the process away from the zero measure.

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