Level-Crossing of Integrated Ornstein-Uhlenbeck Processes

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Abstract

This paper deals with processes of the type $X(t) = X(0) + \int_0^t U(s)\,ds$ where $U(s)$ is an Ornstein-Uhlenbeck process with drift $\mu > 0$, friction parameter β and variance σ^2. In particular, we are interested in the first-passage time of $X(t)$, i.e. in $T(x, v+\mu, \sigma^2) = \inf\{t : X(t) = 0 \mid X(0) = -x,\ V(0) = v + \mu\}$. A system of partial differential equations is derived for the moments of $T(x, v+\mu, \sigma^2)$. These equations are then transformed into Schrödinger-type equations and, using WKB analysis and perturbation methods, asymptotic expansions for the moments of $T(x, v+\mu, \sigma^2)$ are obtained. The first few terms of these asymptotic expansions are used as approximations for the moments of $T(x, v+\mu, \sigma^2)$. A series of simulations (for varies x, v, μ, σ^2,) confirms that these approximations are very accurate for the mean of $T(x, v+\mu, \sigma^2)$. An approximation to the distribution is also given.

1. Introduction

First passage time problems (henceforth called FPT problems) constitute a set of old and rather famous problems of probability theory. They are of importance both in theoretical and applied contexts and they do arise naturally in fields as different as biology, hydrology, seismology, medicine and physics, to name only a few. For example, in stochastic models of neuronal behavior the firing of a neuron may be modelled as the first hitting of some threshold value by the stochastic process representing the membrane potential (see e.g. Holden (1976)), in population biology, the extinction of a population is often interpreted as the first passage through some threshold value of the counting process representing the number of individuals (Maruyama (1977)). In many other instances the stochastic process for which level-crossing properties need to be obtained is a one-dimensional diffusion.

The literature on the subject is extensive and the results are widely scattered through many journals. It is not the purpose of this work to give a comprehensive overview over the various methods, techniques and results. The reader is referred to Blake and Lindsey (1973) and Abrahams (19) where these are expertly surveyed. Most FPT problems are unsolved analytically and a large amount of work therefore concerns itself with finding approximations to the FPT density. Sometimes the problems are solved for moment information only or to within the Laplace transform of the density.

Occasionally, one is not interested in the solution of the FPT problem for the stochastic process itself but for the *integrated* process with drift. This is the concern of the present paper, which studies this problem for an integrated Ornstein-Uhlenbeck process. The following might serve as an example where these type of problems arise naturally: In stochastic models for particle sedimentation in viscuous fluids (see e.g. Pickard and Tory (1977) or Hesse (1987)), particle velocity is modelled as an Ornstein-Uhlenbeck process with drift. Particle position is then the integral over this Ornstein Uhlenbeck process and the question when a particle first travels through a given distance leads to the FPT problem for this integrated Markov process.

We now define this problem rigorously: Let $\{V(s), s \geq 0\}$ be an Ornstein-Uhlenbeck process with drift $\mu \geq 0$, friction parameter β, and variance σ^2 starting at $V(0) = v + \mu$. Then for all $s \geq 0$ the mean and variance of the process are given by

$$\begin{aligned} E(V(s)) &= v \exp(-\beta s) + \mu, \\ \mathrm{Var}(V(s)) &= \frac{\sigma^2}{2\beta}(1 - \exp(-2\beta s)). \end{aligned}$$

Define the integrated process

$$X(t) = X(0) + \int_0^t V(s)\, ds$$

with $X(0) = -x \leq 0$, and let

$$T(x, v + \mu, \sigma^2) = \inf\{t : X(t) = 0 \;|\, X(0) = -x,\ V(0) = v + \mu\}$$

denote the first time $X(t)$ is in state 0. We are interested in the distribution and the moments of $T(x, v+\mu, \sigma^2)$ for fixed x and under one of the following specifications for the initial velocity $V(0)$:

(a) $V(0) = v + \mu$ fixed

(b) $V(0)$ sampled from some velocity distribution

(c) $V(0)$ sampled from the boundary crossing velocity distribution of the process $\{V(s), s \geq 0\}$.

In the present paper we focus on (a), i.e. we consider a fixed starting velocity $V(0)$. In the following section we will briefly review some of the literature on this and related problems.

2. Previous Work

Consider the stochastic differential equation given by

$$\frac{d^2 X(t)}{dt^2} + c_1(X(t)) \frac{dX(t)}{dt} + c_2(X(t)) = \frac{dB(t)}{dt} \tag{2.1}$$

with

$$V(t) = \frac{dX(t)}{dt} \tag{2.2}$$

and

$$X(0) = -x \tag{2.3}$$

$$V(0) = v \tag{2.4}$$

where $\{B(t),\ t \geq 0\}$ is a (one-dimensional) Brownian motion starting at zero, and $\frac{dB(t)}{dt}$ is white noise, the formal derivative of Brownian motion.

Focusing on the simplest possible case only (i.e. $c_1(\cdot) = c_2(\cdot) \equiv 0$, $x = 0$ and $v = 1$) McKean (1963) gives a complete description of the winding of this diffusion about the origin of the coordinate system. He also addresses the problem of finding the distribution of the time between successive returns to 0. If $T'(0,1)$ denotes the time till first *return* to zero (starting from $X(0) = 0$, $V(0) = 1$) and $H = |V(T'(0,1))|$ the hitting velocity, McKean demonstrates that the joint distribution of $T'(0,1)$ and H is given by

$$\text{(2.5)}\ P(T'(0,1) \in dt,\ H \in dh) = \frac{3h}{\pi\sqrt{2}t^2} \exp(-2(h^2 - h + 1)/t) \int_0^{4h/t} \frac{\exp(-3\theta/2)}{\sqrt{\pi\theta}}\, d\theta.$$

The marginal distribution of $T'(0,1)$ is unknown, however, integration with respect to t can be performed and the distribution of the hitting velocity H is given by

$$P(H \in dh) = \frac{3}{2\pi} \frac{h^{3/2}}{1 + h^3}\, dh.$$

Wong (1966) expresses the integral in (2.5) in terms of eta functions.

McKean's and Wong's results make use of the fact that the probabilities $P_{ab}(X(t) \in d\xi,\ V(t) \in d\eta)$, i.e. the transition probabilities of moving from $X(0) = a$, $V(0) = b$ to $X(t) = \xi$ and $V(t) = \eta$ are easily obtained. They also use a renewal-type equation for these probabilities. Wong's (1966) results hold in fact, more generally, for the zero-mean Gaussian process $Y(t)$ with autocorrelation function $\rho(\tau)$ given by

$$\rho(\tau) = \frac{3}{2}\exp(-|\tau|/\sqrt{3})\,(1 - \frac{1}{3}\exp(-(2/\sqrt{3})|\tau|) = 1 - \frac{\tau^2}{2} + \frac{2}{3\sqrt{3}}|\tau|^3 + O(\tau^4)$$

It can be shown that Y (t) has the same probability law as

$$\sqrt{3}\exp(-\sqrt{3}t) \int_0^{\exp((2/\sqrt{3})t)} B(s)\,ds$$

where again B (s) is standard Brownian motion.

Goldman, (1971) who also studies the system (2.1) - (2.2) with $X(0) = 0$, $V(0) \le 0$ gives some expressions related to the FPT to *non-zero* levels of the process X (t). In particular, if $p(t, \xi, \eta, x, y)$ is the transition probability density of $(X(t), V(t))$ and $\phi_v(x,t) = \frac{d}{dt} P(T(x,v) \le t)$, then

(a) $$\phi_v(x,t) = \frac{1}{2}[3/(2\pi t^3)]^{1/2}(3xt^{-1} - v)\exp[-3(x - vt)^2/2t^3] +$$

$$\int_0^\infty d\xi \int_0^t \int_0^\infty \xi\, P(T'(0,\xi) \in ds, |V(T'(0,\xi)| \in dh\,[p(t-s,0,v,x,\xi) - p(t-s,0,v,x,-\xi)]$$

(b) As $t \to \infty$, $\phi_0(x,t) \sim \text{const.} \cdot x^{1/6} t^{-(5/4)}$

(c) $P(V(t) = \max_{0 \le s \le t} V(s)) \approx 0.372$ if $V(0) = 0$ and $X(0) = 0$.

Abrahams (1982), in a generalization of Wong's (1966) results, finds the time to first zero for a special class of second order Gaussian processes of which Wong's process is the only stationary member.

Rogers and Williams (1984) in an insightful paper, obtained a generalization of McKean's (1963) work to situations where $X(t) = \int_0^t g(B(s))\,ds$ with

$$g(s) = \begin{cases} s^\alpha & \text{if } s > 0 \\ -K^{2+\alpha}|s|^\alpha & \text{if } s < 0. \end{cases}$$

Their paper is built on the concepts of excursions and local time. The cases of only one and also finitely many boundary points are carefully analyzed. Other related work is contained in Gor'kov (1976) and Durbin (1985). However, it should be noted that Durbin's Condition (1) is not satisfied by our process $X(t)$.

3. A System of PDEs for the Moments of $T(x, v + \mu, \sigma^2)$

In this section we derive a system of partial differential equations (PDEs) for the moments of the first passage time distribution of an integrated Ornstein-Uhlenbeck process with drift. Problems of this type are complicated by the fact that the integral over a Markov process, i.e.

$$X(t) = X(0) + \int_0^t V(s)\,ds$$

is neither Markovian nor stationary. This implies that the technology involving Kolmogorov backward and forward equations can no longer be used directly. However, the two-dimensional process $(V(t), \int V(s)\,ds)$ *is* Markovian and this provides a starting point for further analysis. Unfortunately, it turns out, as we will see, that the boundary and initial conditions provided by the respective contexts are often insufficient to constitute a well-posed problem with unique solution in two dimensions. This impass can be circumvented by solving a similar problem for a closely related process $X^{(r)}$, say, for which sufficient boundary and initial information *can* be obtained.

Consider the process $\{X(s), V(s), s \geq 0\}$, $X(0) = -x \leq 0$, $V(0) = v + \mu$ with an absorbing boundary at the plane $X = 0$ and let $p(x_t, v_t, t \mid x, v + \mu, \sigma^2)$ be the probability density associated with:

$$X(0) = -x,\ V(0) = v + \mu,\ X(t) = x_t,\ V(t) = v_t \text{ and}$$
$$\{X(s),\ V(s)\} \text{ did not reach the boundary in } [0, t).$$

Although a boundary has been introduced, the density p, of course, still satisfies both the Kolmogorov forward and backward equations. Define also

$$P(b, t \mid x, v + \mu, \sigma^2) = \int_{-\infty}^{b} \int_{-\infty}^{+\infty} p(x_t, v_t, t \mid x, v + \mu, \sigma^2)\, dv_t\, dx_t$$

so that $P(0, t \mid x, v + \mu, \sigma^2)$ is the probability that the boundary has not been reached prior to time t. Hence, if $f(t, x, v + \mu, \sigma^2)$ denotes the density of $T(x, v + \mu, \sigma^2)$ then

$$f(t, x, v + \mu, \sigma^2) = -\frac{\partial}{\partial t} P(0, t \mid x, v + \mu, \sigma^2).$$

Also, write

$$\psi(s, x, v + \mu, \sigma^2) = E(\exp(-s\, T(x, v + \mu, \sigma^2) \mid X(0) = -x,\ V(0) = v + \mu)$$

for the moment generating function of T. Since both $P(0, t \mid x, v + \mu, \sigma^2)$ and $f(t \mid x, v + \mu, \sigma^2)$ still satisfy the backward equation, ψ satisfies the equation obtained by applying the Laplace transform to the backward operator, i.e. ψ is a solution of

$$\frac{1}{2}\left(\alpha_{11} \frac{\partial^2}{\partial x^2} - \alpha_{12} \frac{\partial^2}{\partial x \partial v} - \alpha_{21} \frac{\partial^2}{\partial v \partial x} + \alpha_{22} \frac{\partial^2}{\partial v^2}\right) - \beta_1 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial v} = \frac{\partial}{\partial t}$$

with $\psi(s, o, v+\mu, \sigma^2) = 1$ for all s, σ^2 and $v + \mu > 0$. Note that the factors (-1) in this operator equation are due to the fact that we differentiate with respect to x rather than

$-x$. As usual $\alpha_{ij} = \alpha_{ij}(x, v + \mu, \sigma^2, t)$ and $\beta_i = \beta_i(x, v + \mu, \sigma^2, t)$ for $i = 1,2$, are the infinitesimal parameters, e.g.

$$\begin{aligned}\alpha_{11}(x, v + \mu, \sigma^2, 0) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \operatorname{Var}(X(\Delta t) - X(0) \mid X(0) = -x,\ V(0) = v + \mu) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \operatorname{Var}(\int_0^{\Delta t} (V(s)\, ds \mid V(0) = v + \mu) = 0.\end{aligned}$$

Similarly

$$\begin{aligned}\alpha_{22}(x, v + \mu, \sigma^2, 0) &= \sigma^2 \\ \alpha_{12}(x, v + \mu, \sigma^2, 0) = \alpha_{21}(x, v + \mu, \sigma^2, 0) &= 0 \\ \beta_1(x, v + \mu, \sigma^2, 0) &= v + \mu \\ \beta_2(x, v + \mu, \sigma^2, 0) &= -\beta v.\end{aligned}$$

Hence $\psi(s, x, v + \mu, \sigma^2)$ satisfies the following partial differential equation

$$\frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial v^2} - (v + \mu) \frac{\partial \psi}{\partial x} - \beta v \frac{\partial \psi}{\partial v} - s\psi = 0 \tag{3.1}$$

with the obvious boundary condition

$$\psi(s, 0, v + \mu, \sigma^2) = 1 \quad \text{for all } s, \sigma^2 \text{ and } v + \mu > 0. \tag{3.2}$$

The system (3.1) and (3.2) does not determine ψ uniquely, because an initial condition is missing and cannot be easily obtained from the problem context. What is determined uniquely, as we will see, are the asymptotic expansions (as $v + \mu \to \infty$) of the moments of $T(x, v + \mu, \sigma^2)$.

To get a handle on (3.1) and (3.2) and to generate a closely related problem with sufficient initial and boundary information, define for $r > V(0) = v + \mu$

$$X^{(r)}(t) = \begin{cases} X(t) \text{ if } V(s) < r \text{ for all } s \le t \\ X(t_0) \text{ for all } t \ge t_0 \text{ if } V(t_0) = r. \end{cases}$$

Clearly, the process $\{X^{(r)}(t),\ t \geq 0\}$ has the same sample paths as $X(t)$ except when the velocity $V(s)$ becomes equal to r, then the particle is stopped dead. Also, let $T^{(r)}(x, v + \mu, \sigma^2)$ be the first hitting time of $X^{(r)}$ on 0 and $\psi^{(r)}(s, x, v + \mu, \sigma^2)$ the corresponding Laplace transform. Then $\psi^{(r)}$ clearly satisfies (3.1) and (3.2) and in addition

(3.3) $$\psi^{(r)}(s, x, r, \sigma^2) = 0 \text{ for all } s \text{ and } x > 0.$$

Since $T^{(r)}(x, v + \mu, \sigma^2) \geq T(x, v + \mu, \sigma^2)$ for every sample path, with strict inequality whenever the process $V(t)$ hits r, and since with probability one $T^{(r)}(x, v+\mu, \sigma^2) \to T(x, v+\mu, \sigma^2)$ as $r \to \infty$, it is true that

$$\lim_{r\to\infty} \psi^{(r)}(s, x, v + \mu, \sigma^2) = \psi(s, x, v + \mu, \sigma^2).$$

Now, define

(3.4) $$\hat{\psi}^{(r)}(s, c, v + \mu, \sigma^2) = \int_0^\infty \exp(-cx)\, \psi^{(r)}(s, x, v + \mu, \sigma^2)\, dx,$$

the Laplace-transform of the moment generating function $\psi^{(r)}$. Then $\hat{\psi}^{(r)}$ satisfies

(3.5) $$\frac{\sigma^2}{2} \frac{\partial^2 \hat{\psi}^{(r)}}{\partial v^2} - (v + \mu)(c\hat{\psi}^{(r)} - \psi^{(r)}(s, 0, v + \mu, \sigma^2)) - \beta v \frac{\partial \hat{\psi}^{(r)}}{\partial v} - s\hat{\psi}^{(r)} = 0$$

with

(3.6) $$\hat{\psi}^{(r)}(s, c, r, \sigma^2) = 0$$

Remark: We can also obtain equation (3.1) using stochastic Itô differentials. Towards this end write

$$dX = (V + \mu)\, dt$$

where V is an Ornstein-Uhlenbeck process, i.e.

$$dV = \sigma dB - \beta V dt$$

where W is a standard Brownian motion. Then define

$$\phi(t) = e^{-st}\psi(s, X(t), V(t), \sigma^2)$$

where $\psi(s, x, v+\mu, \sigma^2)$ is, as above, the Laplace transform of T for a process starting at $-x$ with initial velocity $v + \mu$. What are the increments $\Delta\phi$ of the process $\phi(t)$? Ignoring two of the arguments of ψ in our notation we obtain

$$\begin{aligned}
\Delta\phi &= \phi(t + \Delta t) - \phi(t) \\
&= \exp(-s(t + \Delta t))\psi(X(t + \Delta t), V(t + \Delta t)) - \exp(-st)\psi(X(t), V(t)) \\
&= (\exp(-s(t + \Delta t)) - \exp(-st))[\psi(X(t + \Delta t), V(t + \Delta t)) - \psi(X(t), V(t))] \\
&+ (\exp(-s(t + \Delta t)) - \exp(-st))\psi(X(t), V(t)) \\
&+ \exp(-st)[\psi(X(t + \Delta t), V(t + \Delta t)) - \psi(X(t), V(t))]
\end{aligned}$$

To estimate the order of the infinitesimal terms we consider the conditional expectation of $\Delta\phi$ given $X(t) = x$ and $V(t) = v$. Also, we retain terms of order Δt only:

$$\begin{aligned}
\Delta\phi = & -s\exp(-st)\psi(X(t), V(t))\Delta t \\
& + \exp(-st)\left\{-\frac{\partial\psi(X(t), V(t))}{\partial x}(v + \mu) + \frac{\partial\psi(X(t), V(t))}{\partial v}(\sigma\Delta B - \beta v\Delta t)\right. \\
& \left. + \frac{\sigma^2}{2}\frac{\partial^2\psi(X(t), V(t))}{\partial v^2}\right\}\Delta t,
\end{aligned}$$

since $(\Delta B)^2$ is an infinitesimal of order Δt. Replacing the infinitesimal increments by the differentials one obtains the stochastic Itô differential of the process $\phi(t)$, i.e.

$$d\phi = \exp(-st)\sigma\frac{\partial\psi}{\partial v}dB + \exp(-st)\left[\frac{\sigma^2}{2}\frac{\partial^2\psi}{\partial v^2} - (v + \mu)\frac{\partial\psi}{\partial x} - \beta v\frac{\partial\psi}{\partial v} - s\psi\right]dt$$

or using the integral representation

(3.7)

$$\phi(t) = \phi(0) + \int_0^t \exp(-s\tau)\sigma\frac{\partial\psi}{\partial v}dB(\tau) + \int_0^t \exp(-s\tau)\left[\frac{1}{2}\frac{\partial^2\psi}{\partial v^2} - (v+\mu)\frac{\partial\psi}{\partial x} - \beta v\frac{\partial\psi}{\partial v} - s\psi\right]dt.$$

Also, it is clear from stopping time arguments that the process $\phi(t)$ is a martingale, in fact a local martingale. However, for ϕ in (3.7) to be a martingale the argument of the second integral needs to vanish identically. Hence we obtain (3.1)

a. We will first solve the system (3.5), (3.6) for the special case $\beta = 0$ and without loss of generality for $\mu = 0$, i.e. for the first hitting time on 0 of an integrated Wiener process starting at $-x$ with velocity v. The equation in (3.5) simplifies to

$$\frac{\sigma^2}{2}\frac{\partial^2\hat{\psi}^{(r)}}{\partial v^2} - (vc + s)\hat{\psi}^{(r)} + v = 0 \tag{3.8}$$

with (3.6) still holding. The most profitable path is now the substitution $\omega = a_1 v + a_2$ with $a_1 = 2^{1/3}\sigma^{-2/3}c^{1/3}$ and $a_2 = +2^{1/3}\sigma^{-2/3}c^{-2/3}s$ which transforms (3.7) into

$$\frac{\partial^2\hat{\psi}^{(r)}(s,c,\omega,\sigma^2)}{\partial\omega^2} - \omega\cdot\hat{\psi}^{(r)}(s,c,\omega,\sigma^2) = -c^{-1}\omega + 2^{1/3}\sigma^{-2/3}c^{-5/3}s \tag{3.9}$$

with

$$\hat{\psi}^{(r)}(s,c,\omega_r,\sigma^2) = 0$$

where $\omega_r = 2^{1/3}c^{1/3}\sigma^{-2/3}r + 2^{1/3}c^{-2/3}\sigma^{-2/3}s$. The homogeneous part of (3.9) is a one-dimensional Schrödinger equation. We use perturbation methods and methods of global analysis such as WKB analysis and the method of dominant balance on this Schrödinger equation.

If the power series $\sum_{n=0}^{\infty} p(n)\,\omega^n$ is substituted into the homogeneous part of (3.8) then the coefficients satisfy

$$p(3n+2) = 0$$

$$p(3n+1) = \frac{p(1)\cdot\Gamma(4/3)}{9^n\cdot n!\,\Gamma(n+4/3)}$$

$$p(3n) = \frac{p(0)\cdot\Gamma(2/3)}{9^n n!\,\Gamma(n+2/3)}$$

for all $n \geq 0$. Defining the so-called Airy functions

$$\phi_1(\omega) = 3^{-2/3}\sum_{n=0}^{\infty}\frac{\omega^{3n}}{9^n u!\,\Gamma(n+2/3)} - 3^{-4/3}\sum_{n=0}^{\infty}\frac{\omega^{3n+1}}{9^n n!\,\Gamma(n+4/3)}$$

and

$$\phi_2(\omega) = 3^{-1/6}\sum_{n=0}^{\infty}\frac{\omega^{3n}}{9^n n!\,\Gamma(n+2/3)} + 3^{-5/6}\sum_{n=0}^{\infty}\frac{\omega^{3n+1}}{9^n n!\,\Gamma(n+4/3)}$$

then it is clear that any linear combination of $\phi_1(\omega)$ and $\phi_2(\omega)$ formally solves the homogeneous part of (3.8). After variation of parameters and letting $r \to \infty$ the following solution (which vanishes as ω tends to ∞) is obtained

$$\text{(3.10)}\quad \hat{\psi}^{(\infty)}(s,c,\omega,\sigma^2) = -s\pi\, 2^{1/3}\sigma^{-2/3}c^{-5/3}\left(\phi_2(\omega)\int_{\omega}^{\infty}\phi_1(t)\,dt + \phi_1(\omega)\int_0^{\omega}\phi_2(t)\,dt\right) + k(s,c)\,\phi_1(\omega)$$

$$+ \pi c^{-1}\left(\phi_1(\omega)\int_0^{\omega} t\,\phi_2(t)\,dt - \phi_2(\omega)\int_0^{\omega} t\,\phi_1(t)\,dt\right).$$

Note that the Wronskian of $\phi_1(\omega)$ and $\phi_2(\omega)$ is equal to π^{-1}. We also made use of the fact that

$$\int_0^{\omega}\phi_1(t)\,dt = \frac{1}{3} - \int_{\omega}^{\infty}\phi_1(t)\,dt,$$

which can be checked using the Bessel function representation of $\phi_1(\omega)$ and $\phi_2(\omega)$ (see Abramowitz and Stegun (1965)), i.e.

$$\phi_1(\omega) = \frac{1}{3}\omega^{1/2}[I_{-1/3}(\eta) - I_{1/3}(\eta)] = \pi^{-1}(\frac{\omega}{3})^{1/2}K_{1/3}(\eta)$$

$$\phi_2(\omega) = (\frac{\omega}{3})^{1/2}[I_{-1/3}(\eta) + I_{1/3}(\eta)]$$

with $\eta = \frac{2}{3}\omega^{3/2}$ For given s and c, $k(s,c)$ is a constant. Now, to simplify (3.9) observe that

$$\int_0^\omega t\phi_2(t)\,dt = \int_0^\omega \frac{d^2\phi_2(t)}{dt^2}\,dt = \frac{d\phi_2(t)}{dt}\Big/_{t=\omega} + \text{const.}$$

and after some computations, asymptotically as ω tends to infinity

$$\frac{d\phi_2(t)}{dt}\Big/_{t=\omega} \sim \pi^{-1/2}\omega^{1/4}\exp(\frac{2}{3}\omega^{3/2}) \text{ as } \omega \to \infty.$$

Similarly

$$\int_0^\omega t\phi_1(t)\,dt = \int_0^\omega \frac{d^2\phi_1(t)}{dt^2}\,dt = \frac{d\phi_1(t)}{dt}\Big/_{t=\omega} + \text{const.}$$

and

$$\frac{d\phi_1(t)}{dt}\Big/_{t=\omega} \sim -2^{-1}\pi^{-1/2}\omega^{1/4}\exp(-\frac{2}{3}\omega^{3/2}).$$

Hence, asymptotically as ω tends to infinity

$$(3.11)\ \hat{\psi}^\infty(s,c,\omega,\sigma^2) \sim c^{-1} - s\pi 2^{1/3}\sigma^{-2/3}c^{-5/3}(\phi_2(\omega)\int_\omega^\infty \phi_1(t)dt + \phi_1(\omega)\int_0^\omega \phi_2(t)dt) \text{ as } \omega \to \infty.$$

By the notation $f(x) \sim g(x)$, as $x \to x_0$ here is meant that $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$. Note that the summand $k(s,c)$ does not appear in (3.11) since $\phi_1(\omega)$ converges to zero exponentially fast, in fact

$$\phi_1(\omega) \sim \omega^{-1/4} \exp(-\frac{2}{3}\omega^{3/2}) \quad \text{as} \quad \omega \to \infty .$$

We now derive an asymptotic expansion in terms of powers of ω. For this purpose the method of dominant balance is used on (3.9), (3.11). The basic strategy is to first peel off the leading asymptotic behavior then, after having removed this, to determine the leading behavior of the remainder, and so on. For an account of this method see, for example, Bender and Orszag (1978).

As a first step assume

$$\frac{\partial^2 \hat{\psi}^{(\infty)}(s, c, \omega, \sigma^2)}{\partial \omega^2} \sim 0, \quad \text{as} \quad \omega \to \infty$$

where the notation $f(x) \sim g(x)$ has the same meaning as above. Using this in (3.8) we get

$$\hat{\psi}^{(\infty)}(s, c, \omega, \sigma^2) = c^{-1} - 2^{1/3}\sigma^{-2/3}c^{-5/3}\omega^{-1}s, \quad \text{as} \quad \omega \to \infty .$$

Corrections to this leading term are determined as follows: Write

$$\hat{\psi}^{(\infty)}(c, s, \omega, \sigma^2) = c^{-1} - 2^{1/3}\sigma^{-2/3}c^{-5/3}s(\omega^{-1} + R(\omega)), \text{as} \quad \omega \to \infty ,$$

where the correction term $R(\omega)$ is asymptotically of smaller order than ω^{-1}. It is easy to check that $R(\omega)$ in turn satisfies the differential equation

$$\frac{d^2 R(\omega)}{d\omega^2} + 2\omega^{-3} = \omega R(\omega).$$

Again, setting $\dfrac{d^2 R(\omega)}{d\omega^2} \sim 0$ (as $\omega \to \infty$) one obtains that $R(\omega) \sim 2\omega^{-4}$. Continuing in this fashion, one arrives at the full asymptotic power series expansion of $\hat{\psi}^{(\infty)}(s, c, \omega, \sigma^2)$:

$$(3.12)\quad \hat{\psi}^{(\infty)}(s, c, \omega, \sigma^2) \sim c^{-1} - 2^{1/3}\sigma^{-2/3}c^{-5/3}\, s \sum_{n=0}^{\infty} \frac{(3n)!}{3^n n!}\, \omega^{-3n-1} \quad \text{as } \omega \to \infty .$$

Again, it is confirmed that the asymptotic expansion does not depend on $k(s, c)$ due to the fast exponential decay of $\phi_1(\omega)$ as $\omega \to \infty$.

For the purpose of approximation it is sufficient to know the asymptotic behavior of $\hat{\psi}^{\infty}(s, c, \omega, \sigma^2)$ as given in (3.12). Remembering again that $\omega = a_1 v + a_2$ with $a_1 = 2^{1/3}\sigma^{-2/3}c^{1/3}$ and $a_2 = 2^{1/3}\sigma^{-2/3}c^{-2/3}s$ one gets

$$(3.13)\qquad \hat{\psi}^{(\infty)}(s, c, v, \sigma^2) \sim c^{-1} - \sum_{n=0}^{\infty} \frac{(3n)!\,\sigma^{2n}\, s}{6^n n!\, c^{n+2}} (v + c^{-1}s)^{-3n-1} \quad \text{as } v \to \infty .$$

Note that $\hat{\psi}^{(\infty)}(s, c, v, \sigma^2)$ in (3.13) formally satisfies equation (3.8) exactly but the sum does *not* converge. After formally taking the inverse Laplace transform termwise, differentiating (repeatedly) with respect to s and evaluating the derivatives at zero one obtains

$$E^*(T(x, v, \sigma^2)) \sim \sum_{n=0}^{\infty} \frac{(3n)!\,\sigma^{2n} x^{n+1}}{n!\,(n+1)!\,6^n v^{3n+1}} \quad \text{as } v \to \infty$$

$$E^*(T^2(x, v, \sigma^2)) \sim \sum_{n=0}^{\infty} \frac{2\,(3n+1)!\,\sigma^{2n} x^{n+2}}{n!\,(n+2)!\,6^n v^{3n+2}} \quad \text{as } v \to \infty$$

and in general

$$(3.14)\qquad E^*(T^k(x, v, \sigma^2)) \sim \sum_{n=0}^{\infty} \frac{k\,(3n + (k-1))!\,\sigma^{2n} x^{n+k}}{n!\,(n+k)!\,6^n v^{3n+k}}, \quad k = 1,2,\ldots$$

where $E^*(T^k(x, v, \sigma^2))$ denotes the conditional k-th moment of T *given* that the boundary is being reached in finite time. Since $\psi^{(\infty)}$ is the Laplace transform of $T(x,v,\sigma^2)$ it has the representation

$$(3.15) \qquad \psi^{(\infty)}(s,x,v,\sigma^2) \sim \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} E^*(T^k(x,v,\sigma^2)) \quad \text{as} \quad v \to \infty .$$

Substituting (3.15) into (3.1) for $\beta = 0$ and $\mu = 0$ and equating coefficients of powers of s one derives the following system of partial differential equations for $E^*(T^k(x,v,\sigma^2))$

$$(3.16) \qquad \frac{\sigma^2}{2} \frac{\partial^2 E(T^k(x,v,\sigma^2))}{\partial v^2} - v \frac{\partial E(T^k(x,v,\sigma^2))}{\partial x} = -k\,E(T^{k-1}(x,v,\sigma^2))$$

for all $k \geq 1$. It is easily checked by differentiating (3.14) termwise that $E^*(T^k(x,v,\sigma^2))$ formally satisfies (3.16) and the appropriate boundary condition but that the sum in (3.14) does *not* converge for any nonzero value of $\sigma^2 x / v^3$. Although this might be surprising at first it is well-known in theoretical physics and applied mathematics that most problems in perturbation theory or WKB analysis lead to divergent series. It was Poincaré who introduced this concept of divergent asymptotic expansions into mathematics and demonstrated that formal solutions of differential equations are asymptotic expansions of actual solutions. In fact one can even go a step further: Typically, optimally truncated divergent series are very good approximations for these actual solutions. For some information on divergent series and on their optimal truncation we refer the reader to the book by Bender and Orszag (1978). We chose to use the first 3 terms of the asymptotic expansion.

Hence we obtain for the conditional mean

$$E^* (T(x, v, \sigma^2)) \approx \frac{x}{v} + \frac{\sigma^2}{2} \frac{x^2}{v^4} + \frac{5\sigma^4}{3} \frac{x^3}{v^7}$$

Intuitively $\frac{x}{v}$, the first term of the series, provides a deterministic approximation to $T(x, v, \sigma^2)$ (when $\sigma^2 = 0$ so that $V(s) = v =$ const. for all s), while $\sigma^2 x v^{-3}$ measures the effect of Brownian fluctuations. A probabilistic explanation for this can be obtained in the following way. By definition

$$T(x, v, \sigma^{2)} = \min\{t : t \geq 0, vt + \int_0^t W(s)\,ds = 0 \mid X(0) = -x, \ V(0) = v\}$$

where here $W(s)$ is a Brownian motion process starting at zero with variance σ^2. Clearly,

$$T(x, v, \sigma^2) = \min\{t : t \geq 0, vt + \int_0^t W(s)\,ds = x \mid X(0) = 0, \ V(0) = v\}$$

$$= \min\{t : t \geq 0, \ vt + \int_0^t c\,\sigma B(s/c^2)\,ds = x \mid X(0) = 0, \ V(0) = v\}$$

where c is a positive constant and $B(s)$, $s \geq 0$ is a standard Brownian motion, i.e. with mean 0, and variance 1, and starting at 0. We also made use of the fact that $c\,\sigma B(s/c^2)$ is identical in law to $W(s)$. We can further simplify to

$$= \min\{t : t \geq 0, \ c^2 vt/c^2 + c^3 \sigma \int_0^{t/c^2} W(z)\,d(z) = x \mid X(0) = 0, \ V(0) = v\}$$

$$= c^2 \min\{t : t \geq 0, \ c^2 vt + c^2 \sigma \int_0^t W(z)\,d(z) = x \mid X(0) = 0, \ V(0) = v\}.$$

Now, taking $c = v/\sigma$ we get

$$v^2 \sigma^{-2} \min \{t : t \geq 0, \ t + \int_0^t W(z)\, d(z) = \frac{x\sigma^2}{v^3} \mid X(0) = 0, \ V(0) = v\}.$$

And this implies that

$$T(x, v, \sigma^2) \ \text{ and } \ \sigma^{-2} v^2 T(x\sigma^2 v^{-3}, 1, 1)$$

have the same distribution.

To make an effort to also find an approximation for the distribution of T consider again (3.13) and perform a Laplace inversion with respect to s. Then

$$\hat{\psi}^{(\infty)}(t, c, v, \sigma^2) \sim c^{-1}\delta(t) - \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \frac{\sigma^{2n} c^{2n-1} t^{3n} \exp(-vct)}{6^n n!} \quad \text{as } v \to \infty$$

and hence

$$(3.17) \qquad \hat{\psi}^{(\infty)}(t, c, v, \sigma^2) \sim c^{-1}\delta(t) - \frac{\partial}{\partial t} \frac{1}{c} \exp\left(-vct + \frac{\sigma^2}{6} c^2 t^3\right) \quad \text{as } v \to \infty$$

$\hat{\psi}^{(\infty)}(t,c,v,\sigma^2)$ in (3.17) satisfies exactly the following PDE (obtained e.g. by applying the Laplace transform with respect to x to the Kolmogorov backward equation which $\psi(t,x,v,\sigma^2)$ satisfies):

$$\frac{\sigma^2}{2} \frac{\partial^2 \hat{\psi}^{(\infty)}(t,c,v,\sigma^2)}{\partial v^2} - vc\,\hat{\psi}^{\infty}(t,c,v,\sigma^2) = \frac{\partial \hat{\psi}^{(\infty)}(t,c,v,\sigma^2)}{\partial t}.$$

Again, applying the inverse Laplace transform but now with respect to c and after some calculations we get

$$f(t,x,v,\sigma^2) \sim (3x - vt) / ((8/3)\pi\sigma^2 t^5)^{1/2} \exp(-3(x - vt)/2t^3) \quad \text{as } v \to \infty$$

Notice that the argument of the integral is the density of a normal distribution with mean vt and variance $\frac{\sigma^2}{3} t^3$.

Hence for the distribution function of T

$$(3.18) \qquad F(t,x,v,\sigma^2) \approx 1 - \phi\left[\frac{x-vt}{(\sigma^2 t^3/3)^{1/2}}\right] + \text{const.}(x,v,\sigma^2)$$

where ϕ is the distribution function of the standard normal.

This approximation is expected to be accurate for not too large values of t if negative velocities occur with small probabilities. Notice, however, that $\frac{\partial}{\partial t} F(t,x,v,\sigma^2) \approx -\frac{\partial}{\partial t}\psi\left[\frac{x-vt}{(\sigma^2 t^3/3)^{1/2}}\right]$ has no moments of any order (and is not even a density) and so we are unable to obtain the asymptotic expansions of the conditional moments and their approximations from (3.19) directly. We are presently working on refinements of the approximation (3.18).

b. We will now consider the case where V(s) is an Ornstein-Uhlenbeck process with drift i.e. we will try to solve (3.1) with $\beta > 0$ and $\mu > 0$. Towards this end perturbation methods will be used. Again, working with the representation of ψ as in (3.14) and (3.15) we obtain a partial differential equation for the mean $m_\beta(x, v + \mu, \sigma^2)$ say:

$$\frac{\sigma^2}{2}\,\frac{\partial^2 m_\beta(x,v+\mu,\sigma^2)}{\partial v^2} - (v+\mu)\frac{\partial m_\beta(x,v+\mu,\sigma^2)}{\partial x} - \beta v\,\frac{\partial m_\beta(x,v+\mu,\sigma^2)}{\partial v} = -1$$

with $m_\beta(0, v + \mu, \sigma^2) = 0$ whenever $v + \mu > 0$. Defining Laplace transforms,

$$\hat{m}_\beta(c, v+\mu,\sigma^2) = \int_0^\infty \exp(-cx)\, m_\beta(x, v+\mu,\sigma^2)\,dx$$

one has

$$\frac{\sigma^2}{2}\frac{\partial^2 \hat{m}_\beta}{\partial v^2} - (v+\mu)c\,\hat{m}_\beta - \beta v\frac{\partial \hat{m}_\beta}{\partial v} = -c^{-1},$$

so that by making the substitution $\omega = 2^{1/3}c^{1/3}\sigma^{-2/3}(v+\mu)$ one arrives at

$$(3.20)\quad \frac{\partial^2 \hat{m}_\beta}{\partial \omega^2} - \omega\,\hat{m}_\beta - \beta(2^{1/3}c^{-2/3}\sigma^{-2/3}\omega - 2^{2/3}c^{-1/3}\sigma^{-4/3}\mu)\frac{\partial \hat{m}_\beta}{\partial \omega} = -2^{1/3}c^{-5/3}\sigma^{-2/3}.$$

Now, set

$$(3.21)\qquad \hat{m}_\beta(x, v+\mu, \sigma^2) = \sum_{i=0}^{\infty}\beta^i m^{(i)}(x, v+\mu, \sigma^2) \quad \text{for small } \beta,$$

where $m^{(0)}(x, v+\mu, \sigma^2)$ is the solution of (3.1) for $\beta = 0$, i.e. $m^{(0)}(x, v+\mu, \sigma^2) = E^*(T(x, v+\mu, \sigma^2)$. Using (3.21) in (3.20) and setting $a = 2^{1/3}c^{-2/3}\sigma^{-2/3}$ and $b = -2^{2/3}c^{-1/3}\sigma^{-4/3}$ one obtains the following system to be solved:

$$\frac{\partial^2 \hat{m}^{(i)}}{\partial \omega^2} - \omega\,\hat{m}^{(i)} - (a\omega + b\mu)\frac{\partial \hat{m}^{(i-1)}}{\partial \omega} = 0 \quad \text{for } i = 1, 2, \ldots$$

We will be content with first order corrections to the mean. For i = 1,

$$\frac{\partial^2 \hat{m}^{(1)}}{\partial \omega^2} - \omega\hat{m}^{(1)} - (a\omega + b\mu)\frac{\partial \hat{m}^{(0)}}{\partial \omega} = 0.$$

Again using the method of dominant balance as described above, it is easy to show that the first two terms in the asymptotic expansion of $m^{(1)}$ are given by

$$m^{(1)}(x, v+\mu, \sigma^2) \approx \frac{1}{2}\frac{x^2 v}{(v+\mu)^3} + \frac{(7v-3\mu)\sigma^2 x^3}{6(v+\mu)^6}.$$

and hence we get

$$m_\beta(x, v+\mu, \sigma^2) \approx \frac{x}{v+\mu} + \frac{\sigma^2}{2}\frac{x^2}{(v+\mu)^4} + \frac{5}{3}\sigma^4\frac{x^3}{(v+\mu)^7} + \beta\left(\frac{1}{2}\frac{x^2 v}{(v+\mu)^3} + \frac{(7v-3\mu)\sigma^2 x^3}{6(v+\mu)^6}\right)$$

as an approximation to the mean FPT of an integrated Ornstein-Uhlenbeck process with drift μ, variance σ^2 and friction parameter β. Again we ignored powers $(v+\mu)^{-8}$

and smaller. We will study the quality of our approximations via a series simulations.

4. Simulations.

It is quite complicated to find an analytic bound for the error term introduced by the various approximations and heuristic arguments which led to (3.14). Therefore we performed an extensive series of simulations for different values of σ^2 and several distances x and initial velocities v obtaining 224 different simulated mean first-passage times. These simulations show that the approximation is excellent whenever $\sigma^2 z v^{-3} < 1/4$. For the given values of σ^2 this is the case for all the values of z and v in Tables 1,2,3,4 at the end of this section.

The basis for simulations is provided by the following relation between the velocity process $V(s)$ and the position process $X(t)$:

$$(4.1) \qquad X(t+\Delta t) = X(t) + \int_t^{t+\Delta t} V(s)\,ds.$$

Of course, it is impossible to obtain complete (for all $s \geq 0$) realizations of the velocity process $V(s)$. Instead, we deduce the entire trajectory from the velocities $V(k \cdot \Delta t)$, $k = 1, \ldots, n$. Then an assumption is necessary for the behavior of the velocity process between the discrete time points $k \cdot \Delta t$. The only assumption which makes both physical and analytic sense is the assumption of constant acceleration during $[k \cdot \Delta t, (k+1) \cdot \Delta t)$ for all $k < n$. This assumption leads to position being a quadratic spline (and hence to a quadratic interpolation scheme for the exact first passage-time) and to the approximation of the integral in (4.1) by the trapezoidal rule. The simulations were performed in the Statistical Laboratory at Queens University in Kingston, Canada. At time $t = 0$ we started 2000 particles with $V(0) = v$. The time increment Δt was chosen in such a way that always of the order one hundred steps were needed for the particle to reach the boundary. This is a necessary compromise between accuracy and computation cost. For each particle, the time was determined (via the quadratic interpolation scheme in the Appendix) when it crossed several boundaries at $x = 1,2,...$ and ensemble averages were taken.

For $\sigma = .1, .3, .5, .7$ the results of the simulations are reported in Tables 1,2,3,4 at the end of this section. The following notation is used (for convenience the dependence on x, v, σ^2 will not be explicitly indicated):

m_s: sample averages of simulated first-passage times for given z, v, σ^2.

$$A_1 = \frac{x}{v}, \quad A_2 = \frac{\sigma^2}{2} \frac{x^2}{v^4}, \quad A_3 = \frac{5 \cdot \sigma^4}{3} \frac{x^3}{v^7}$$

$$M_1 = 10^3 \cdot \frac{m_s - A_1}{A_1}, \quad M_2 = 10^3 \cdot \frac{m_s - (A_1 + A_2)}{A_1}, \quad M_3 = 10^3 \cdot \frac{m_s - (A_1 + A_2 + A_3)}{A_1}$$

$SD(M_i)$: sample standard deviation of M_i, $i = 1,2,3$.

$$t_i = \frac{M_i}{SD(M_i)}, \quad i = 1,2,3.$$

The findings are summarized as follows:

1. In 105 out of 224 cases, $|t_1| \leq 2.00$ and in 21 cases $t_1 \leq 0.00$ indicating clearly that the 1^{st} order approximation A_1 tends to underestimate the mean first-passage time. This is confirmed by the following summary statistics of t_1:

   Mean = 2.57, STDEV = 4.15, SEMEAN = 0.14

   Max = 8.00, Min = -0.87, Q_3 = 4.15, Q_1 = 0.85.

   Q_1 and Q_3 are the first and third quantiles, respectively.

2. In 197 out of 224 cases, $|t_2| \leq 2.00$ and in 118 cases $t_2 \leq 0.00$. So, in the great majority of experiments (i.e. 88%), the 2^{nd} order approximation is within 2 standard errors of the simulated mean. Also, the frequencies of underestimation and overestimation (relative to the simulated mean are about equal. The summary statistics of t_2 are:

   MEAN = -0.04, STDEV = 1.27, SEMEAN = 0.09

   MAX = 3.48, MIN = -3.63, Q_3 = 0.85, Q_1 = –0.91

3. In 199 out of 224 cases, $|t_3| \leq 2.00$ and in 126 cases $t_3 \leq 0.00$. The 3^{rd} order terms in the approximation and hence the difference between t_2 and t_3 are sometimes very small. The precision of the experiments (i.e., 2000 particles for given σ^2, x, v) is not sufficiently high to determine whether the 3^{rd} order approximation improves over the 2^{nd} order approximation. The summary statistics of t_3 are:

MEAN = -0.22, STDEV = 1.32, SEMEAN = 0.09

MAX = 2.69, MIN = -6.85, $Q_3 = 0.70$, $Q_1 = -1.02$

Appendix: Boundary Crossing by Quadratic Interpolation.

A particle which is observed in discrete time crosses a barrier at 0 during the time interval $((k-1)\cdot\Delta t, k\cdot\Delta t]$ if $X(k\cdot\Delta t) \geq 0$ and $X((k-1)\cdot\Delta t) < 0$. In this work we made the assumption of constant acceleration during times $k\cdot\Delta t$, $k = 1, \ldots, n$. This leads to position being a quadratic spline and hence suggests a quadratic interpolation scheme, compare also Pickard et. al. (1985). Since particles can have negative velocities, first passage can also take place if $X((k-1)\cdot\Delta t) < 0$ and $X(k\cdot\Delta t) < 0$. The following interpolation scheme can also handle this case but it cannot distinguish between two or more crossings during the time increment.

Let $X((k-1)\cdot\Delta t) = x < 0$, $V((k-1)\cdot\Delta t) = u$, $V(k\cdot\Delta t) = v$. The position at intermediate times is given by

$$X((k-1)\cdot\Delta t + \tau) = x + u\cdot\tau + \left[\frac{v-u}{\Delta t}\right]\cdot\frac{\tau^2}{2} \text{ for } 0 \leq \tau \leq \Delta t$$

and interpolation involves finding the smallest real nonnegative zero τ_0 of f, say, where $f(\tau) = X((k-1)\cdot\Delta t + \tau)$. If the discriminant $D = u^2 - 2\left[\frac{v-u}{\Delta t}\right]\cdot x$ is nonnegative then

I. $$\tau_0 = \frac{(-u + \sqrt{D})\cdot\Delta t}{v-u}.$$

Since $f(\tau)$ is a quadratic function, the maximum is attained either at $\tau = \Delta t$ in which case $f(\Delta t) = x + \frac{u+v}{2}\Delta t$ or at the critical point $c = (u \cdot \Delta t)/(u-v)$ if $u > 0 > v$ and then $f(c) = \frac{u^2 \Delta t}{2(u-v)} + x$. Clearly, both $f(\Delta t) \geq 0$ and $f(c) \geq 0$ imply $D \geq 0$ and hence first-passage.

In summary, first passage occurs during an increment at time $((k-1) \cdot \Delta t + \tau_0)$ with τ_0 as in Equation I. if either $\frac{u+v}{2} \cdot \Delta t \geq -x$ or $D \geq 0$ with $u > 0 > v$.

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