

Equivalent Martingale Measures and No-Arbitrage in Stochastic Securities Market Models

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Technical Report No. 182
November 1988

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ABSTRACT

We characterize those vector-valued stochastic processes (with a finite index set and defined on an arbitrary stochastic base) which can become a martingale under an equivalent change of measure.

This solves a problem which arises in the study of finite period securities markets with one riskless bond and a finite number of risky stocks. In this setting, our characterization provides necessary and sufficient conditions for the absence of arbitrage opportunities ("free lunches"). This result can be interpreted as saying "if one cannot win betting on a process, then it must be a martingale under an equivalent measure", and provides a converse to the intuitive notion that "one cannot win betting on a martingale".

1. INTRODUCTION

Classical martingale systems theorems (Halmos (1939), Doob (1953)) formalize the intuitive idea behind martingales, namely that “one can’t win betting on a martingale”. More precisely, using Burkholder’s martingale transforms (Burkholder (1966)), we have the following. Let $X = (X_t : t = 0, 1, \dots, T)$ be an R^d -valued martingale ($1 \leq d < \infty$) on some stochastic base (Ω, \mathcal{F}, P) and let $V = (V_t : t = 1, 2, \dots, T)$ denote an R^d -valued \mathcal{F} -predictable stochastic process (“betting strategy”). Then the martingale transform $V \circ X = ((V \circ X)_t : t = 0, 1, \dots, T)$ of V with respect to X is defined by

$$(V \circ X)_t = V_1 \cdot X_0 + \sum_{s=1}^t V_s \cdot (X_s - X_{s-1})$$

where $(V \circ X)_t$ represents the “accumulated gain up to time t ” when following the strategy V ($V_s \cdot X_s$ is the euclidean scalar product of the vectors V_s and X_s). If $V \circ X$ is integrable then classical martingale systems results show that $V \circ X$ is again a martingale with respect to P and \mathcal{F} . Thus there are no “smart” betting strategies which can change the character of the fair game X in favor of the gambler. Put differently, if X and V are as above then the following condition (1.1) holds.

$$(1.1) \quad \text{for } t = 1, 2, \dots, T \text{ and for } \mathcal{F}\text{-predictable } V, \\ V_t \cdot (X_t - X_{t-1}) \geq 0 \text{ } P\text{-a.s.} \Rightarrow V_t \cdot (X_t - X_{t-1}) = 0 \text{ } P\text{-a.s.}$$

Observe that (1.1) still holds if P is replaced by an equivalent probability measure Q on (Ω, \mathcal{F}) (i.e. P and Q have the same null sets), but that the martingale-property of $V \circ X$ (under P) will in general be destroyed when P is replaced by Q .

In this paper, we show that condition (1.1) is not only sufficient but also necessary for a process X to be a martingale under an equivalent probability measure Q on (Ω, \mathcal{F}) ; such a Q is called an *equivalent martingale measure* for (\mathcal{F}, X) . The results in this paper can thus be viewed as a converse to the classical martingale systems theorem and can be interpreted as saying: “if one can’t win betting on a process then it must be a martingale under an equivalent change of measure”. Our approach is based on a pathwise

analysis of condition (1.1) and extends two recent developments in this area: (i) a similar but more elementary sample path investigation of (1.1) when there are only finitely many states of nature (see Taquu and Willinger (1987)), and (ii) an analysis of the same change of measure problem in the case of a single-period random process X (see Willinger and Taquu (1988)). Willinger and Taquu (1988) have also solved the problem of existence of a *unique* martingale measure. Whereas the uniqueness problem can be solved using only elementary probability tools, our extension relies on some abstract measurable selection theorems. Our results also include the special case $d = 1$ studied by Back and Pliska (1987). Their proof, however, does not generalize to higher dimensions.

Characterizing stochastic processes which can be transformed into martingales by means of an equivalent change of measure is of particular interest in the analysis of stochastic models of securities markets (see, for example, Harrison and Kreps (1979), Harrison and Pliska (1981), Taquu and Willinger (1987), Back and Pliska (1987), Willinger and Taquu (1988)). In this setting, our results (in particular, Theorem 2.9) extend those of Harrison and Pliska (1981) and Taquu and Willinger (1987) who consider finite-period, frictionless securities market models when there are only finitely many states of nature. Here we allow an arbitrary probability space and show that the economically meaningful assumption of "no arbitrage" is both necessary and sufficient for the existence of an equivalent martingale measure for the securities price processes (modelling the prices of one riskless bond and $1 \leq d < \infty$ risky stocks over time). Intuitively, an arbitrage opportunity ("free lunch") represents a riskless plan for making profits without initial investments and, therefore, has to be ruled out for an economic equilibrium to exist.

The problem of proving the existence of an equivalent martingale measure for general securities price processes defined on a general stochastic base from such primitive economic considerations as "no arbitrage" was posed by Harrison and Pliska (1981) (see also Harrison and Kreps (1979)). The work of Back and Pliska (1987) solves the one-dimensional finite period case (i.e., one risky stock and one riskless bond). Although their method of proof does not generalize to higher dimensions, they conjectured that the same result holds for finite-period, vector-valued securities price processes. This

conjecture is proved in our Theorem 2.9. Also note that Back and Pliska work with a more restrictive class of “feasible” trading strategies than we do (we impose no “positive wealth constraints”, see Section 3.1); in the present finite-period setting, it is easy to see that this restriction is not essential.

A pathwise analysis along the lines suggested in this paper of the no-arbitrage assumption for continuous-time price processes, where trading in stocks and bonds can take place continuously in time, remains an open problem. See, however, Harrison and Pliska (1981) and Back and Pliska (1987) for examples of what can go wrong when trading continuously.

The paper is structured as follows. In Section 2 we prove that condition (1.1) is necessary and sufficient for the existence of an equivalent martingale measure Q for X and briefly mention the known results concerning uniqueness of such a Q . In Section 3 we apply our results in the context of stochastic modelling of finite-period, frictionless securities markets and show how the existence of an equivalent martingale measure for the securities price process is related to the economically meaningful “no arbitrage” assumption.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF AN EQUIVALENT MARTINGALE MEASURE

The purpose of this section is to give necessary and sufficient conditions under which a discrete-time, \mathbb{R}^d -valued process with finite time horizon has an equivalent martingale measure. We first show that condition (1.1) is sufficient for the existence of an equivalent martingale measure for a “one-step” process $X = (X_0, X_1)$. This special case contains most of the technical difficulties and the problem of a finite-period process $X = (X_t : t = 0, 1, \dots, T)$ follows as a corollary of the results for the “one-step” process. Our proofs are essentially self-contained, using only standard results from convex analysis, and some general results concerning measurable selection. In particular, the proofs do not rely on results of the finite-probability space setting (see Taqqu and Willinger (1987)), nor on the one-dimensional result of Back and Pliska (1987). Both are special cases of our proof, but a reader only interested in these settings

should consult these references. (Note that Back and Pliska (1987) assume integrability of the process, an apparently unnatural condition since it is not preserved under a change of equivalent measure; see, however, Remark 3.4).

We would like to point out that most of the technical difficulties in our proof come from the fact that with $d > 1$, an explicit construction as in Back and Pliska (1987) is no longer feasible. However, if in addition the process were assumed to be bounded, a discrete approximation argument as in Willinger and Taqqu (1988, Theorem 2.3.1), together with appropriate use of measurable selection would be possible.

2.1 The one-step case

Let (Ω, \mathcal{F}, P) be a (complete) probability space. If \tilde{P} is a probability measure on (Ω, \mathcal{F}) , then P and \tilde{P} are *equivalent* (on \mathcal{F}) provided for all $F \in \mathcal{F}$, $P(F) = 0$ if and only if $\tilde{P}(F) = 0$. Note that if \mathcal{G} is a sub σ -algebra of \mathcal{F} , then P and \tilde{P} may be equivalent on \mathcal{G} but not on \mathcal{F} . When P and \tilde{P} are equivalent, $d\tilde{P}/dP$ denotes the Radon-Nikodym derivative of \tilde{P} with respect to P . In this case, $d\tilde{P}/dP > 0$, $P - a.s.$

In order to simplify the notation, the expectation of Y under the probability measure P will be denoted $P(Y)$ and the conditional expectation of Y given \mathcal{G} will be denoted $P(Y | \mathcal{G})$. If P and \tilde{P} are equivalent on \mathcal{F} and Y is \tilde{P} -integrable, recall that

$$(2.1) \quad \tilde{P}(Y | \mathcal{G}) = P(Y \frac{d\tilde{P}}{dP} | \mathcal{G}) / P(\frac{d\tilde{P}}{dP} | \mathcal{G}).$$

The set \mathbf{R}^d with its euclidean norm $\| \cdot \|$ will be equipped with its usual topology and its Borel σ -algebra $\mathcal{B}(\mathbf{R}^d)$. We will add to \mathbf{R}^d an element ∞ , and set $\bar{\mathbf{R}}^d = \mathbf{R}^d \cup \{\infty\}$. If the open neighborhoods of ∞ are the complements of compact sets then $\bar{\mathbf{R}}^d$ is compact and metrisable. We equip $\bar{\mathbf{R}}^d$ with this topology.

Given two elements $x, y \in \mathbf{R}^d$, $x \cdot y$ will denote their euclidean scalar product. Each $\alpha \in \mathbf{R}^d$ defines a hyperplane $H^\alpha = \{x \in \mathbf{R}^d : \alpha \cdot x = 0\}$. We also define $H_{\geq}^\alpha = \{x \in \mathbf{R}^d : \alpha \cdot x \geq 0\}$, $H_{>}^\alpha = \{x \in \mathbf{R}^d : \alpha \cdot x > 0\}$. H_{\leq}^α and $H_{<}^\alpha$ are defined analogously.

The following lemma gives a slightly sharper result than Theorem 2.3.1 of Willinger and Taqqu (1988); indeed, it is not difficult to see that the two statements would be equivalent if the function g below were only required to be measurable instead of continuous.

2.1. Lemma. Let ν be a probability measure on \mathbf{R}^d with *bounded support* (i.e. $\nu(K) = 1$, for some compact set K). Then the following two statements are equivalent.

(a) For all $\alpha \in \mathbf{R}^d$, $\nu(H_\alpha^\alpha) = 1$ implies $\nu(H^\alpha) = 1$.

(b) There exists a continuous (strictly positive) function $g : \mathbf{R}^d \rightarrow (0, 1]$ such that

$$\int_{\mathbf{R}^d} x \cdot g(x) \nu(dx) = 0.$$

Proof. We do not need the implication (b) \Rightarrow (a), so its easy proof is omitted (it is similar to the first few lines of the proof of Theorem 2.6). So suppose (a) holds. Then for each $\alpha \in \mathbf{R}^d$, we have either $\nu(H^\alpha) = 1$ or

$$(2.2) \quad \nu(H^\alpha) < 1 \text{ and } (\nu(H_\alpha^>) > 0 \text{ and } \nu(H_\alpha^<) > 0).$$

Now if $\nu(H^\alpha) = 1$, define two functions $g_\alpha^>$ and $g_\alpha^<$ by

$$g_\alpha^>(x) = 1, \quad g_\alpha^<(x) = 1, \text{ for all } x \in \mathbf{R}^d.$$

If $\nu(H^\alpha) < 1$, then by (2.2),

$$\int_{\mathbf{R}^d} (\alpha \cdot x) I_{H_\alpha^>}(x) \nu(dx) > 0 \text{ and } \int_{\mathbf{R}^d} (\alpha \cdot x) I_{H_\alpha^<}(x) \nu(dx) < 0.$$

Approximating the indicator function $I_{H_\alpha^>}$ (resp. $I_{H_\alpha^<}$) by piecewise-linear continuous functions, we see that there is a function $g_\alpha^> : \mathbf{R}^d \rightarrow (0, 1]$ (resp. $g_\alpha^< : \mathbf{R}^d \rightarrow (0, 1]$) such that

$$\int_{\mathbf{R}^d} (\alpha \cdot x) g_\alpha^>(x) \nu(dx) > 0 \text{ (resp. } \int_{\mathbf{R}^d} (\alpha \cdot x) g_\alpha^<(x) \nu(dx) < 0).$$

Regardless of the value of $\nu(H^\alpha)$, we set

$$m_{>}^{\alpha} = \int_{\mathbb{R}^d} x g_{>}^{\alpha}(x) v(dx), \quad m_{<}^{\alpha} = \int_{\mathbb{R}^d} x g_{<}^{\alpha}(x) v(dx)$$

(observe that since the support of v is bounded, $m_{>}^{\alpha}$ and $m_{<}^{\alpha}$ are (finite) elements of \mathbb{R}^d). Consider

$$M = \{m_{>}^{\alpha} : \alpha \in \mathbb{R}^d\} \cup \{m_{<}^{\alpha} : \alpha \in \mathbb{R}^d\}.$$

Recall that a set C is *properly separated* from the origin by a hyperplane H^{α} provided H_{\geq}^{α} contains C but H^{α} does not. The set M has the following property: no hyperplane properly separates M from the origin. Indeed, if $\beta \in \mathbb{R}^d$ is such that $v(H^{\beta}) = 1$, then for all $\alpha \in \mathbb{R}^d$,

$$\beta \cdot m_{>}^{\alpha} = \int_{\mathbb{R}^d} (\beta \cdot x) g_{>}^{\alpha}(x) v(dx) = 0,$$

and similarly $\beta \cdot m_{<}^{\alpha} = 0$. Thus M would be included in H^{β} , and so H^{β} would not properly separate M from the origin. On the other hand, if $v(H^{\beta}) < 1$, then by the definition of $g_{<}^{\beta}$,

$$\beta \cdot m_{<}^{\beta} = \int_{\mathbb{R}^d} (\beta \cdot x) g_{<}^{\beta}(x) v(dx) < 0,$$

so H_{\geq}^{β} does not contain M .

Since no hyperplane properly separates M from the origin, the same holds for the convex hull $\text{conv}(M)$ of M . By Theorem 11.3 of Rockafellar (1970), the origin belongs to $\text{conv}(M)$. This means that there are finitely many points $m_1, \dots, m_k \in M$ and real numbers $\lambda_1 \geq 0, \dots, \lambda_k \geq 0$ such that $\lambda_1 + \dots + \lambda_k = 1$ and $\lambda_1 m_1 + \dots + \lambda_k m_k = 0$. Now set $g(x) = \lambda_1 g_1(x) + \dots + \lambda_k g_k(x)$, where g_i is the continuous function which defines $m_i \in M$ ($i = 1, 2, \dots, k$). Then g is continuous, $0 < g(x) \leq 1, \forall x \in \mathbb{R}^d$, and

$$\int_{\mathbb{R}^d} x g(x) v(dx) = \sum_{i=1}^k \lambda_i m_i = 0.$$

Thus (b) holds, and the proof is complete. ■

2.2. Theorem. Let v be an arbitrary probability measure on \mathbb{R}^d . Then the following two conditions are equivalent.

(a) For all $\alpha \in \mathbb{R}^d$, $v(H_{\geq}^{\alpha}) = 1$ implies $v(H^{\alpha}) = 1$.

(b) There is a continuous function $g : \bar{\mathbf{R}}^d \rightarrow [0, 1]$, such that $g(x) > 0, \forall x \in \mathbf{R}^d$, and

$$\int_{\mathbf{R}^d} \|x\| g(x) \nu(dx) < +\infty \text{ and } \int_{\mathbf{R}^d} x g(x) \nu(dx) = 0.$$

Proof. Again, we do not need the implication (b) \Rightarrow (a), so its easy proof is omitted. So we show that (a) implies (b).

Define a one-to-one transformation $\psi : \mathbf{R}^d \rightarrow B(0, 1)$ (the open unit ball centered at the origin) by $\psi(x) = x/(1 + \|x\|)$, and let $\tilde{\nu}$ be the image of ν under this transformation. Then $\tilde{\nu}$ defines in fact a probability measure on \mathbf{R}^d with bounded support, and $\tilde{\nu}$ satisfies condition (a) in Lemma 2.1. So let $\tilde{g} : \mathbf{R}^d \rightarrow (0, 1]$ satisfy condition (b) of Lemma 2.1, and define $g : \bar{\mathbf{R}}^d \rightarrow [0, 1]$ by

$$g(x) = \tilde{g}(\psi(x)) / (1 + \|x\|) \text{ if } x \neq \infty,$$

and $g(\infty) = 0$. This function satisfies all the required conditions, since

$$\begin{aligned} \int_{\mathbf{R}^d} \|x\| g(x) \nu(dx) &= \int_{\mathbf{R}^d} \|\psi(x)\| \tilde{g}(\psi(x)) \nu(dx) \\ &= \int_{\mathbf{R}^d} \|x\| \tilde{g}(x) \tilde{\nu}(dx) \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbf{R}^d} x g(x) \nu(dx) &= \int_{\mathbf{R}^d} \psi(x) \tilde{g}(\psi(x)) \nu(dx) \\ &= \int_{\mathbf{R}^d} x \tilde{g}(x) \tilde{\nu}(dx) \\ &= 0. \quad \blacksquare \end{aligned}$$

2.3. Remark. The function g need not be a density, since its integral with respect to ν need not be 1. We could of course rescale g appropriately, but it is the theorem as stated which will be useful in the sequel.

Let \mathcal{G} be a sub σ -algebra of \mathcal{F} . Recall (see, for example, Ash (1972)) that Y has a regular

conditional probability distribution given \mathcal{G} , that is, there exists a function $\mu : \Omega \times \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that

(a) $\omega \rightarrow \mu(\omega, B)$ is \mathcal{G} -measurable, $\forall B \in \mathcal{B}(\mathbb{R}^d)$;

(b) for P -almost all $\omega \in \Omega$, $B \rightarrow \mu(\omega, B)$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$;

(c) $\mu(\cdot, B) = P(Y \in B \mid \mathcal{G})$ a.s., $\forall B \in \mathcal{B}(\mathbb{R}^d)$.

2.4. Lemma. Let Y be an arbitrary \mathbb{R}^d -valued random variable. Then the following conditions are equivalent:

(2.3) For all \mathcal{G} -measurable \mathbb{R}^d -valued random variables Z ,

$$Z \cdot Y \geq 0 \text{ } P\text{-a.s.} \Rightarrow Z \cdot Y = 0 \text{ } P\text{-a.s.}$$

(2.4) For almost all $\omega \in \Omega$, for all $\alpha \in \mathbb{R}^d$, $\mu(\omega, H_\Sigma^\alpha) = 1 \Rightarrow \mu(\omega, H^\alpha) = 1$.

In order to prove Lemma 2.4, we need the following technical result.

2.5. Lemma.

(a) Let (S, \mathcal{S}) be a measure space. Suppose $F : \Omega \times \mathbb{R}^d \times S \rightarrow \mathbb{R}$ is $\mathcal{G} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{S}$ -measurable and non-negative. Then the map $F^* : \Omega \times S \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$F^*(\omega, s) = \int_{\mathbb{R}^d} F(\omega, x, s) \mu(\omega, dx)$$

is $\mathcal{G} \times \mathcal{S}$ -measurable.

(b) Suppose $h : \Omega \rightarrow S$ is \mathcal{G} -measurable. Set $U(\omega) = F(\omega, Y(\omega), h(\omega))$ and $V(\omega) = F^*(\omega, h(\omega))$.

Then V is P -integrable if and only if U is, and in this case $V = P(U \mid \mathcal{G})$ a.s.

(c) For $K \subset \Omega \times \mathbb{R}^d$, set $K^\omega = \{x \in \mathbb{R}^d : (\omega, x) \in K\}$. Now suppose $K \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^d)$. Then the map $\omega \rightarrow \mu(\omega, K^\omega)$ is \mathcal{G} -measurable and

$$P\{\omega \in \Omega : Y(\omega) \in K^\omega\} = \int_{\Omega} \mu(\omega, K^\omega) dP(\omega).$$

Proof. Statement (c) is a special case of (b): just set $F(\omega, x, s) = I_K(\omega, x)$. So we prove (a) and (b).

Suppose $F(\omega, x, s) = I_G(\omega) I_B(x) I_R(s)$, where $G \in \mathcal{G}$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $R \in \mathcal{S}$. Then

$$F^*(\omega, s) = I_G(\omega) \mu(\omega, B) I_R(s),$$

which is $\mathcal{G} \times \mathcal{S}$ -measurable. Furthermore, $U(\omega) = I_G(\omega) I_{\{Y \in B\}}(\omega) I_R(h(\omega))$, so

$$P(U | \mathcal{G})(\omega) = I_G(\omega) \mu(\omega, B) I_R(h(\omega)) = F^*(\omega, h(\omega)).$$

Thus (a) and (b) hold for these special F , and so by applying the Monotone Class Theorem (see Dellacherie and Meyer (1978 Th. I.19)), (a) holds for all bounded $\mathcal{G} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{S}$ -measurable functions F . The following truncation argument will yield (a) and (b) as stated. Set $F_n = \min(F, n)$, $U_n(\omega) = F_n(\omega, Y(\omega), h(\omega))$, $F_n^*(\omega, s) = \int_{\mathbb{R}^d} F_n(\omega, x, s) \mu(\omega, dx)$, and $V_n = F_n^*(\omega, h(\omega))$. Then F_n is bounded and $\mathcal{G} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{S}$ -measurable, and $F_n \uparrow F$, $U_n \uparrow U$, and $V_n \uparrow V$, as $n \rightarrow \infty$. By the above, $V_n = P(U_n | \mathcal{G})$. Since the monotone convergence theorem for conditional expectations of increasing sequences of non-negative random variables holds whether or not the variables have finite expectations (see Ash (1972, Theorem 6.5.3)), we get $V = P(U | \mathcal{G})$. Using Theorem 6.4.3 of Ash (1972), this shows that U is P -integrable if and only if V is and completes the proof of (a) and (b). ■

Proof of Lemma 2.4.

(2.4) \Rightarrow (2.3). Let Z be an \mathbb{R}^d -valued \mathcal{G} -measurable random variable such that $Z \cdot Y \geq 0$ P -a.s.. Note that

$$\{Z \cdot Y \geq 0\} = \{\omega \in \Omega : Y(\omega) \in H_{\Sigma}^{Z(\omega)}\},$$

$$\{Z \cdot Y = 0\} = \{\omega \in \Omega : Y(\omega) \in H^{Z(\omega)}\}.$$

Using Lemma 2.5(c) we see that

$$1 = P\{Z \cdot Y \geq 0\} = \int_{\Omega} \mu(\omega, H_{\Sigma}^{Z(\omega)}) dP(\omega),$$

and so $\mu(\omega, H_{\Sigma}^{Z(\omega)}) = 1$, for $\omega \in \Omega \setminus N$, where N is a P -null set. By (2.4), for almost all $\omega \in \Omega \setminus N$,

$\mu(\omega, H^{Z(\omega)}) = 1$. Thus by Lemma 2.5(c)

$$P \{ Z \cdot Y = 0 \} = \int_{\Omega} \mu(\omega, H^{Z(\omega)}) dP(\omega) = 1 .$$

This proves (2.3).

(2.3) \Rightarrow (2.4). Set

$$U = \{ (\omega, \alpha) \in \Omega \times \mathbb{R}^d : \mu(\omega, H_{\Sigma}^{\alpha}) = 1 \text{ and } \mu(\omega, H^{\alpha}) < 1 \} .$$

Then $U \in \mathcal{G} \times \mathcal{B}(\mathbb{R}^d)$.

Let $pr : \Omega \times \mathbb{R}^d \rightarrow \Omega$ be the canonical projection: $pr(\omega, x) = \omega$. To prove (2.4), we must show that $pr(U)$ has P -probability zero (since (Ω, \mathcal{F}, P) is complete, U is \mathcal{F} -measurable: cf. Dellacherie and Meyer (1978, Theorem III. 44-45)).

Suppose $P(pr(U)) > 0$. We shall show that this leads to a contradiction. Using measurable selection (cf. for example, the theorem of Dellacherie and Meyer mentioned above) we see that there is a \mathcal{G} -measurable $\bar{\mathbb{R}}^d$ -valued random variable \tilde{Z} such that

$$P \{ \omega \in \Omega : (\omega, \tilde{Z}(\omega)) \in U \} = P(pr(U)) > 0 .$$

Set $Z(\omega) = \tilde{Z}(\omega)$ if $(\omega, \tilde{Z}(\omega)) \in U$, $Z(\omega) = 0$ otherwise. Now we shall show that $P \{ Z \cdot Y \geq 0 \} = 1$ but $P \{ Z \cdot Y = 0 \} < 1$, contradicting (2.3). Indeed, we have

$$\begin{aligned} P \{ Z \cdot Y \geq 0 \} &= \int_{\Omega} \mu(\omega, H_{\Sigma}^{Z(\omega)}) dP(\omega) \\ &= \int_{pr(U)} 1 dP(\omega) + \int_{(pr(U))^c} 1 dP(\omega) \\ &= 1 , \end{aligned}$$

but

$$\begin{aligned}
 P\{Z \cdot Y = 0\} &= \int_{\Omega} \mu(\omega, H^Z(\omega)) dP(\omega) \\
 &= \int_{pr(U)} \mu(\omega, H^Z(\omega)) dP(\omega) + \int_{(pr(U))^c} 1 dP(\omega) \\
 &< P(pr(U)) + P((pr(U))^c) \\
 &= 1 \quad \blacksquare
 \end{aligned}$$

2.6. Theorem. Let $\mathcal{G} \subset \mathcal{H}$ be two (complete) sub σ -algebras of \mathcal{F} , and let Y be an arbitrary \mathcal{H} -measurable random variable. Then (2.3) is equivalent to the following: there exists an \mathcal{H} -measurable real random variable D such that $0 < D \leq 1$ a.s., $P(\|Y\| D) < +\infty$ and $P(Y D \mid \mathcal{G}) = 0$.

The proof of this theorem uses the following technical lemma.

2.7. Lemma. Let $C[\bar{\mathbb{R}}^d]$ be the space of continuous real functions on $\bar{\mathbb{R}}^d$, with the norm $\|g\|_{\infty} = \sup_{x \in \bar{\mathbb{R}}^d} |g(x)|$, and let $\mathcal{B}(C[\bar{\mathbb{R}}^d])$ be its Borel σ -algebra. Then the following properties hold.

- (a) $\{g \in C[\bar{\mathbb{R}}^d] : 0 < g(x) \leq 1, \forall x \in \mathbb{R}^d\} \in \mathcal{B}(C[\bar{\mathbb{R}}^d])$.
- (b) The map $(x, g) \rightarrow g(x)$ is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(C[\bar{\mathbb{R}}^d])$ -measurable.
- (c) Suppose $F : \Omega \times \mathbb{R}^d \times C[\bar{\mathbb{R}}^d] \rightarrow \mathbb{R}^d$ is $\mathcal{G} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(C[\bar{\mathbb{R}}^d])$ -measurable. Set

$$\tilde{F}^*(\omega, g) = \int_{\mathbb{R}^d} \|F(\omega, x, g)\| \mu(\omega, dx),$$

and

$$F^*(\omega, g) = \begin{cases} \int_{\mathbb{R}^d} F(\omega, x, g) \mu(\omega, dx) & , \text{ if } \tilde{F}^*(\omega, g) < +\infty \\ \infty & , \text{ otherwise.} \end{cases}$$

Then F^* is $\mathcal{G} \times \mathcal{B}(C[\bar{\mathbb{R}}^d])$ -measurable.

- (d) Suppose $h : \Omega \rightarrow C[\bar{\mathbb{R}}^d]$ is \mathcal{G} -measurable. Set $U(\omega) = F(\omega, Y(\omega), h(\omega))$, $V(\omega) = F^*(\omega, h(\omega))$, $\tilde{U}(\omega) = \|F(\omega, Y(\omega), h(\omega))\|$, $\tilde{V}(\omega) = \tilde{F}^*(\omega, h(\omega))$.

If \tilde{V} is P -integrable, then U and V are P -integrable and $V = P(U \mid \mathcal{G})$ P -a.s..

Proof. To see (a), it is sufficient to observe that

$$\{g \in C[\bar{\mathbb{R}}^d] : 0 < g(x) \leq 1, \forall x \in \mathbb{R}^d\} = \bigcap_{N \in \mathbb{N}} \bigcup_{r \in \mathbb{Q}_+} \bigcap_{\substack{x \in \mathbb{Q}^d \\ \|x\| \leq N}} \{g \in C[\bar{\mathbb{R}}^d] : r < g(x) \leq 1\}$$

where \mathbb{Q} is the set of rational numbers and $\mathbb{Q}_+ = \{r \in \mathbb{Q} : r > 0\}$. As for (b), note that $(x, g) \rightarrow g(x)$ is continuous for the product topology on $\mathbb{R}^d \times C[\bar{\mathbb{R}}^d]$, hence measurable with respect to the Borel σ -algebra of the product topology. But this σ -algebra coincides with $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(C[\bar{\mathbb{R}}^d])$, since both \mathbb{R}^d and $C[\bar{\mathbb{R}}^d]$ are separable (see Billingsley (1968, App. II, p. 225)). This proves (b).

If $\|F\|$ is bounded, then a Monotone Class argument similar to that of Lemma 2.5 yields (c) and (d). Now if \tilde{V} is only P -integrable, then Lemma 2.5(b) implies that $\|U\| = \tilde{U}$ is P -integrable, so a dominated convergence argument completes the proof. ■

Proof of Theorem 2.6 Suppose there exists a random variable D with the properties stated in the theorem: we show that this implies (2.3). Indeed, let Z be an \mathbb{R}^d -valued random variable such that $Z \cdot Y \geq 0$ P -a.s. We must show that $P\{Z \cdot Y > 0\} = 0$. Now if $P\{Z \cdot Y > 0\} > 0$, we would have $P\{Z \cdot (DY) > 0\} > 0$, and thus

$$0 < P(Z \cdot (DY)) = P(Z \cdot P(DY | \mathcal{G})) = 0,$$

a contradiction.

Now suppose (2.3) holds, or equivalently, by Lemma 2.4, that (2.4) holds. We shall prove the existence of a random variable D with the desired properties.

Set $F(\omega, x, g) = x \cdot g(x)$, $\tilde{F}(\omega, x, g) = \|x\| |g(x)|$. Using Lemma 2.7, we see that the set

$$H = \{(\omega, g) \in \Omega \times C[\bar{\mathbb{R}}^d] : 0 < g(x) \leq 1, \forall x \in \mathbb{R}^d, \text{ and}$$

$$\int_{\mathbb{R}^d} \|x\| |g(x)| \mu(\omega, dx) \leq 1, \int_{\mathbb{R}^d} x \cdot g(x) \mu(\omega, dx) = 0\}$$

is $\mathcal{G} \times \mathcal{B}(C[\bar{\mathbb{R}}^d])$ -measurable. Since (2.4) holds, we use Theorem 2.2 to see that for P -almost all $\omega \in \Omega$, there is a $g_\omega \in C[\bar{\mathbb{R}}^d]$ such that $(\omega, g_\omega) \in H$. This means that the projection of H on Ω has P -

probability one.

Now since $\bar{\mathbf{R}}^d$ is compact and metrisable, $C[\bar{\mathbf{R}}^d]$ is a separable and complete, and thus we can apply a measurable selection theorem (see, for example, Dellacherie and Meyer (1978, Theorem III. 44-45)) to get a \mathcal{G} -measurable map $G : \Omega \rightarrow C[\bar{\mathbf{R}}^d]$ such that $(\omega, G(\omega)) \in H$ for P -almost all $\omega \in \Omega$. We write $G(\omega, x)$ instead of $G(\omega)(x)$. The map $(\omega, x) \rightarrow G(\omega, x)$ is $\mathcal{G} \times \mathcal{B}(\mathbf{R}^d)$ -measurable, since it is the composition of the two measurable maps $(\omega, x) \rightarrow (x, G(\omega))$ and $(x, g) \rightarrow g(x)$ (see Lemma 2.7 (b)).

Set $D(\omega) = G(\omega, Y(\omega))$: D is \mathcal{H} -measurable (again since it is the composition of two \mathcal{H} -measurable maps), and $0 < D \leq 1$ a.s. Furthermore, since

$$\bar{F}^*(\omega, G(\omega)) = \int_{\mathbf{R}^d} \bar{F}(\omega, x, G(\omega)) \mu(\omega, dx) = \int_{\mathbf{R}^d} \|x\| G(\omega, x) \mu(\omega, dx) \leq 1,$$

$\bar{U}(\omega) = \bar{F}(\omega, Y(\omega), G(\omega)) = \|Y(\omega)\| D(\omega)$ is P -integrable and since $\bar{F}^*(\omega, G(\omega)) = 0$ P -a.s., Lemma 2.7 (d) with $U(\omega) = \bar{F}(\omega, Y(\omega), G(\omega)) = Y(\omega) D(\omega)$ implies $P(YD | \mathcal{G}) = 0$. This concludes the proof. ■

2.2 Discrete time, finite horizon

Let (Ω, \mathcal{F}, P) be a complete probability space, $\mathbf{F} = (\mathcal{F}_k : k = 0, \dots, n)$ a *filtration*, that is, each \mathcal{F}_k is a complete sub- σ -algebra of \mathcal{F} and $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, $k = 0, \dots, n-1$. Let $X = (X_k : k = 0, \dots, n)$ be an \mathbf{R}^d -valued stochastic process which is *adapted* to $(\mathcal{F}_k : k = 0, \dots, n)$, that is, X_k is \mathcal{F}_k -measurable, $k = 0, \dots, n$.

Recall that \bar{P} is called an *equivalent martingale measure* for X if $\|X_k\|$ is \bar{P} -integrable ($k = 0, 1, \dots, n$) and $\bar{P}(X_{k+1} | \mathcal{F}_k) = X_k$ a.s. ($k = 0, 1, \dots, n-1$). A consequence of Theorem 2.6 is a necessary and sufficient condition for the existence of an equivalent martingale measure for X .

2.9 Theorem. The following two conditions are equivalent.

(2.5) For $k = 1, \dots, n$, for all \mathcal{F}_{k-1} -measurable random variables Z ,

$$Z \cdot (X_k - X_{k-1}) \geq 0 \text{ } P\text{-a.s.} \Rightarrow Z \cdot (X_k - X_{k-1}) = 0 \text{ } P\text{-a.s.}$$

(2.6) There exists an equivalent martingale measure \tilde{P} for X .

Proof. The implication (2.6) \Rightarrow (2.5) is an immediate consequence of Theorem 2.6. Before proving the converse implication, recall that standard properties of conditional expectation for real random variables, such as $P(D_1 D_2 | \mathcal{G}) = D_1 P(D_2 | \mathcal{G})$ when D_1 is \mathcal{G} -measurable, or $P(P(D | \mathcal{H}) | \mathcal{G}) = P(D | \mathcal{G})$ when $\mathcal{H} \subset \mathcal{G}$, are valid when $P(D_1 D_2)$, $P(D_2)$ and $P(D)$ exist, but are not necessarily finite (see Ash (1972, Theorems 6.5.12 and 6.5.10)); in particular, they always hold when D_1 , D_2 and D are non-negative.

Now suppose (2.5) holds, and set $\mathcal{F}_{n+1} = \mathcal{F}_n$, $X_{n+1} = X_n$, $D_{n+1} = 1$, $Y_{n+1} = 0$. Fix $k \leq n$ and suppose by backwards induction that we have defined D_{k+1}, \dots, D_{n+1} and Y_{k+1}, \dots, Y_{n+1} in such a way that for $k+1 \leq l \leq n+1$

(2.7) D_l is \mathcal{F}_l -measurable and $0 < D_l \leq 1$ P -a.s.,

and for $k+1 \leq l \leq n$

(2.8) $Y_l = (X_l - X_{l-1}) P(D_{l+1} \dots D_{n+1} | \mathcal{F}_l)$ P -a.s.,

and

(2.9) $P(\|Y_l\| | D_l) < +\infty$ and $P(Y_l | \mathcal{F}_{l-1}) = 0$.

Using Theorem 2.6, we then construct an \mathcal{F}_k -measurable random variable D_k with $0 < D_k \leq 1$ P -a.s., in such a way that if Y_k is defined by (2.8) with $l = k$, then (2.9) holds with $l = k$. Indeed, since $0 < D_l \leq 1$ P -a.s., we have $0 < P(D_{k+1} \dots D_{n+1} | \mathcal{F}_k) \leq 1$ P -a.s., and so by (2.5), Y_k satisfies (2.3). Since Y_k is \mathcal{F}_k -measurable, Theorem 2.6 implies the existence of an \mathcal{F}_k -measurable random variable D_k such that (2.9) holds with $l = k$.

By backwards induction, we thus have defined random variables D_1, \dots, D_{n+1} and Y_1, \dots, Y_{n+1} such that (2.8) and (2.9) hold for $1 \leq l \leq n+1$. Finally, we set $D_0 = 1/(1 + \|X_0\|)$, and $D = D_0 \dots D_{n+1}$. Observe that $0 < D \leq 1$ P -a.s. Let \tilde{P} be the probability measure which is equivalent to P and whose Radon-Nikodym derivative $d\tilde{P}/dP$ equals D . We shall show that \tilde{P} is a martingale measure for X .

Indeed,

$$\begin{aligned}\tilde{P}(\|X_0\|) &= P(\|X_0\| D) = P(\|X_0\| D_0 \dots D_{n+1}) \\ &\leq P(\|X_0\| D_0) \leq 1,\end{aligned}$$

and

$$\begin{aligned}\tilde{P}(\|X_l - X_{l-1}\|) &= P(\|X_l - X_{l-1}\| D) = P(D_0 \dots D_l \|X_l - X_{l-1}\| D_{l+1} \dots D_{n+1}) \\ &= P(D_0 \dots D_l \|Y_l\|) \\ &\leq P(D_l \|Y_l\|) \\ &< +\infty\end{aligned}$$

by (2.9), $1 \leq l \leq n$. Thus $\|X_l\|$ is \tilde{P} -integrable, $0 \leq l \leq n$.

Finally, to check $\tilde{P}(X_l | \mathcal{F}_{l-1}) = X_{l-1}$, it is sufficient by (2.1) to show that $P((X_l - X_{l-1}) D | \mathcal{F}_{l-1}) = 0$.

Now

$$\begin{aligned}P((X_l - X_{l-1}) D | \mathcal{F}_{l-1}) &= D_0 \dots D_{l-1} P((X_l - X_{l-1}) D_l \dots D_{n+1} | \mathcal{F}_{l-1}) \\ &= D_0 \dots D_{l-1} P((X_l - X_{l-1}) D_l P(D_{l+1} \dots D_{n+1} | \mathcal{F}_l) | \mathcal{F}_{l-1}) \\ &= D_0 \dots D_{l-1} P(D_l Y_l | \mathcal{F}_{l-1}) \\ &= 0\end{aligned}$$

by (2.9), $1 \leq l \leq n$. ■

2.10. Remark. In the case of a discrete-time stochastic process $X = (X_k : k \in \mathbb{N})$ with infinite time horizon, condition (2.5) of Theorem 2.9 does not necessarily guarantee the existence of an equivalent martingale measure \tilde{P} for X . This is illustrated by the following simple example. Suppose $X_k = Y_1 + \dots + Y_k$ where for some $0 < p < 1$, $p \neq 1/2$, $(Y_k : k \in \mathbb{N})$ is a sequence of i.i.d. random variables with $P\{Y_k = 1\} = p$, $P\{Y_k = -1\} = 1 - p$. Set $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$, $k \in \mathbb{N}$. It is then easy to see that if \tilde{P} is a martingale measure for X which is equivalent to P , then $\tilde{P}\{Y_k = 1\} = \tilde{P}\{Y_k = -1\} = 1/2$. But then the strong law of large numbers shows that \tilde{P} and P are not equivalent on $\mathcal{F} = \sigma(\bigcup_{k \geq 0} \mathcal{F}_k)$ (in

fact, they are mutually singular).

For the purpose of being complete, we finish this section by summarizing the results obtained in Willinger and Taqqu (1988) concerning the uniqueness problem of an equivalent martingale measure \bar{P} for X . In contrast to the existence problem, Willinger and Taqqu (1988) show that uniqueness can be dealt with using elementary probability techniques for it requires an essentially finite and rather restrictive probabilistic setting. More precisely, they proved the following

2.11. Corollary. The following two conditions are equivalent.

(2.10) For $k = 0, 1, \dots, n - 1$, there exists a finite minimal partition \mathcal{P}_k of Ω with $\mathcal{F}_k = \sigma(\mathcal{P}_k)$ (up to P -null sets) and such that for all $A \in \mathcal{P}_k$, $\dim(\text{span}(\{X_{k+1}(\omega) - X_k(\omega) : \omega \in A\})) = \text{cardinality}(A' \in \mathcal{P}_{k+1} : A' \subseteq A) - 1$ where without loss of generality, we assume $P\{A\} > 0$ for all $A \in \mathcal{P}_k$.

(2.11) There exists a unique equivalent martingale measure \bar{P} for X .

Whereas Theorem 2.9 imposes no restrictions on the underlying filtration F and is exclusively concerned with the proper “geometry” of X , Corollary 2.11 explicitly depicts the fundamental role of the fine structure of F . In fact, Corollary 2.11 not only implies that if \bar{P} is unique then F is necessarily minimal (i.e., $F = F^X = (\mathcal{F}_k^X : k = 0, 1, \dots, n)$ with $\mathcal{F}_k^X = \sigma(X_0, X_1, \dots, X_k)$, $0 \leq k \leq n$) but it also imposes stringent constraints of the form (2.10) on the relationship between X and F . It is this lack of a tight control on X and F that requires the use of measurable selections in the general case (see Theorem 2.6).

3. The analysis of finite-period stochastic securities markets

In this section we illustrate the main results of Section 2 in the context of a stochastic model for the buying and selling of securities in discrete and finite time. The model was introduced by Harrison and Pliska (1981) and further discussed in the setting of finite probability spaces by Taqqu and Willinger (1987). Here we show that condition (2.5) of Theorem 2.9 arises naturally from and is equivalent to the economically meaningful assumption of “no arbitrage”. Moreover, uniqueness of an equivalent

martingale measure for the underlying securities price process is related to the so-called “completeness”-property of the market which enables one to uniquely price any financial instrument in the market.

3.1 The stochastic model

For a fixed time horizon $T < \infty$ (terminal date of all economic activities), consider an \mathbf{R}^{d+1} -valued ($1 \leq d < \infty$) stochastic process $S = (S_t : t = 0, 1, \dots, T)$ defined on some complete probability space (Ω, \mathcal{F}, P) . Each component-process $S^k = (S_t^k : t = 0, 1, \dots, T)$, $0 \leq k \leq d$, is assumed to be strictly positive so that $S_t^k(\omega)$ can be interpreted as the price of security k at time t if $\omega \in \Omega$ represents the true state of nature. The 0^{th} security is called the *bond* and without loss of generality (see Harrison and Kreps (1979)), we set $S_t^0 = 1$ for all t ; that is, we assume that the stock prices have been discounted by the price of the riskless bond. S is also assumed to be adapted to a given filtration $\mathbf{F} = \{\mathcal{F}_t : t = 0, 1, \dots, T\}$ and for convenience, we take $\mathcal{F}_T = \mathcal{F}$. \mathbf{F} describes how information is revealed to the investors when securities are traded over time; starting with an initial knowledge \mathcal{F}_0 about the true state of nature, investors learn without forgetting ($\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t = 0, 1, \dots, T - 1$) until they have complete information by time T ($\mathcal{F}_T = \mathcal{F}$).

The buying and selling of securities over time must be done according to certain trading strategies. A *trading strategy* is an \mathbf{F} -predictable, \mathbf{R}^{d+1} -valued stochastic process $\phi = (\phi_t : t = 1, 2, \dots, T)$ with components $\phi^0, \phi^1, \dots, \phi^d$. $\phi_t^k(\omega)$ represents the number of shares of stock k held by an investor between times $t - 1$ and t , namely during the time period $[t - 1, t)$ if $\omega \in \Omega$ occurs. Thus, the vector ϕ_t denotes the investor's *portfolio at time t* and the components of ϕ_t may assume positive as well as negative values. When investors readjust their portfolio ϕ_t at time t , that is, buy and sell securities so as to form a new portfolio ϕ_{t+1} , they must do so without any knowledge of the future since ϕ is required to be \mathbf{F} -predictable (i.e., $\phi_{t+1} \in \mathcal{F}_t$). The *value-process* $V(\phi) = (V_t(\phi) : t = 0, 1, \dots, T)$ associated with a trading strategy ϕ is defined by

$$V_t(\phi) = \begin{cases} \phi_1 \cdot S_0 = \sum_{k=0}^d \phi_1^k S_0^k & \text{if } t = 0 \\ \phi_t \cdot S_t = \sum_{k=0}^d \phi_t^k S_t^k & \text{otherwise,} \end{cases} \quad P - a.s.$$

Thus, $V_t(\phi)$ represents the value of the portfolio ϕ_t at time t and before any changes are made at that time. A trading strategy ϕ is called *self-financing* if all changes in the value of ϕ_t are due to net gains realized on investments; that is, if

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t \quad P - a.s., \quad t = 1, \dots, T - 1.$$

We denote by Φ the set of all self-financing trading strategies.

Finally, we state the following assumptions commonly found in the economics literature: (i) there are no transaction costs, (ii) all securities are perfectly divisible, (iii) the securities do not pay dividends in $[0, T]$, and (iv) short sales of all securities are allowed without any restrictions. Subsequently, the stochastic model corresponding to the stochastic base (Ω, \mathcal{F}, P) , the price process S , and the set Φ of allowable trading strategies, and satisfying conditions (i) - (iv) will be denoted by (T, \mathcal{F}, S) and called a *(finite-period, frictionless) securities market*, where $T = \{0, 1, \dots, T\}$ denotes the set of all trading dates.

3.1. Remark. We do not impose any kind of wealth constraint as, for example, in Harrison and Pliska (1981) or Back and Pliska (1987) but allow for unbounded short sales. In discrete time, restrictions on short sales have little effect on subsequent results and are not needed from a mathematical point of view (see also the comment in Back and Pliska (1987, p.3)).

3.2 The “no-arbitrage-assumption” and the martingale-property

An arbitrage opportunity (“free lunch”) represents a riskless plan for making profit without any investment. Prohibiting arbitrage opportunities is, therefore, economically reasonable and is necessary for any kind of economic equilibrium to exist. More formally, we have

3.2. Definition. An *arbitrage opportunity* is a self-financing trading strategy $\phi \in \Phi$ such that $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$ with probability one, and $V_T(\phi) > 0$ with positive probability. The market model (T, F, S) is said to contain *no arbitrage opportunities* if for all $\phi \in \Phi$ with $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$ $P - a.s.$, we have $V_T(\phi) = 0$ $P - a.s.$

Although an arbitrage opportunity as described above is defined “globally” (that is, it involves the trading dates 0 and T only), “no arbitrage” also holds “locally”, namely at any trading date $t = 1, 2, \dots, T$, as we will see below. In addition to illustrating this pathwise nature of the “no-arbitrage”-property, Theorem 3.3 below relates “no arbitrage” to condition (2.5) of Theorem 2.9 and hence to the martingale property of the price process S under a new equivalent probability measure \bar{P} . Let \mathbf{P} denote the set of all equivalent martingale measures for S and let $\bar{S} = (\bar{S}_t : t = 0, 1, \dots, T)$ be the \mathbf{R}^d -valued, F -adapted process obtained from S by deleting the 0^{th} component-process $S_t^0 \equiv 1$ (i.e., $S = (1, \bar{S})$).

3.3 Theorem. The following three conditions are equivalent.

(3.1) The market model (T, F, S) contains no arbitrage opportunities.

(3.2) For all $t \in \{1, 2, \dots, T\}$ and all \mathbf{R}^d -valued \mathcal{F}_{t-1} -measurable random variables α ,

$$\alpha \cdot (\bar{S}_t - \bar{S}_{t-1}) \geq 0 \quad P - a.s. \Rightarrow \alpha \cdot (\bar{S}_t - \bar{S}_{t-1}) = 0 \quad P - a.s.$$

(3.3) $\mathbf{P} \neq \emptyset$; that is, there exists an equivalent martingale measure \bar{P} for S .

Proof.

1) The proof of (3.1) \Rightarrow (3.2) is similar to that of Taqqu and Willinger (1987, Lemma 3.2) except that the probability space is no longer finite. Assume that there exists $t \in \{0, 1, \dots, T-1\}$ and $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^d) \in \mathcal{F}_t$ such that $\alpha \cdot (\bar{S}_{t+1} - \bar{S}_t) \geq 0$ $P - a.s.$ and $\alpha \cdot (\bar{S}_{t+1} - \bar{S}_t) > 0$ with positive probability. Set $W = \{\omega \in \Omega : P\{\alpha \cdot (\bar{S}_{t+1} - \bar{S}_t) > 0 \mid \mathcal{F}_t\}(\omega) > 0\}$ and note that, by assumption, $P(W) > 0$. We will construct a trading strategy $\phi \in \Phi$ with $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$ $P - a.s.$, and such

that $V_T(\phi) > 0$ with positive probability; that is, ϕ defines an arbitrage opportunity, contradicting the assumption that the market model (T, F, S) is "arbitrage-free".

In order to construct ϕ with the desired properties, define ϕ_s at every point in time and for each $\omega \in \Omega$ as follows:

for $s \leq t : \phi_s(\omega) = 0$ for all $\omega \in \Omega$,

$$\text{for } s = t + 1 : \text{ on } W, \text{ set } \phi_{t+1}^k(\omega) = \begin{cases} \alpha^k(\omega) & \text{if } k \in \{1, 2, \dots, d\}, \\ -\sum_{k=1}^d \alpha^k(\omega) S_t^k(\omega) & \text{if } k = 0, \end{cases}$$

and on W^c , set $\phi_{t+1}(\omega) = 0$,

$$\text{for } t + 1 < s < T : \phi_s^k(\omega) = \begin{cases} V_{t+1}(\phi)(\omega) & \text{if } k = 0 \text{ and } \omega \in W \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ϕ is F -predictable. To see that ϕ is self-financing, we check the relation $\phi_s \cdot S_s = \phi_{s+1} \cdot S_s$ which clearly holds for $s < t$ and $s > t$. For $s = t$, we have $\phi_t(\omega) \cdot S_t(\omega) = 0$ for all $\omega \in \Omega$; for $\omega \in W$,

$$\phi_{t+1}(\omega) \cdot S_t(\omega) = -\left(\sum_{k=1}^d \alpha^k S_t^k\right)(\omega) + \left(\sum_{k=1}^d \alpha^k(\omega) S_t^k(\omega)\right) = 0.$$

Since $\phi_{t+1} \cdot S_t = 0$ on W^c , $\phi_{t+1} \cdot S_t = \phi_t \cdot S_t$ holds true for all $\omega \in \Omega$.

Next observe that $V_0(\phi) = 0$ P -a.s. and $V_T(\phi) \geq 0$ P -a.s. In fact, for all $s > t + 1$,

$$V_s(\phi)(\omega) = \begin{cases} V_{t+1}(\phi)(\omega) = \alpha(\omega) \cdot (\bar{S}_{t+1}(\omega) - \bar{S}_t(\omega)) \geq 0 & \text{if } \omega \in W, \\ 0 & \text{otherwise,} \end{cases}$$

and hence,

$$P\{V_T(\phi) \geq 0\} = P\{\alpha \cdot (\bar{S}_{t+1} - \bar{S}_t) \geq 0\} = 1.$$

Moreover, by the definition of W ,

$$\begin{aligned} P \{ V_T(\phi) > 0 \} &= P \{ \{ V_T(\phi) > 0 \} \cap W \} \\ &= P(1_W P \{ \alpha \cdot (\tilde{S}_{t+1} - \tilde{S}_t) > 0 \mid \mathcal{F}_t \}) > 0, \end{aligned}$$

which shows that ϕ is an arbitrage opportunity.

2) The equivalence (3.2) \Leftrightarrow (3.3) holds by Theorem 2.9.

3) Finally, in order to prove (3.3) \Rightarrow (3.1), let $\tilde{P} \in \mathbf{P}$ and let $\phi \in \Phi$ be such that $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$ $P - a.s.$ Then $V_T(\phi) = 0$ $P - a.s.$ because repeated applications of (i) the properties of conditional expectations mentioned at the beginning of the proof of Theorem 2.9 (recall that S is positive), (ii) the martingale property of S under \tilde{P} , and (iii) the self-financing property of $\phi \in \Phi$ enable us to write

$$\tilde{P}(V_T(\phi)) = \tilde{P}(V_0(\phi)) = 0. \quad \blacksquare$$

3.4. Remarks. 1) To our knowledge, Theorem 3.3 is the first of its kind that proves the existence of an equivalent martingale measure for a given finite-period, \mathbf{R}^d -valued process from such primitive economic considerations as “no arbitrage”. In particular, we make no assumptions concerning the integrability of S under P . On the one hand, such a requirement seems somewhat unnatural since (i) it is, in general, not preserved under an equivalent change of measure, and (ii) for the formulation of an arbitrage opportunity (see Definition 3.2), integrability of S under P is irrelevant. This absence of integrability assumptions on S is in contrast to existing results (see, for example, Harrison and Kreps (1979) who require square-integrability of S , or Back and Pliska (1987) who assume P -integrability of the one-dimensional price process which turns out to be crucial for proving their one-dimensional version of our Theorem 3.3). On the other hand one can always assume the existence of an equivalent probability measure P' on (Ω, \mathcal{F}) with bounded Radon-Nikodym derivative dP'/dP such that S is integrable under P' (see Dellacherie and Meyer (1982, Thm. VII. 57)). Assuming integrability of S under P thus becomes a modeling issue and is not necessary from a mathematical point of view.

2) Theorem 3.3 not only provides a probabilistic characterization (condition (3.3)) but also a geometric characterization (condition (3.2)) of “no arbitrage”. Indeed, condition (3.2) states implicitly that along almost all sample paths of \tilde{S} (or S), the support of the conditional distribution of the increment $\tilde{S}_{t+1} - \tilde{S}_t$ given \mathcal{F}_t cannot be concentrated on only one “side” of any \mathcal{F}_t -measurable hyperplane in \mathbb{R}^d (or \mathbb{R}^{d+1}).

We conclude this section with a brief discussion concerning the question of uniqueness of \tilde{P} and its economic interpretation. To this end, let X denote a non-negative, \mathcal{F} -measurable random variable (*contingent claim*) and interpret X as representing a contract that pays $X(\omega)$ dollars if, at time T , $\omega \in \Omega$ denotes the true state of nature. We would like to know what prices at time zero are “reasonable” for X if the market model (T, F, S) contains no arbitrage opportunities. Clearly, if X is *attainable*; that is, there exists $\phi \in \Phi$ such that $X = V_T(\phi)$ $P - a.s.$, then “no arbitrage” implies that the (time zero) price $\pi(X)$ is given by $\pi(X) = V_0(\phi)$. But which claims are attainable? The market model (T, F, S) is called *complete* if all contingent claims are attainable. The following characterization of the economically desirable completeness-property of the market model (T, F, S) is an immediate consequence of Corollary 2.11 and the finite securities market analysis in Taqqu and Willinger (1987, Section 4).

3.5. Corollary. If the market model contains no arbitrage opportunities (or, equivalently, $P \neq \emptyset$) then the following are equivalent.

(3.4) The market model (T, F, S) is complete.

(3.5) For every $t = 0, 1, \dots, T - 1$, there exists a finite minimal partition \mathcal{P}_t of Ω with $\mathcal{F}_t = \sigma(\mathcal{P}_t)$ (up to P -null sets) such that for all $A \in \mathcal{P}_t$, $\dim(\text{span}(\{S_{t+1}(\omega) - S_t(\omega) : \omega \in A\})) = \text{cardinality}(A' \in \mathcal{P}_{t+1} : A' \subseteq A) - 1$ (where without loss of generality, we assume $P\{A\} > 0$ for all $A \in \mathcal{P}_t$, $0 \leq t < T$).

(3.6) $|P| = 1$, that is, there exists a unique equivalent martingale measure \tilde{P} for (F, S) .

For further discussions concerning complete markets, we refer the interested reader to Taqqu and Willinger (1987) where, among other things, completeness of (T, F, S) and uniqueness of $\tilde{P} \in \mathbf{P}$ are treated as a family of primal-dual pairs of linear programs.

Acknowledgment. Freddy Delbaen suggested to one of us the use of continuous densities in relation with Measurable Selection.

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