Intersection Local Times for Infinite Systems of Planar Brownian Motions and for the Brownian Density Process

By

Robert J. Adler[1] & Raisa Epstein[2]

Faculty of Industrial Engineering & Management
Technion - Israel Institute of Technology

and

Department of Statistics
University of California, Berkeley

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Department of Statistics
University of California
Berkeley, California

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Abstract

Let $X_t^1, X_t^2, \ldots,$ be a sequence of independent, planar Brownian motions starting at the points of a planar Poisson process of intensity λ. Let $\sigma^1, \sigma^2, \ldots,$ be independent, ± 1 random variables. Let $L_t(X^i, X^j)$ be the intersection local time of X^i and X^j up to time t. We study the limit in distribution of $\lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j L_t(X^i, X^j)$ as $\lambda \to \infty$.

The resulting process is called the intersection local time for the Brownian density process, and its existence was established in a companion paper by Adler and Lewin (1988). The current paper concentrates on establishing the above limit theorem, and, as a bonus, obtains a Tanaka-like formula giving an evolution equation representation of the Brownian density's intersection local time.

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1. INTRODUCTION

Let Π^λ be a Poisson point process on $\Re^2$ of intensity λ, i.e. the number of points of Π^λ in a Borel set $A \subset \Re^2$ is a Poisson random variable with parameter $\lambda|A$, and the numbers in disjoint sets are independent. Since the probability that any two points of Π^λ lie exactly the same distance from the origin is zero we can order them by magnitude, and shall denote them by $X_0^1, X_0^2, \ldots$.

Let $X_t^1, X_t^2, \ldots,\quad t \geq 0$ be a sequence of independent, planar Brownian motions, with initial values given by $X_0^1, X_0^2, \ldots$, and let $\sigma^1, \sigma^2, \ldots$ denote a sequence of independent Rademacher random variables. $(P\{\sigma^i = +1\} = P\{\sigma^i = -1\} = \frac{1}{2})$. The two sequences and Π^λ are assumed independent of one another except for the fact that Π^λ determines the initial values of the X^i.

For $\phi \in S_2 = S(\Re^2)$, the Schwartz space of infinitely differentiable functions on $\Re^2$ decreasing rapidly at infinity, let η_t^λ be the S_2' valued random process defined by

$$\eta_t^\lambda(\phi) = \lambda^{-1/2} \sum_{i=1}^{\infty} \sigma^i \phi(X_t^i). \tag{1.1}$$

(If one thinks of the random signs as signed particles moving throughout space according to independent Brownian motions, then $\eta_t^\lambda(1_A)$ would describe the average net charge in the set A at time t, if only it were true that indicator functions belonged to S_2.)

The $\lambda \to \infty$ behaviour of η_t^λ has been a subject of some considerable interest, and the most complete results can be found in Walsh (1986), where it is shown that η_t^λ converges in distribution in the Skorohod space $D([0,1], S_2'))$ to the solution of the stochastic partial differential equation (SPDE)

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{1}{2}\Delta\eta + \nabla \cdot W \\ \eta_0 &= \tilde{\Pi}, \end{aligned} \tag{1.2}$$

where $\tilde{\Pi}$, a Gaussian white noise on $\Re^2$, is the weak limit in S_2' of $\lambda^{-1/2} \sum_i \sigma^i \phi(X_0^i)$, and W is an $\Re^2$ valued Gaussian white noise on $\Re^2 \times \Re_+$. Equation (1.2) should be understood in the weak form developed in Walsh (1986): i.e. for every $\phi \in S_2$,

$$\eta_t(\phi) = \frac{1}{2}\int_0^t \eta_s(\Delta\phi)ds + \int_0^t \int_{\Re^2} \langle \nabla\phi(x), W(dx, ds) \rangle.$$

The solution η_t of (1.2) is called the *Brownian density process*. It can also be defined as a simple S_2' valued process via its covariance function (c.f. Definition 2.1 in Adler and Lewin (1988)), but the above formulation will be more natural for us.

Another process of considerable interest, also defined in the same setup, is the following:

$$(1.3) \qquad \mu_t^\lambda(\phi) = \lambda^{-1/2}\sum_{i=1}^{\infty}\sigma^i\int_0^t \phi(X_s^i)\,ds, \\ = \int_0^t \eta_s^\lambda(\phi)\,ds.$$

for $\phi \in S_2$. By taking $\phi = 1_A$, where $A \subset \Re^2$, (and ignoring the fact that 1_A is not an element of S_2) we see that $\mu_t^\lambda(1_A)$ describes the average "net charge" of the X^i in A up until time t, and so we shall refer to μ_t^λ, and its $\lambda \to \infty$ limit as ***Brownian occupation processes.***

Since integration is a continuous functional in $D([0,1], S_2)$ the limiting distribution of $\mu_t^\lambda(\phi)$, as a process in t, is clearly that of $\int_0^t \eta_s(\phi)\,ds$. It then follows (as, in fact, it does from a simple central limit theorem) that the limiting marginal distribution of μ_t^λ is that of a centered S_2' valued Gaussian random variable, which we shall denote by μ_t, whose covariance kernel, which will not actually concern us in this paper, has a natural representation in terms of the transition density of a planar Brownian motion, viz.

$$(1.4) \qquad p_t(x,y) = p_t(x-y) = \frac{1}{2\pi t}e^{-\|x-y\|^2/2t},$$

where, hopefully, our use if the same function p_t to denote both a function on $\Re^2 \times \Re^2$ and $\Re^2$ will not lead to too much confusion. Limit theorems of this form (modulo some minor technical differences - see the comments at the end of this section) were discussed in detail in Adler and Epstein (1987), where we also discussed more complicated, and more interesting, limit theorems for sums of additive functionals of quite general symmetric Markov processes. A particular, and important, special case was a limit theorem for sums of intersection local times of planar Brownian motions. The intersection local time of two such processes X and Y is defined formally as the continuous S_2' valued process

$$(1.5) \qquad L_t(\phi; X, Y) = \int_0^t\int_0^u \phi(Y_v)\delta(X_u - Y_v)\,du\,dv,$$

where δ is the Dirac delta function, or, rigorously, as the $\mathcal{L}^2(P)$ limit, as $\epsilon \to 0$, of

$$(1.6) \qquad L_t^\epsilon(\phi; X, Y) = \int_0^t\int_0^u \phi(Y_v)\,e^{-\epsilon/2}\,p_\epsilon(X_u - Y_v)\,du\,dv,$$

where p_ϵ was defined above. The limit theorem studied, for fixed t, sums of the form

$$(1.7) \qquad \Psi_t^\lambda(\phi) = \lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j L_t(\phi; X^i, X^j),$$

which were shown to converge in distribution, as $\lambda \to \infty$, to certain S_2' valued random variables which could be represented as double Wiener-Ito integrals of the Gaussian process μ_t which appeared as the limit of μ_t^λ. Denote the $\lambda \to \infty$ limit of Ψ_t^λ as Ψ_t.

Our aim in the current paper is to study the temporal development of Ψ_t, as a process in t. In particular, we shall derive a result somewhat akin to Tanaka's formula for the temporal development of the local time of a single Brownian motion, so that we shall be able to write Ψ_t in terms of an evolution equation driven by Gaussian white noises.

The result is interesting from two points of view. Firstly, the process Ψ_t appears naturally in a model of interacting, signed particles as the limit of Ψ_t^λ, (c.f. Adler (1989)) and the evolution equation formulation adds insight into that model. Since the infinite particle limit in that model is a Euclidean field theory, the added insight extends beyond the specific prescriptions of that paper. Secondly, rather than defining Ψ_t as the limit of the Ψ_t^λ, it can be defined directly as a sort of self-intersection local time for the S_2' valued process η_t via the formal relationship

$$\Psi_t(\phi) = \int_0^t \int_0^u \int_{\Re^2} \int_{\Re^2} \phi(y)\delta(x-y)(\eta_u(dx)\times\eta_v(dy))\,du dv, \tag{1.8}$$

where a certain renormalisation is required to keep (1.8) finite.

(A proper formulation of Ψ_t is given in the following section. A detailed and careful proof of its existence is given in the companion paper by Adler and Lewin (1988), which is meant to be read concurrently with the present paper.)

Intersection local times of this form, for measure valued process, have been the subject of intense recent interest, commencing with the work of Perkins (1986) and followed by Dynkin (1987) and others. The emphasis in Perkins' work was on the dimensionality of the support of the intersection local time, while Dynkin was primarily concerned with integral representations of the intersection local time itself. We shall obtain a representation for Ψ_t as a *process* in t, as a multiple stochastic integral over $\Re^2 \times \Re_+ \times \Re_+$. This representation is, perhaps, the deepest result of this paper.

The most interesting result of the paper, however, is the fact that the convergence, as $\lambda \to \infty$, of Ψ_t^λ to Ψ_t justifies the consideration of the latter as a candidate for the intersection local time of the Brownian density process. Up to this point of time, expressions such as (1.8) have been taken at face value as representing a generalisation of the intersection local time concept to distribution or measure valued processes, with no real justification other than "things seemed to work". We shall discuss this point in more detail in the following section.

In the following section we shall present all our major results, with some discussion but without proofs. These follow in Section 3. The main technical tool used there is the weak convergence theory developed in Walsh (1986), and we are grateful to John Walsh for long ago providing us with a prepublication copy of his excellent set of notes.

The results we present here are restricted to the two dimensional case; i.e. to planar Brownian motions. It is not too hard to see that most of our arguments also extend to three dimensions, although there are some non-trivial technical problems to overcome on the way. The important fact to note however is that although some details change, the same intuition developed by the results of this paper for the intersection local time of the planar Brownian density process carries over qualitatively to the three dimensional one (whose existence was established in Adler and Lewin (1988)) as well.

We close the Introduction with two technical notes, explaining why the processes considered here are directly comparable neither with those treated in Adler and Epstein (1987) nor with the measure valued processes referred to above. The reader not interested in this can skip immediately to the following section without loss of continuity.

Technical asides In Adler and Epstein (1987) and Adler (1989) our basic processes were general Markov processes with symmetric transition densities. Here we treat only Brownian motion, and so are, in essence, obtaining much more detail for a much smaller class of processes. In Adler and Epstein (1987) our processes were not started according to the points of a Poisson process, but either according to a (non-probability but σ-finite) uniform measure on $\Re^2$, or according to a rather awkward probabilistic way of spreading points out through a partition of $\Re^2$, (also used in Adler (1989)). We could have saved ourselves a substantial amount of trouble had we used the Poissonization trick, for no lack of realism in the model. Thus we do so here.

Furthermore, objects like $L_t(\phi; X, Y)$ were replaced by intersection local times of the form

$$(1.9) \qquad L^*(\phi; X, Y) = \int_0^\infty \int_0^\infty e^{-u} e^{-v} \phi(Y_v) \delta(X_u - Y_v)\, du dv,$$

so that the corresponding limit theorems for objects like Ψ_t^λ are somewhat different. Because of the exponential weighting in (1.9), and the fact that the integration on u and v is over the entire real line, the results of this and the previous paper are not strictly comparable. Both, however, show similar phenomena. We emphasise again, however, that the results of the current paper, in so far as planar Brownian motion are concerned, give substantially more detailed information.

Finally, it is worthwhile to note that our processes and those of Perkins and Dynkin mentioned above are not strictly comparable, and, despite the fact that they are distributions rather than measures, are actually somewhat simpler to work with. (In the Perkins/Dynkin formulation, each one of our Brownian motions must be replaced with a branching Brownian motion with a branching rate that goes to infinity as $\lambda \to \infty$.) Nevertheless, we thought we might start with the simpler case, both because of our original interest in it resulting from the interacting particle results described above and because, as will become clear in future sections, even this case is not all that easy.

2. MAIN RESULTS

We retain the notation and general setting of the introduction, so that $X_t^1, X_t^2, \ldots$ is a sequence of planar Brownian motions, started at the points of a homogeneous Poisson process Π^λ of intensity λ. Let X and Y be two generic processes from this collection.

Our first task is to define the *intersection local time* between X and Y, and to represent it via a Tanaka-like formula. To this end, let g be the Green's function of X, given by either

$$(2.1) \qquad g(x) = \int_0^\infty e^{-t/2} p_t(x)\, dt, \quad \text{or} \quad g(x,y) = \int_0^\infty e^{-t/2} p_t(x-y)\, dt,$$

where p_t is the Brownian transition density (1.4). Then, as noted in the Introduction, the intersection local time, $L_t(\phi; X, Y)$ between X and Y, up to time t, and weighted by the test function $\phi \in S_2$, is defined as the $\mathcal{L}^2$ limit, as $\epsilon \to 0$, of

$$(2.2) \qquad L_t^\epsilon(\phi; X, Y) = \int_0^t \int_0^u e^{-\epsilon/2} p_\epsilon(X_u - Y_v)\phi(Y_v)\, du dv.$$

The existence of this limit, which gives a precise meaning to the formal expression (1.5), follows from results in Dynkin (1981). A Tanaka-like formula also holds for L_t, and is given in the following result.

THEOREM 2.1. *The following equality holds for all $\phi \in S_2$:*

$$\begin{aligned} L_t(\phi; X, Y) = \; & \frac{1}{2}\int_0^t \int_0^u g(X_u - Y_v)\phi(Y_v)\, dudv + \int_0^t g(X_u - Y_u)\phi(Y_u)\, du \\ & - \int_0^t g(X_t - Y_u)\phi(Y_u)\, du + \int_0^t \int_0^u \nabla g(X_u - Y_v)\phi(Y_v)\, dX_u dv. \end{aligned}$$

We shall indicate how to establish this result in the following section.

IMPORTANT REMARK ON NOTATION: Note that since $X_t \in \Re^2$ we should really write the last integrand above as the inner product $\langle \phi(Y_v)(\nabla g(X_u - Y_v)), dX_u \rangle\, dv$, using the second option in (2.1) to define g. To keep our formulae reasonably neat, however, we shall use the more ambiguous formulation above throughout the paper, and the reader will do well to keep this in mind later on.

The next result incorporates the central limit theorem for η_t^λ described in the Introduction. To formulate it, we need to define two (orthogonal) martingale (signed) measures. The first is defined for each $\lambda > 0$ as

$$(2.3) \qquad W^\lambda(A,t) = \lambda^{-1/2}\sum_{i=1}^{\infty}\sigma^i\int_0^t 1_A(X_u^i)\,dX_u^i,$$

where $A \subset \Re^2$ and from now on we restrict t to the interval $[0,1]$. The σ^i are, of course, the random Rademacher signs of the Introduction. The second measure is the $\Re^2$-valued Gaussian white noise W on $\Re^2 \times \Re_+$, defined by the requirement that the two components $W_i(A,t)$, $i = 1,2$ of $W(A,t)$ be independent, zero mean Gaussian random variables, and for all $A, B \subset \Re^2$ and $s<t, u<v$

$$(2.4) \quad E\{[W_i(A,t) - W_i(A,s)]\cdot[W_j(B,v) - W_j(B,u)]\} = \delta_{ij}\mid A\cap B\mid\cdot\mid[s,t]\cap[u,v]\mid,$$

where $|\cdot|$ is two dimensional Lebesgue measure.

The following result is a consequence of Proposition 8.16 of Walsh (1986). We associate with the Poisson point process Π^λ giving the initial points of the Brownian motions a signed version defined, in distribution form, by

$$\tilde{\Pi}^\lambda(\phi) = \sum_i \sigma^i\phi(X_0^i),$$

so that $\lambda^{-1/2}\tilde{\Pi}^\lambda(\phi) = \eta_0^\lambda(\phi)$. (Recall that the points of Π^λ are the X_0^i.) Weak convergence is denoted by $\Rightarrow$.

THEOREM 2.2. *As* $\lambda \to \infty$

$$(\lambda^{-1/2}\tilde{\Pi}^\lambda, W^\lambda, \eta^\lambda) \Rightarrow (\tilde{\Pi}, W, \eta),$$

where $\tilde{\Pi}$ *is Gaussian white noise on* $\Re^2$, W *is defined above,* η *is defined as the solution of the SPDE (1.2) with initial condition* $\eta_0 = \tilde{\Pi}$, *and the weak convergence is on the Skorohod space* $D([0,1], S_2' \times S_2' \times S_2')$.

The next step is to set up a central limit like result for sums of intersection local times. As noted in the introduction, results of this kind were established previously in Adler and Epstein (1987), but in the formulation of that paper there was an automatic integration over the t parameter (The integration was over $t \in [0,\infty)$, and finiteness of this integral was assured by an exponential damping of the intersection local time not applied in (2.2).) Consequently, the next result contains far more structure (for the Brownian motion case) than was achieved there.

Let Ψ_t^λ be as defined at (1.7), and let Ψ_t be the S_2' valued stochastic process defined formally by

$$(2.5) \qquad \Psi_t(\phi) = \int_0^t du\int_0^u dv(\eta_u\otimes\eta_v)(\delta(x-y))\phi(x),$$

where $\eta_u \otimes \eta_v$ is the centered version of the product $\eta_u \times \eta_v$ of distributions, so that for $\phi \in S_4$

$$(\eta_u \otimes \eta_v)(\phi) = (\eta_u \times \eta_v)(\phi) - E\{(\eta_u \times \eta_v)(\phi)\}$$

and $\eta_u \times \eta_v(\phi)$ is defined by setting $(\eta_u \times \eta_v)(\phi) = \sum_{k=1}^{N} \eta_u(\phi_k)\eta_v(\psi_k)$ for all functions in S_4 of the form $\phi(x,y) = \sum_{k=1}^{N} \phi_k(x)\psi_k(y)$, $x, y \in \Re^2$, and extending by continuity to all $\phi \in S_4$. (See Adler and Lewin(1988) for details.)

The existence of Ψ_t as an $\mathcal{L}^2$ limit, along the lines of (2.2), was also established in Adler and Lewin (1988), where it was called the "intersection local time process" for the Brownian density process η_t. The only motivation provided there to justify the fact that it was indeed an appropriate candidate for an intersection local time was the fact that it "seemed the right thing to do at the time". In fact, the same rather weak justification is all that has ever been offered in all the previous studies of intersection local times for measure valued diffusions listed in the Introduction. The following result provides a somewhat more substantial justification, and is one of the two central results of this paper.

THEOREM 2.3. $\Psi^\lambda \Rightarrow \Psi$ *as* $\lambda \to \infty$ *on the Skorohod space* $D([0,1], S_2')$.

NOTE: One immediate consequence of Theorem 2.3 is the fact that Ψ takes values in a Skorohod space. While in Adler and Lewin (1988) we succeeded in establishing the existence of Ψ_t as an $\mathcal{L}^2$ limit, the extra cadlag property implicit in Theorem 2.3 did not come out of that proof.

The reason why this result provides the required justification is due to the fact that we know that L_t is precisely what we want to serve as the intersection local time of two Brownian motions, and so Ψ_t^λ, as a sum of such local times, is well understood, and has support exactly on the intersections of the paths of the individual Brownian motions. It seems reasonable, therefore, that the $\lambda \to \infty$ limit of Ψ_t^λ should be an appropriate candidate for the intersection local time of the Brownian density process. Since Theorem 2.3 gives us that this limit is the Ψ_t of (2.5), we have the required justification.

At first inspection, Theorem 2.3 should be a "straightforward" consequence of Theorem 2.2, via an appropriate version of the Continuous Mapping Theorem. Since we know by Theorem 2.2 that $\eta^\lambda \Rightarrow \eta$, that Ψ_t^λ is a functional defined on η_t^λ and that Ψ_t is a similar functional on η, Theorem 2.3 seems to almost be an immediate consequence of Theorem 2.2. The difficulty with this line of argument, however, is that functionals based on intersection local times, whether they be of Brownian motions or the Brownian density process, are generally not smooth enough to apply continuity arguments of this kind. (A related, but somewhat different problem, was studied by Dynkin (1988).)

As a consequence of this, the proof of Theorem 2.3 is unfortunately somewhat circuitous, and will rely on Theorem 2.5 below, which is somewhat more involved and somewhat less interesting. Fortunately, however, another consequence of Theorem 2.5 will be the next theorem, which gives a Tanaka-like representation of the intersection local time process Ψ_t. The stochastic evolution equation given there is essentially the main result of this paper.

We shall require, however, a little more notation, and one lemma, which will aid in formulating this result. This lemma gives an easy Tanaka-like representation for Ψ_t^λ, the average intersection local times of the component Brownian motions defined at (1.7).

For $\phi \in S_4$ set

$$A_{st}^\lambda(\phi) = \lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j \phi(X_s^i, X_t^j). \tag{2.6}$$

Note that the sum here does *not* include the diagonal $i = j$.

To help out in the following, if $\Phi \in S_d'$ is a distribution, and $\phi \in S_d$ a test function, we shall often write $\Phi(\phi(x))$ to denote $\Phi(\phi)$.

LEMMA 2.1. *For every $t \in [0,1]$, $\lambda > 0$, $\phi \in S_2$,*

$$\begin{aligned} \Psi_t^\lambda(\phi) &= \frac{1}{2} \int_0^t du \int_0^u dv\, A_{uv}^\lambda(g(x-y)\phi(y)) \\ &+ \int_0^t du\, A_{uu}^\lambda(g(x-y)\phi(y)) - \int_0^t du\, A_{tu}^\lambda(g(x-y)\phi(y)) \\ &+ \int_0^t \int_{\Re^2} \mu_u^\lambda(\nabla g(x-\cdot)\phi(\cdot))\, W^\lambda(dx, du) - R_t^\lambda(\phi), \end{aligned} \tag{2.7}$$

where

$$R_t^\lambda(\phi) := \lambda^{-1} \sum_i \int_0^t \int_0^u \nabla g(X_u^i - X_v^i)\phi(X_v^i)\, dv dX_u^i,$$

and $R_t^\lambda(\phi) \to_p 0$ as $\lambda \to \infty$, for all $t \in [0,1]$ and all $\phi \in S_2$.

To save on notation, we shall write A_{st} to denote the centered product $\eta_s \otimes \eta_t$. Equality in law is denoted by $\stackrel{\mathcal{L}}{=}$. Then Lemma 2.1 indicates that the following theorem, which is the second of our major results, should be true.

THEOREM 2.4. *The following equality holds for all $\phi \in S_2$:*

(2.8)

$$\begin{aligned} \Psi_t(\phi) \stackrel{\mathcal{L}}{=} &\frac{1}{2} \int_0^t du \int_0^u dv\, A_{uv}(g(x-y)\phi(y)) + \int_0^t du\, A_{uu}(g(x-y)\phi(y)) \\ &- \int_0^t du\, A_{tu}(g(x-y)\phi(y)) + \int_0^t \int_{\Re^2} \mu_u(\nabla g(x-\cdot)\phi(\cdot))\, W(dx, du) \end{aligned}$$

where the stochastic integral is of the type studied by Walsh (1986).

Once again, it rather looks as if any "easy" proof of Theorem 2.4 would be to apply the Continuous Mapping Theorem via Theorem 2.2 and Lemma 2.1. In particular, for the reader familiar with the weak convergence theorems of Walsh (1986), which we shall heavily rely on in the following section, it would seem that virtually all the work has already been done in that paper. The difficulty in following this direct route, however, lies in the fact that the functions g and ∇g appearing above are not always the best behaved (e.g. $g(x,x) \equiv \infty$, $\nabla g(x-y)\phi(y) \notin \mathcal{L}^2(\Re^2 \times \Re^2)$) and so substantial technical difficulties arise.

The key to proving both Theorems 2.3 and 2.4 is the following result, based on the random distributions

$$\Psi_t^{\lambda\epsilon}(\phi) \;:=\; \lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j\, L_t^\epsilon(\phi; X^i, X^j), \tag{2.9}$$

where L_t^ϵ was defined at (2.2), and

$$\Psi_t^\epsilon(\phi) \;:=\; \int_0^t\int_0^u (\eta_u\otimes\eta_v)(e^{-\epsilon/2}p_\epsilon(x-y)\phi(y))\,du dv. \tag{2.10}$$

In the terminolgy of Dynkin (1988), $\Psi_t^{\lambda\epsilon}$ and Ψ_t^ϵ provide "links" between Ψ_t^λ and Ψ_t.

THEOREM 2.5. $\Psi_t^{\lambda\epsilon} \Rightarrow \Psi_t^\epsilon$ as $\lambda \to \infty$, *for every* $\epsilon > 0$, *on the Skorohod space* $D([0,1], S_2')$.

Since it is (essentially) the main result of Adler and Lewin (1988) that $\Psi_t^\epsilon \to_{\mathcal{L}^2} \Psi_t$ as $\epsilon \to 0$, and it follows from the definition of $\Psi_t^{\lambda\epsilon}$ and the $\mathcal{L}^2$ convergence of L_t^ϵ to L_t that $\Psi_t^{\lambda\epsilon} \to_{\mathcal{L}^2} \Psi^\lambda$ as $\epsilon \to 0$ for every $\lambda > 0$, it is now easy to see how to prove Theorem 2.3 from Theorem 2.5. We shall give details in the next section.

In closing this section, however, we note that results similar to Theorems 2.3 and 2.4 hold also for the intersection local time of the Brownian density process on $\Re^3$ defined by Adler and Lewin (1988). The technical differences in this case, referred to briefly in the introduction, arise primarily from the fact that the corresponding summands in the remainder term $R_t^\lambda(\phi)$ of (2.7) do not exist in this case. Nevertheless, approximations to these, analagous to the summands of $R_t^{\lambda\epsilon}(\phi)$ of (3.4) in the following section do exist, and judicious handling of the $\lambda \to \infty$ and $\epsilon \to 0$ limits overcomes the difficulties. We shall not go into the details here, however.

3. PROOFS

We commence by trying to get a result like Theorem 2.4, but for the process $\Psi_t^{\lambda\epsilon}$, i.e. an evolution equation representation for $\Psi_t^{\lambda\epsilon}$. Recall that $\Psi_t^{\lambda\epsilon}$ was defined at (2.8), as a sum of "approximate" intersection local times. For $\epsilon > 0$ set

$$K^\epsilon(x) \;=\; \int_\epsilon^\infty e^{-t/2} p_t(x)\, dt. \tag{3.1}$$

Note that $K^\epsilon \to_{\mathcal{L}^2} g$ as $\epsilon \to 0$, where g is the Green's function (2.1). Unlike g however, K^ϵ is well behaved in that it is C^∞, everywhere finite and $\nabla K^\epsilon \in \mathcal{L}^2(\Re^2)$. Furthermore,

$$\frac{1}{2}(-\Delta - 1)\, K^\epsilon(x) \;=\; e^{-\epsilon/2}\, p_\epsilon(x), \tag{3.2}$$

as is easily checked by direct differentiation. This leads us to

LEMMA 3.1. *For all* $\lambda, \epsilon > 0$, $\;t \in [0,1]$, *and* $\phi \in S_2$

$$\begin{aligned}
\Psi_t^{\lambda\epsilon}(\phi) \;=\; & \frac{1}{2}\lambda^{-1} \sum_{i\neq j} \sigma^i\sigma^j \int_0^t du \int_0^u dv\, K^\epsilon(X_u^i - X_v^j)\, \phi(X_v^j) \\
& + \lambda^{-1} \sum_{i\neq j} \sigma^i\sigma^j \int_0^t du\, K^\epsilon(X_u^i - X_u^j)\, \phi(X_u^j) \\
& - \lambda^{-1} \sum_{i\neq j} \sigma^i\sigma^j \int_0^t du\, K^\epsilon(X_t^i - X_u^j)\, \phi(X_u^j) \\
& + \lambda^{-1} \sum_{i\neq j} \sigma^i\sigma^j \int_0^t \int_0^u \nabla K^\epsilon(X_u^i - X_v^j)\, \phi(X_v^j)\, dv dX_u^i.
\end{aligned} \tag{3.3}$$

PROOF: Apply Ito's formula (using (3.2)) to the C^∞ function

$$f(t,x) \;=\; \int_0^t K^\epsilon(x - X_v^j)\phi(X_v^j)\, dv,$$

replace x by X_u^i, multiply by $\sigma^i\sigma^j$ and sum over $i \neq j$. (A similar argument, used in establishing the original Tanaka formula for Brownian motion intersection local time, can be found in Rosen (1986).)

PROOF OF THEOREM 2.1: In the proof of the above lemma we actually established, *en passant*, a version of Theorem 2.1 with L_t^ϵ replacing L_t and K^ϵ replacing g. Sending $\epsilon \to 0$ to obtain $\mathcal{L}^2$ limits on both sides of the equation is not trivial, but, fortunately, has already been done for us in Rosen (1986). (Rosen actually treats *self*-intersections of Brownian motions, so his proof is a little harder, and the precise terms in the Tanaka formula slightly different. Nevertheless, the proofs carry over almost *verbatim*.)

To convert (3.3) to a form more reminiscent of an evolution equation, we need to make optimal use of the notation set up in the previous section. We need, furthermore, one more piece of notation, and so for $\phi \in S_2$ we set

$$R_t^{\lambda\epsilon}(\phi) := \lambda^{-1} \sum_i \int_0^t \int_0^u \nabla K^\epsilon(X_u^i - X_v^i)\, \phi(X_v^i)\, dv dX_u^i. \tag{3.4}$$

LEMMA 3.2.

$$\begin{aligned} \Psi_t^{\lambda\epsilon}(\phi) &= \frac{1}{2}\int_0^t du \int_0^u dv\, A_{uv}^\lambda (K^\epsilon(x-y)\phi(y)) \\ &\quad + \int_0^t du\, A_{uu}^\lambda (K^\epsilon(x-y)\phi(y)) - \int_0^t du\, A_{tu}^\lambda (K^\epsilon(x-y)\phi(y)) \\ &\quad + \int_0^t \int_{\Re^2} \mu_u^\lambda (\nabla K^\epsilon(x-\cdot)\phi(\cdot))\, W^\lambda(dx,du) - R_t^{\lambda\epsilon}(\phi). \end{aligned}$$

PROOF: The above is basically a rewrite of (3.3). The first three terms are easily seen to be equivalent to the first three terms of (3.3) on applying the definition (2.6) of A_{st}^λ. To obtain the last two terms, we write the last expression in (3.3) as

$$\lambda^{-1} \sum_i \sum_j \sigma^i \sigma^j \int_0^t \int_0^u \nabla K^\epsilon(X_u^i - X_v^j)\, \phi(X_v^j)\, dv dX_u^i$$
$$- \lambda^{-1} \sum_i \int_0^t \int_0^u \nabla K^\epsilon(X_u^i - X_v^i)\, \phi(X_v^i)\, dv dX_u^i.$$

Consider the first term here. (The second is much easier, and it is easy to see that it is equal to $R_t^{\lambda\epsilon}$.) By Fubini's theorem for stochastic integrals (e.g. Revuz and Yor (1987), Section VI, Lemma 1.4) this is equal to

$$\lambda^{-1/2} \sum_j \sigma^j \int_0^t dv\, [\lambda^{-1/2} \sum_i \sigma^i \int_v^t \nabla K^\epsilon(X_u^i - X_v^j)\phi(X_v^j)\, dX_u^i].$$

From the definition of the measure W^λ, it follows that this is equal to

$$\lambda^{-1/2} \sum_j \sigma^j \int_0^t dv\, [\int_v^t \int_{\Re^2} \nabla K^\epsilon(x - X_v^j)\phi(X_v^j)\, W^\lambda(dx,du)].$$

(This follows from Proposition 8.3 of Walsh (1986). The proof there proceeds by establishing a result like the above first for indicator functions ϕ, and then via passage to the limit for general $\phi \in \mathcal{L}^2$. It is precisely at this point that we need $\nabla K^\epsilon \in \mathcal{L}^2(\Re^2)$.)

A stochastic Fubini theorem for worthy martingale measures (Walsh (1986), Theorem 2.6) implies that the above equals

$$\lambda^{-1/2}\sum_j \sigma^j \int_0^t \int_{\Re^2} [\int_0^u \nabla K^\epsilon(x - X_v^j)\phi(X_v^j)\,dv]\,W^\lambda(dx,du).$$

Interchanging summation and integration yields

$$\int_0^t \int_{\Re^2} \{\int_0^u \lambda^{-1/2}\sum_j \sigma^j [\nabla K^\epsilon(x - X_v^j)\phi(X_v^j)\,dv]\}\,W^\lambda(dx,du)$$
$$= \int_0^t \int_{\Re^2} \mu_u^\lambda(\nabla K^\epsilon(x - \cdot)\phi(.))\,W^\lambda(dx,du),$$

where the last line follows from the definition of μ_t^λ at (1.3).

This completes the proof of the lemma. The next step is to show that $R_t^{\lambda\epsilon}$, the remainder term in (3.4), goes to 0 as $\lambda \to \infty$.

LEMMA 3.3. *$R_t^{\lambda\epsilon}(\phi) \to_p 0$ as $\lambda \to \infty$, for each $t \in [0,1]$, $\epsilon > 0$, and $\phi \in S_2$.*

PROOF: Note first that if X denotes a generic term of our sequence $X_1, X_2, \ldots$, B a standard Brownian motion on $\Re^2$ starting at the origin, and F a functional on $C([0,1],\Re^2)$, then

$$E\{F(X) \mid X_0 = x\} = E\{F(B + x)\}.$$

Consequently, using the fact that the sequence of planar Brownian motions $X^1, X^2, \ldots$ commence at the points of a Poisson process of intensity λ, it follows that

$$E\{\sum_i F(X^i)\} = \lambda \int_{\Re^2} E\{F(B + x)\}\,dx. \tag{3.5}$$

Define now

$$r_t(X) = r_t(X:\phi,\epsilon) = \int_0^t \int_0^u \nabla K^\epsilon(X_u - X_v)\phi(X_v)\,dv dX_u,$$

so that $R_t^{\lambda\epsilon}(\phi) = \lambda^{-1}\sum_i r_t(X^i)$. Thus

$$E\{[R_t^{\lambda\epsilon}(\phi)]^2\} = \lambda^{-2} E\{\sum_i [r_t(X^i)]^2\}$$
$$= \lambda^{-1} \int_{\Re^2} E[\int_0^t \int_0^u \nabla K^\epsilon(B_u - B_v)\phi(B_v + x)\,dv dB_u]^2\,dx,$$

the last line following from (3.5). Standard inequalities for the moments of stochastic integrals (e.g. Ikeda and Watanabe (1981), page 110), combined with the fact that ϕ decays rapidly at infinity give us that the triple integral above is finite, and so $R_t^{\lambda\epsilon} \to_P 0$ as $\lambda \to \infty$, and the lemma is established.

For $\phi \in S_2$ let $U_t^{\lambda}(\phi)$ be the second to last term in the representation of $\Psi_t^{\lambda\epsilon}$ given by Lemma 3.2; i.e.

$$U_t^{\lambda}(\phi) := \int_0^t \int_{\Re^2} \mu_u^{\lambda}(\nabla K^{\epsilon}(x-\cdot)\phi(\cdot))\, W^{\lambda}(dx, du). \tag{3.6}$$

Furthermore, set

$$U_t(\phi) := \int_0^t \int_{\Re^2} \mu_u(\nabla K^{\epsilon}(x-\cdot)\phi(\cdot))\, W(dx, du). \tag{3.7}$$

The next result is the key step in establishing all of our main results.

THEOREM 3.1. *Let $\lambda \to \infty$ along a countable sequence. (We should therefore really replace λ by λ_n in what follows, but there is a limit to how many subscripts and superscripts the human mind can absorb.) Then, for all $\phi_1, \phi_2, \phi_3, \phi_4 \in S_2$, and $\phi_5 \in S_4$,*

$$\begin{aligned} &\langle \lambda^{-1/2}\tilde{\Pi}(\phi_1), W^{\lambda}(\phi_2), \eta^{\lambda}(\phi_3), U^{\lambda}(\phi_4), A_{\cdot t}^{\lambda}(\phi_5)\rangle \\ &\qquad \Rightarrow \langle \tilde{\Pi}(\phi_1), W(\phi_2), \eta(\phi_3), U(\phi_4), A_{\cdot t}(\phi_5)\rangle, \end{aligned} \tag{3.8}$$

as $\lambda \to \infty$ along this sequence.

PROOF: The general theory of weak convergence that we require to prove this result is developed and expounded in detail in Walsh (1986). To give a fully detailed, self-contained version of the proof of (3.8), we would have to copy two pages of definitions from Walsh's notes, and then go through a number of pages of rather detailed, and essentially uninteresting, calculations. Since this seems to be somewhat unjustified, we shall assume that the reader is familiar with Walsh's notes, and merely point out how Theorem 3.1 follows from the results and techniques developed there.

Note, firstly, that the weak convergence of the triple $\langle \lambda^{-1}\tilde{\Pi}, W^{\lambda}, \eta^{\lambda}\rangle$ is a special case of Theorem 2.2, since here we are only taking $\lambda \to \infty$ through a countable sequence. Appending the convergence of $A_{\cdot t}^{\lambda}$ and U_t^{λ} follows as in the proof of Proposition 8.17 of Walsh's notes, once we have checked that the individual limits are as claimed. We shall check this only for $A_{\cdot t}$, this being the harder of the two, and somewhat different to the example treated by Walsh.

Consider firstly $\phi \in S_4$ of the form $\phi(x,y) = \psi(x)\hat{\psi}(y)$, $\quad \psi, \hat{\psi} \in S_2$. Then

$$(3.9) \qquad A^{\lambda}_{ts}(\phi) = \eta^{\lambda}_t(\psi)\eta^{\lambda}_s(\hat{\psi}) - \lambda^{-1}\sum_i \psi(X^i_t)\hat{\psi}(X^i_s).$$

By computing moments as in the proof of Lemma 3.3, it is easy to check that the last term in (3.9) converges in probability to the deterministic expression

$$\begin{aligned} E\{\psi(X^i_t)\hat{\psi}(X^i_s)\} &= \int \psi(x)dx \int \hat{\psi}(y)dy\, p_{|t-s|}(x-y) \qquad t \neq s, \\ &= \int \psi(x)\hat{\psi}(x)\, dx \qquad t = s. \end{aligned}$$

Since this expression is equivalent to $E\{(\eta_t \times \eta_s)(\phi)\}$ (c.f. Theorem 2.1 of Adler and Lewin (1988)) it follows from (3.9), the convergence of η^{λ} to η, and the continuous mapping theorem that for ϕ of product form

$$(3.10) \qquad A^{\lambda}_{ts}(\phi) \Rightarrow A_{ts}(\phi) = (\eta_t \otimes \eta_s)(\phi) = (\eta_t \times \eta_s)(\phi) - E\{(\eta_t \times \eta_s)(\phi)\}.$$

Using now the fact that sums of the form $\sum_{k=1}^{N} \psi_k(x)\hat{\psi}_k(y)$ are $\mathcal{L}^2$ dense in S_4, the extension of (3.10) to all $\phi \in S_4$ is standard.

This fact, together with the comments made above, completes the proof of the theorem.

REMARK: It is important to note that since the proof of Theorem 3.1 relies on results proved by Walsh, the weak convergence in (3.8) can only be shown at this stage, to hold for nice functions ϕ_k, and not for functions like g and ∇g, which is what we need. To handle these functions we need the extra work of the folowing arguments.

PROOF OF THEOREM 2.5: We need to show that $\Psi^{\lambda\epsilon}_t \Rightarrow \Psi^{\epsilon}_t$ as $\lambda \to \infty$, for every $\epsilon > 0$, on $D([0,1], S_2')$. Recall that

$$(3.11) \qquad \Psi^{\lambda\epsilon}_t(\phi) = \int_0^t \int_0^u A^{\lambda}_{uv}(e^{-\epsilon/2}p_\epsilon(x-y)\phi(y))\, dv du,$$

and Ψ^{ϵ}_t is given by an identical expression with A_{uv} replacing A^{λ}_{uv}.

Note firstly that since $e^{-\epsilon/2}p_\epsilon(x-y)\phi(y) \in S_4$ it follows that for every $\phi \in S_2$ we have $A^{\lambda}_{uv}(e^{-\epsilon/2}p_\epsilon(x-y)\phi(y)) \Rightarrow A_{uv}(e^{-\epsilon/2}p_\epsilon(x-y)\phi(y))$. Since integration with respect to the two time parameters is a continuous functional on $D([0,1]\times[0,1], S_4')$, it follows that $\Psi^{\lambda\epsilon}_t(\phi) \Rightarrow \Psi^{\epsilon}_t(\phi)$ for every $\phi \in S_2$ and each fixed $t \in [0,1]$. By the Cramér-Wold device (to handle the t parameter) and linearity (to handle the ϕ parameter) this convergence can be lifted to that of finite-dimensional distributions. The problem now is to establish tightness in t for fixed ϕ. Theorem 6.15 of Walsh (1986) then gives us that $\Psi^{\lambda\epsilon}_t \Rightarrow \Psi^{\epsilon}_t$ on $D([0,1], S_2')$, which completes the proof of the theorem.

For fixed ϕ, however, tightness in t follows easily from standard moment conditions, using the integral form of (3.11) and the fact that $A^{\lambda}_{uv}(\phi)$ has moments of all orders (c.f. Rosen (1986).)

PROOF OF THEOREM 2.3: We want to prove that $\Psi^{\lambda} \Rightarrow \Psi$ on the appropriate Skorohod space. Note firstly that

$$\Psi^{\lambda}_t(\phi) - \Psi^{\lambda\epsilon}_t(\phi) \;=\; \lambda^{-1}\sum_{i\neq j}\sigma^i\sigma^j[L_t(\phi;X^i,X^j) - L^{\epsilon}_t(\phi;X^i,X^j)],$$

and so

$$E\{|\,\Psi^{\lambda\epsilon}_t(\phi) - \Psi^{\lambda}_t(\phi)\,|^2\} \;=\; E\{|\,(L_t - L^{\epsilon}_t)(\phi,X^1,X^2)\,|^2\} \;=\; C_{\epsilon}(\phi),$$

where $C_{\epsilon}(\phi) \to 0$ as $\epsilon \to 0$, by the very definition of L_t as the $\mathcal{L}^2$ limit of L^{ϵ}_t. Consequently, by Tchebychev's inequality,

$$P\{|\,\Psi^{\lambda\epsilon}_t(\phi) \;-\; \Psi^{\lambda}_t(\phi)\,| \geq \delta\} \;\leq\; \frac{C_{\epsilon}(\phi)}{\delta}\text{for every}\quad \phi\in S_2, \tag{3.12}$$

where $C_{\epsilon}(\phi) \to 0$ as $\epsilon \to 0$, for every $\delta > 0$.

We now want to show that for fixed $t\in[0,1]$ and $\phi\in S_2$ that $\Psi^{\lambda}_t(\phi) \to_{\mathcal{D}} \Psi_t(\phi)$ as $\lambda\to\infty$. For $x\in\Re$ and $\delta>0$,

$$\begin{aligned}
\lim_{\lambda\to\infty} P\{\Psi^{\lambda}_t(\phi) \leq x\} \;&=\; \lim_{\epsilon\to 0}\lim_{\lambda\to\infty} P\{\Psi^{\lambda}_t(\phi) \leq x\}\\
&= \lim_{\epsilon\to 0}\lim_{\lambda\to\infty} P\{[\Psi^{\lambda}_t(\phi) \;-\; \Psi^{\lambda\epsilon}_t(\phi)] \;+\; \Psi^{\lambda\epsilon}_t(\phi) \;\leq\; x\}\\
&\leq\; \lim_{\epsilon\to 0}\lim_{\lambda\to\infty} [P\{|\,(\Psi^{\lambda}_t(\phi) - \Psi^{\lambda\epsilon}_t)(\phi)\,| \geq \delta\} \;+\; P\{\Psi^{\lambda\epsilon}_t(\phi) \leq x+\delta\}]\\
&\leq\; \lim_{\epsilon\to 0}[\,\frac{C_{\epsilon}(\phi)}{\delta} \;+\; P\{\Psi^{\epsilon}_t(\phi) \;\leq\; x+\delta\}\,]\\
&=\; P\{\Psi_t(\phi) \;\leq\; x+\delta\,\},
\end{aligned}$$

where the first line follows from the independence of both the right and left hand sides of ϵ, the second and third are trivial, the fourth is a consequence of Theorem 2.5, and the last from the fact that $\Psi^{\epsilon}_t \xrightarrow{\mathcal{L}^2} \Psi_t$ by Adler and Lewin (1988). (For the sake of precision, we should note that the definition of Ψ^{ϵ}_t in the current paper and in Adler and Lewin (1988) differs by the fact that the factor of $e^{-\epsilon/2}$ appearing in (3.11) is replaced by $e^{-\epsilon}$ there. It is easy to check, however, that this makes no difference to the limit process Ψ_t.)

A similar argument shows that $\lim_{\lambda\to\infty} P\{\Psi^{\lambda}_t(\phi) \leq x\} \geq P\{\Psi_t(\phi) \;\leq\; x-\delta\}$, and since δ was arbitrary we have that $\Psi^{\lambda}_t(\phi) \xrightarrow{\mathcal{D}} \Psi_t(\phi)$ for fixed t and ϕ. A similar argument to that used to prove Theorem 2.5 can be applied again here to obtain full weak convergence on $D([0,1],\mathcal{S}_2)$, and so complete the proof of the theorem.

PROOF OF THEOREM 2.4: It follows from Theorems 2.5, 3.1, and Lemmas 3.2, 3.3, that for all $\epsilon > 0$, $t \in [0,1]$, and $\phi \in \mathcal{S}_2$

(3.13)
$$\begin{aligned}\Psi_t^\epsilon(\phi) \stackrel{\mathcal{L}}{=} & \frac{1}{2}\int_0^t du \int_0^u dv\, A_{uv}(K^\epsilon(x-y)\phi(y)) \\ & + \int_0^t du\, A_{uu}(K^\epsilon(x-y)\phi(y)) - \int_0^t du\, A_{tu}(K^\epsilon(x-y)\phi(y)) \\ & + \int_0^t \int_{\Re^2} \mu_u(\nabla K^\epsilon(x-\cdot)\phi(\cdot))\, W(dx,du).\end{aligned}$$

All we need to do to prove Theorem 2.4 is to show that each term in (3.13) has a well defined limit as $\epsilon \to 0$. The $\mathcal{L}^2$ convergence of Ψ_t^ϵ to Ψ_t is established in Adler and Lewin (1988), and has already been noted above. Thus we need only work on the four terms on the right hand side of (3.13). Consider the first of these, which, modulo a factor of $\frac{1}{2}$, equals

$$I_\epsilon(\phi) = \int_0^t du \int_0^u dv\, A_{uv}(K^\epsilon(x-y)\phi(y)).$$

In order to show that I_ϵ converges in $\mathcal{L}^2$, it suffices to show that $E\{I_\epsilon(\phi)I_\delta(\phi)\}$ tends to a limit as $\epsilon,\delta \to 0$. By Theorem 2.1 of Adler and Lewin (1988)

(3.14)
$$\begin{aligned}E\{I_\epsilon(\phi)I_\delta(\phi)\} = & \int_D \int_{\Re^8} p_{|u-u'|}(x,z)p_{|v-v'|}(y,w)K^\epsilon(x-y)\phi(y)K^\delta(z-w)\phi(w) \\ & + \int_D \int_{\Re^8} p_{|u-v'|}(x,w)p_{|v-u'|}(y,z)K^\epsilon(x-y)\phi(y)K^\delta(z-w)\phi(w),\end{aligned}$$

where $D = \{0 \le v \le u \le t; 0 \le v' \le u' \le t\}$ and we have neglected to write the eight differentials in each of the multiple integrals.

Consider the first term in (3.14). (The second is handled in an almost identical fashion.) By the definition of K^ϵ this is equal to

$$\begin{aligned}&\int_D \int_{\Re^8} p_{|u-u'|}(x,z)p_{|v-v'|}(y,w)\int_\epsilon^\infty e^{-\alpha/2}p_\alpha(x,y)\phi(y)\,d\alpha \int_\delta^\infty e^{-\beta/2}p_\beta(z,w)\phi(w)\,d\beta \\ &\quad = \int_\epsilon^\infty e^{-\alpha/2}\,d\alpha \int_\delta^\infty e^{-\beta/2}\,d\beta \int_D \int_{\Re^4} dydw\,\phi(y)\phi(w) \\ &\qquad\quad \cdot \int_{\Re^2} dz\, p_\beta(z,w) \int_{\Re^2} dx\, p_{|u-u'|}(x,z)p_\alpha(x,y)p_{|v-v'|}(y,w).\end{aligned}$$

Integrating over x and then z, by applying the Chapman-Kolmogorov equation twice, we obtain that this is equal to

$$\int_\epsilon^\infty e^{-\alpha/2}\,d\alpha \int_\delta^\infty e^{-\beta/2}\,d\beta \int_D \int_{\Re^4} dydw\, \phi(y)\phi(w)p_{|v-v'|}(y,w)p_{(|u-u'|+\alpha+\beta)}(y,w).$$

By the Lebesgue dominated convergence theorem this will converge, as $\epsilon, \delta \to 0$ to the finite constant

$$4\int_D \int_{\Re^4} dydw\, \phi(y)\phi(w)p_{|v-v'|}(y,w)p_{|u-u'|}(y,w),$$

as long as $\int_D \int_{\Re^4} dydw\, \phi(y)\phi(w)p_{|v-v'|}(y,w)p_{|u-u'|+\alpha+\beta}(y,w)$ is bounded uniformly in ϵ and δ for each $\phi \in S_2$. Calculations similar to those made in the proof of Theorem 3.1 of Adler and Lewin (1988) easily show this to be the case.

This establishes the required convergence for the first term on the right hand side of (3.13). The next two terms are handled similarly, and we leave the details to the reader. The last term is somewhat different, however, so we treat it in detail. Set

$$J_\epsilon(\phi) = \int_0^t \int_{\Re^2} \mu_u(\nabla K^\epsilon(x-\cdot)\phi(\cdot))W(dx,du).$$

Then

$$\begin{aligned}
E\{[(J_\epsilon - J_\delta)(\phi)]^2\} &= E\{[\int_0^t \int_{\Re^2} \mu_u(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot))W(dx,du)]^2\} \\
&\le E\{\int_0^t \int_{\Re^2\times\Re^2} \mid \mu_u(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot)) \mid \\
&\qquad \cdot \mid \mu_u(\nabla(K^\epsilon - K^\delta)(y-\cdot)\phi(\cdot)) \mid \ \delta(x-y)\,dxdydu\},
\end{aligned}$$

where the last line follows from Theorem 2.5 of Walsh (1986), and is, in turn, equal to

$$\begin{aligned}
&E\{[\int_0^t \int_{\Re^2} \mid \mu_u(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot)) \mid^2 \ dxdu \\
&= \int_0^t \int_{\Re^2} dxdu\, E \mid \mu_u(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot)) \mid^2 \\
&= \int_0^t \int_{\Re^2} dxdu\, E \mid \int_0^u \eta_v(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot))\,dv \mid^2 \\
&= \int_0^t \int_{\Re^2} dxdu \int_0^u \int_0^u E[\eta_{v_1}(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot)) \\
&\qquad \cdot \eta_{v_2}(\nabla(K^\epsilon - K^\delta)(x-\cdot)\phi(\cdot))]\,dv_1 dv_2 \\
&= \int_0^t \int_{\Re^2} dxdu \int_0^u \int_0^u E[(\nabla(K^\epsilon - K^\delta)(x-X_{v_1})\phi(X_{v_1})) \\
&\qquad \cdot (\nabla(K^\epsilon - K^\delta)(x-X_{v_2})\phi(X_{v_2}))]\,dv_1 dv_2,
\end{aligned}$$

the last line following from Walsh (1986, p389). Note that for each $\epsilon > 0$

$$K^{\epsilon}(x) = \frac{2}{(2\pi)^2}\int_{\Re^2} e^{ip\cdot x}\frac{e^{-\epsilon(1+\|p\|^2)/2}}{1+\|p\|^2}\,dp.$$

Substitute this into the above to obtain that

(3.15)

$$E\{[(J_\epsilon - J_\delta)(\phi)]^2\} \leq Ct\int_0^t dv_1\int_0^{v_1} dv_2\int_{\Re^2} dx\int_{\Re^2} dp_1\int_{\Re^2} dp_2\prod_{k=1}^{2}\frac{\|p_k\|}{1+\|p_k\|^2}$$
$$\cdot\left[e^{-\epsilon(1+\|p_k\|^2)/2} - e^{-\delta(1+\|p_k\|^2)/2}\right]$$
$$\cdot E\{e^{ip_1(x-X_{v_1})}\phi(X_{v_1})e^{ip_2(x-X_{v_2})}\phi(X_{v_2})\}.$$

We now proceed much as Rosen (1986) argued when dealing with similar expressions that arose in studying Brownian motion self-intersections. The expectation in (3.15) is a simple Gaussian calculation, and is easily seen to be equal to

$$e^{i(p_1+p_2)x}\int_{\Re^2}\int_{\Re^2}\phi(y)\phi(z)e^{-ip_1y-ip_2z}p_{|v_1-v_2|}(y-z)\,dydz. \tag{3.16}$$

Note the elementary inequality $|e^{-\epsilon a^2} - e^{-\delta a^2}| \leq C_\alpha(a^2|\epsilon-\delta|)^\alpha$, for every $\alpha < 1$, and the fact that $\int e^{i(p_1+p_2)x}\,dx = \delta(p_1+p_2)$, where δ here represents the Dirac delta function. Thus, substituting (3.16) into (3.15), integrating out x, and applying these two facts we obtain that (3.15) is bounded above by

(3.17)

$$C_\alpha|\epsilon-\delta|^{2\alpha}t\int_0^t dv_1\int_0^{v_1} dv_2\int_{\Re^2} dp\frac{\|p\|^2}{(1+\|p\|^2)^2}(1+\|p\|^2)^{2\alpha}$$
$$\cdot\int_{\Re^2}\int_{\Re^2}\phi(y)\phi(z)p_{|v_1-v_2|}(y-z)e^{ip(y-z)}\,dydz$$
$$= C_\alpha|\epsilon-\delta|^{2\alpha}t\int_0^t dv\int_v^t dw\int_{\Re^2} dp\frac{\|p\|^2}{(1+\|p\|^2)^{2-2\alpha}}$$
$$\cdot\int_{\Re^2} dz\,p_v(z)e^{ipz}\int_{\Re^2} dy\,\phi(y)\phi(y-z)$$
$$\leq C_\alpha|\epsilon-\delta|^{2\alpha}t^2\int_{\Re^2} dp\frac{\|p\|^2}{(1+\|p\|^2)^{2-2\alpha}}\int_{\Re^2} dz\,e^{ipz}f(z)g_t(z),$$

where C_α may change from line to line, and the functions f and g are given by

$$f(z) = \phi*\phi'(z), \qquad g_t(z) = \int_0^t p_v(z)\,dv,$$

$\phi'(x) = \phi(-x)$ and $*$ represents convolution.

Denoting Fourier transforms in the usual way, and writing $h_\alpha(p)$ for

$$\|p\|^2/(1+\|p\|^2)^{2-2\alpha},$$

we can write the integrals in the last line of (3.17) as

$$\int_{\Re^2} h_\alpha(p)\,(\hat{f} * \hat{g}_t(p))\,dp. \tag{3.18}$$

Clearly, if we can establish that this integral is finite for some $\alpha < 1$, then the $\mathcal{L}^2$ convergence of $J_\epsilon(\phi)$ follows from (3.15)-(3.17). A standard inequality on convolutions, (e.g. Reed and Simon (1975) p29) gives us that

$$\int_{\Re^2} h_\alpha(p)\,(\hat{f} * \hat{g}_t(p))\,dp \;\le \|h_\alpha\|_s\,\|f\|_q\,\|g\|_1, \tag{3.19}$$

where $\frac{1}{s}+\frac{1}{q}=1$. We leave it to the reader to show that if we choose $\alpha < \frac{1}{2}$ and $s > (1-2\alpha)^{-1}$ then each of the three norms in (3.19) is finite. This completes the proof of Theorem 2.4.

Only one task remains to complete our work:

PROOF OF LEMMA 2.1: The proof centers on noting that the equivalence (2.7), which is what we must establish, is almost identical to that established in Lemma 3.2, with the function g in the former replaced by K^ϵ in the latter. If we can send $\epsilon \to 0$, and show that all terms in Lemma 3.2 converge in $\mathcal{L}^2$ to the corresponding term in (2.7), then we shall be done. This, however, is not too difficult, since at this stage the density parameter λ, is still finite.

We shall consider only one term of Lemma 3.2, and, following our established practice, shall choose the most difficult term. Set

$$F_\epsilon^\lambda(\phi) \;=\; \int_0^t \int_0^u A_{uv}^\lambda\,(K^\epsilon(x-y)\phi(y))\,dudv.$$

We need to show that as $\epsilon \to 0$ this converges in $\mathcal{L}^2$ to first term of (2.7). Note that for $\epsilon, \delta > 0$

$$
\begin{aligned}
&E\{|F_\epsilon^\lambda(\phi) - F_\delta^\lambda(\phi)|^2\} \\
&\quad = E\Big[\int_0^t du \int_0^u dv\, \lambda^{-1} \sum_{i \neq j} \sigma^i \sigma^j (K^\epsilon - K^\delta)(X_u^i - X_v^j) \cdot \phi(X_v^j)\Big]^2 \\
&\quad \le \int_0^t du \int_0^u dv\, \lambda^{-2} E\Big[\sum_{i \neq j} \sigma^i \sigma^j (K^\epsilon - K^\delta)(X_u^i - X_v^j) \cdot \phi(X_v^j)\Big]^2 \\
&\quad \le \int_0^t du \int_0^u dv\, \lambda^{-2} \lambda^2 E\{|(K^\epsilon - K^\delta)(X_u^i - X_v^j)|^2 \cdot |\phi(X_v^j)|^2\} \\
&\quad = \int_0^t du \int_0^u dv \int_{\Re^2} dy \int_{\Re^2} dz (K^\epsilon - K^\delta)^2 (y - z) \cdot \phi^2(z) \\
&\quad \le t^2 \int_{\Re^2} \phi^2(z)\, dz \int_{\Re^2} (K^\epsilon - K^\delta)^2(y)\, dy.
\end{aligned}
$$

The important thing to note at this stage is that we have managed to reduce the computations to the point that they are of the same nature as those which prove the existence of intersection local time for two independent Brownian motions, as in Rosen (1986). In fact, they are very similar to those at the end of the previous proof. We thus leave it to the interested reader to satisfy himself that it is now not hard to show that

$$
\lim_{\epsilon, \delta \to 0} E\{|F_\epsilon^\lambda(\phi) - F_\delta^\lambda(\phi)|^2\} = 0,
$$

which completes the proof of the convergence of the first term of the representation of Lemma 3.2 to that of (2.7). The remaining terms can be handled similarly, and this completes the proof of Lemma 2.1.

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