Exponential Bounds in Vapnik-Červonenkis Classes of Index 1

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ABSTRACT

The smallest classes $\mathscr{C}$ which satisfy the Vapnik-Červonenkis combinatorial condition are assigned an index of 1. We show that over all classes $\mathscr{C}$ of index 1, the classical exponential inequalities for empirical processes are optimal.

Keywords and phrases. Empirical process, exponential bound, Vapnik-Červonenkis class, Brownian bridge, set-indexed gaussian process, partial order.

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Abbreviated Title: Vapnik - Červonenkis classes of index 1

1. Introduction

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution P on a measurable space $(\Omega, \mathscr{A})$. We define the empirical measure

$$P_n = n^{-1} \sum_{i=1}^{n} \delta_{X_i}$$

and the normalized empirical process on $\mathscr{A}$ defined by

$$\nu_n = n^{1/2}(P_n - P), \; n \geq 1.$$

For a given class of events $\mathscr{C} \subset \mathscr{A}$, let $D_n(\mathscr{C}) = \sup_{A \in C} |\nu_n(A)|$. Under certain conditions on the class $\mathscr{C}$, exponential bounds for the probability that $D_n(\mathscr{C})$ exceeds M, $M \geq 1$ have been obtained. The inequalities which arise have the general form

$$\Pr\{D_n(\mathscr{C}) > M\} < ae^{-\gamma M^2} \tag{1.1}$$

where a, γ are positive constants that depend on $\mathscr{C}$ but not on M. The results of Dvoretzky (1956), Kiefer (1961), Devroye (1982) and Alexander (1984) are all of this form, for various $\mathscr{C}$.

The classical example comes from considering the class of intervals on the real line. Here, we take $\mathscr{C}_1 = \{[0,t] : t \in [0,1]\}$. The fundamental result (Kolmogorov, 1932) gives

$$\Pr\{\sup_{A \in C_1} \nu_n(A) > M\} \sim e^{-2M^2}$$

as $n \to \infty$, for all M.

In Euclidean space we may consider another well known example. Let $\mathscr{C} = E_d : \{\{x : x_j \leq t_j, j = 1, \ldots, d\}, \mathbf{t} \in \mathbf{R}^d\}$. When $d > 1$, Kiefer (1961) established (1.1) with $\gamma = 2(1-\varepsilon)$, for all $\varepsilon > 0$ and for some $a = a(\varepsilon, d)$. Unlike the case $d = 1$, the form of the limiting distribution of $D_n(\mathscr{C})$ depends on P.

A new direction was set when Vapnik and Červonenkis (1968, 1971) introduced combinatorial ideas that lead to results for general families $\mathscr{C}$ of geometric regions, defined as follows. Let X be a set and $\mathscr{C}$ a class of subsets of X. A finite set $A \subset X$ is *shattered* by $\mathscr{C}$ if every subset $E \subset A$ is of the form $A \cap C$ for some $C \in \mathscr{C}$. Now $\mathscr{C}$ is called a *Vapnik-Červonenkis class* (or VC class) if for some $n \geq 1$, no n-element subset of X is shattered by $\mathscr{C}$. Let $S(\mathscr{C})$ equal the cardinality of the largest subset $A \subset X$ which is shattered by $\mathscr{C}$. We will call $S(\mathscr{C})$ the index of $\mathscr{C}$.

Several familiar classes of geometric regions are VC classes. These include the classes of all rectangles, all closed balls, all polyhedra with at most m faces, and quadrants of the form $(-\infty, \mathbf{t}]$, in $\mathbf{R}^d$. If $\mathscr{C}$ is a VC class, then $\{C \cap D : C, D \in \mathscr{C}\}$,

$\{C \cup D : C, D \in \mathscr{C}\}$ and $\{A^c : A \in \mathscr{C}\}$ are also VC classes. The class of all closed convex sets in $\mathbb{R}^2$ is not a VC class. Additional facts about VC classes are presented in Dudley (1978, 1984, etc.).

Assuming suitable measurability conditions Alexander (1984) in the more general setting of VC classes of functions shows that for VC classes with index d and $M \geq 8$, a bound in (1.1) of the form $a \exp(-(2-\varepsilon) M^2)$ results. The exponent $(2-\varepsilon) M^2$ cannot be improved; a bound of the form $a \exp(-2M^2)$ is the best possible. (Even a simple case where $\mathscr{C} = \{A\}$ with $P(A) = 1/2$ has this exact bound, Hoeffding (1963)). However, Alexander's work (intended for asymptotic use) yields a constant $a = a(\varepsilon, d)$ which is impractically large. Is further refinement possible?

Bounds such as those desired for (1.1) yield faster rates of convergence when ν_n is indexed by some small family of sets, such as the VC classes with index 1. But we shall give an example on $[0, 1]$ that shows:

(1.2) *there is no constant* $\beta < \infty$ *which will satisfy the inequality*

$$\Pr\{D_n(\mathscr{C}) > M\} < \beta e^{-2M^2} \text{ for } \textit{all classes with } S(\mathscr{C}) = 1.$$

2. An Asymptotic Distribution when $S(\mathscr{C}) = 1$

We introduce a collection of subsets on $[0,1]$. Let $X = [0,1]$ and define k disjoint subintervals

$$A_1 = [0, t_1]$$
$$A_{i+1} = (t_i, t_{i+1}],\ i = 1, 2, \ldots, k-1$$

with parameters $t_1 < \frac{1}{2}$; $0 < t_1 < t_2 < \cdots < t_k = 1$ such that $X = \bigcup_{i=1}^{k} A_i$. Let $\mathscr{C}_k$ be a collection of the k sets, $\mathscr{C}_i = A_1 \cup A_{i+1}$, $0 \le i \le k-1$. It is easy to check that $S(\mathscr{C}_k) = 1$ since no 2-point set $\{a, b\} \subset [0,1]$ is shattered by $\mathscr{C}_k$. In this section we show that as $M \to \infty$

(2.1) $$\Pr\{\sup_{A \in C_k} \nu_n(A) > M\} \sim (k-1)e^{-2M^2} \text{ and the}$$

and the result (1.2) follows.

In order to prove (2.1), we now consider appropriate Gaussian limit processes. Let $(\Omega, \mathscr{A}, P)$ be a probability space and let W_P denote the *isonormal* Gaussian process with $EW_P(A) = 0$ and $EW_P(A)W_P(B) = P(A \cap B)$ for *all* $A, B \in \mathscr{A}$. We define $G_P(A) := W_P(A) - P(A)W_P(\Omega)$ so that G_P is a Gaussian process indexed by a class of measurable sets $\mathscr{C} \subset \mathscr{A}$ with mean 0 and covariance

$$EG_P(A)G_P(B) = P(A \cap B) - P(A)P(B).$$

Now Dudley (1978) [under suitable measurability conditions] gives the weak convergence result

$$\underset{A \in \mathscr{C}}{\nu}(A) \to G_P(A) \quad \text{as } n \to \infty$$

which holds under different conditions on $\mathscr{C}$, and in particular for Vapnik-Červonenkis classes.

Remark 2.1. When P is uniform we have $G_P([0,t]) = Y_t$, $0 \le t \le 1$, where Y_t is the Brownian bridge. We will use $G_P(t)$ to denote $G_P([0,t])$.

Remark 2.2. Instead of the usual linear ordering of the parameter sets A the collection $\mathscr{C}_K$ is constructed by a treelike partial ordering. See Dudley (1984): Let $\mathscr{C} \subset 2^X$ satisfy $S(\mathscr{C}) = 1$ and $\phi \in \mathscr{C}$. Then the partial ordering of $\mathscr{C}$ by inclusion is treelike.

We note that a partial order $(\mathscr{C}, \subseteq)$ is called linear if for all $A, B \in \mathscr{C}$, either $A \subseteq B$ or $B \subseteq A$. A partial order will be called treelike iff for all $B \subseteq \mathscr{C}$ and $L(B) := \{A : A \subseteq B\}$, the restriction of $\subseteq$ to $L(B)$ is linear.

In the treelike p.o. $(\mathscr{C}_k, \subseteq)$ we may successfully compare events for which $t \in A_1$ and $t_i \in A_{i+1}$, $i = 1, \ldots, k-1$ but events for which $t_i \in A_i$ and $t_j \in A_j$ for $i \neq j \neq 1$ are not comparable for inclusion.

Remark 2.3. If A and B are disjoint measurable sets then $G_P(A \cup B) = G_P(A) + G_P(B)$ a.s. since the variance of $G_P(A \cup B) - G_P(A) - G_P(B) = 0$. Thus, if $X = \bigcup_{i \le k} A_i$ for disjoint A_i we have the linear relationship $G_P(X) = \sum_{i \le k} G_P(A_i) = 0$.

In the subsequent analysis, we will require several well-known [Doob (1965, 1949)] facts about the Brownian bridge. One such result is given here. We omit the straightforward proof.

Lemma 2.4. Let Y_t denote the Brownian bridge, $0 \le s \le t \le 1$.
Then $\Pr(\sup_{0 \le s \le t} Y_s > M \mid Y_t = y) = \exp(-2M(M-y)/t)$ where $M \ge \max(0, y)$.

We now state our first theorem.

Theorem 2.5. Let $(X, \mathscr{A}, P)$ be a probability space, P Lebesgue measure, $X = [0,1]$ and $\mathscr{C}_k$ the collection $\{\mathscr{C}_i, i = 0, \ldots, k-1\}$. Then as $M \to \infty$,

$$\Pr\{\sup_{A \in \bigcup \mathscr{C}_i} G_P(A) > M\} \text{ is less than or asymptotic to}$$

$$(k-1)\, e^{-2M^2}.$$

Proof. The proof of theorem 2.5 is done in two stages. First we determine an upper bound for $\Pr\{\sup G_P(A) > M\}$ when $A \in \mathscr{C}_k$; then we determine the form of the asymptotic distribution. For the class $\mathscr{C}_k$ we have $X = \bigcup_{i \le k} A_i$ and assign probabilities to the disjoint A_i as follows:

$$P(A_j) = p_j,\ j = 1, \ldots, k \text{ where } p_1 < \tfrac{1}{2} \tag{2.2}$$

and assume $P(A_1 \cup A_j) > \frac{1}{2}$ for each $j > 1$. In terms of the parameters t we have

$$p_1 = t_1,\ p_j = t_j - t_{j-1},\ j > 1.$$

Define events

$$B_i := \{\sup_{t \in A_i} G_P([0,t]) > M\}, \quad 1 \le i \le k.$$

We will indicate a method for obtaining bounds for $\Pr(B_i)$ over the collection $\mathscr{C}_k$ by considering the following inequality.

(2.3) $$\Pr(\sup_{t \in \bigcup A_i} G_P([0,t]) > M)$$

(2.4) $$= \Pr(\bigcup_i B_i) \le \Pr(B_1) + \sum_{j>1} \Pr(B_j \backslash B_1).$$

Now let $Z_t := G_P([0,t])$. When $t \in A_1$, we have $Z_t \equiv Y_t$ when Y_t denotes the Brownian bridge.

We wish to find the distribution of the supremum of the Brownian bridge indexed by the parameter sets A_i. We begin by computing the distribution on the set A_1. Later we will consider the general case.

Now

(2.5) $$\Pr(B_1) = \Pr(Y_{t_1} > M) + \Pr(\sup Y_t > M \text{ for } t \in [0,t_1) \text{ and } Y_{t_1} < M).$$

The probability given in the first term is simply the tail distribution of the Brownian bridge with $E\,Y_{t_1} = 0$ and variance $\sigma_1^2 = t_1(1-t_1)$. To evaluate the second term we integrate the conditional probability $\Pr(\sup_{0 \le t < t_1} Y_t > M \mid Y_{t_1} = y)$ with respect to the distribution of Y_{t_1}. Using Lemma 2.4 and after some calculation the expression in (2.5) reduces to

(2.6) $$\Pr(B_1) = K(M/\sigma_1) + \exp(-2M^2)\,K[M(1-2t_1)/\sigma_1].$$

where $K(\lambda) = 1 - \Phi(\lambda)$, and Φ is the standard normal distribution function.

Next, to evaluate the second term in the inequality given by (2.4) we start with

(2.7) $$\Pr(B_j \backslash B_1) = \Pr[\sup_{t \in C_j} Z_t > M] - \Pr[\sup_{t \in A_1} Z_t > M].$$

Define $\nu_j = t_1 + t_j - t_{j-1}$. Note that $\nu_j > ½$. We consider the first probability in (2.7)

(2.8) Write $$\Pr[\sup_{t \in C_j} Z_t > M] = \Pr[\sup_{t \in [0,\nu_j]} Y_t > M]$$

$$= \Pr[Y_{\nu_j} \ge M] + \Pr[\sup_{t \in [0,\nu_j]} Y_t > M \text{ and } Y_{\nu_j} < M].$$

$$= \Pr_1 + \Pr_2$$

The distribution for Y_{ν_j} has the same form as in (2.6) with t_1 replaced by ν_j and variance $\sigma_j^2 := \nu_j(1-\nu_j)$. The case for the parameter ν_j can be found using analysis similar to that used in computing (2.6). Therefore, we find that

$$\Pr_1 = K(M/\sigma_j), \text{ and}$$

$$\Pr_2 = \int_{-\infty}^{M} e^{-2M(M-y)/v_j} \frac{1}{\sqrt{2\pi}\,\sigma_j} e^{-y^2/(2\sigma_j^2)}\,dy$$

and after combining the exponential terms and completing the square this integral reduces to

$$= e^{-2M^2}[1 - K[M(2v_j - 1)/\sigma_j]]$$

where $v_j > ½$.

The inequality (2.5) may be evaluated by substituting (2.5), (2.8) and rearranging the terms to yield the upper bound:

$$\text{(2.9)} \qquad \Pr(\bigcup_i B_i) \le \sum_{j=2}^{k} \exp(-2M^2) + K[M/\sigma_1] + \sum_{j>1} [K[M/\sigma_j] - K[M/\sigma_1]]$$

$$+ \exp(-2M^2)\,K[M(1 - 2t_1)/\sigma_1]$$

$$- \sum_{j>1} \exp(-2M^2)\,[K[M(2v_j - 1)/\sigma_j] + K[M(1 - 2t_1)/\sigma_1]]$$

$$= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) - (\mathrm{V}).$$

We complete the proof by finding the dominant terms in the asymptotic expression for (2.9) as $M \to \infty$.

In (II) + (III) let

$$\gamma_j := (2\sigma_j^2)^{-1} - 2, \quad j = 1,\ldots,k$$

Choosing $\gamma = \min(\gamma_1, \ldots, \gamma_k) > 0$ will condense the notation.

The Mills' ratio expansion for the tail-end area $K(\cdot)$ gives (II) + (III) $= O(\frac{1}{M}\exp(-(2 + \gamma)M^2))$. Computing along the same lines we obtain for

$$\delta_m := \begin{cases} (1 - 2t_1)^2/\sigma_1^2 & \text{if } m = 1 \\ (2v_j - 1)^2/\sigma_j^2 & \text{if } m = j > 1 \end{cases}$$

that

$$(\mathrm{IV}) = O[\frac{1}{M}\exp(-(2 + \delta_1)M^2)]$$

$$\text{and } (\mathrm{V}) = O[\sum_{j=1}^{k} \frac{1}{M}\exp(-(2 + \delta_j)M^2)].$$

The appropriate substitutions allow us to conclude that $\Pr[\cup B_i] \le (k-1)\exp(-2M^2) + O[\frac{1}{M}\exp(-(2+\delta)M^2)]$ where $\delta := \min(\gamma, \delta_1, \ldots, \delta_k) > 0$. This inequality yields the desired result. □

We consider now the remaining result of this paper.

Theorem 2.6. Let $(X, \mathscr{A}, P)$ be a probability space, P Lebesgue measure, $X = [0,1]$ and $\mathscr{C}_k = \{\mathscr{C}_i; i = 0, \ldots, k-1\}$. Then as $M \to \infty$,

$$\Pr[\sup_{A \in \bigcup C_i} G_P(A) > M] \sim (k-1)e^{-2M^2}.$$

Proof. To prove theorem 2.6 we make the assumption used in the computation of the upperbound namely, $P(A_1) < ½$ and $P(A_1 \cup A_j) > ½$ for each $j > 1$. We will determine a lower bound for (2.3) by using the inequality

(2.10) $$\Pr[\bigcup_i B_i] \ge \Sigma \Pr(B_i) - \sum_{i<j} \Pr(B_i \cap B_j).$$

For $j > 1$, we begin our evaluation with

(2.11) $$\Pr(B_j) \ge \Pr(B_1 \cup B_j) - \Pr(B_1).$$

In view of (2.8) and (2.6) it is easy to verify that

$$\begin{aligned} \Pr(B_j) \ \ge\ & K[M/\sigma_j] + \exp(-2M^2)[1 - K(M(2v_j - 1)/\sigma_j)] \\ & - K[M/\sigma_1] - \exp(-2M^2)K[M(1-2t_1)/\sigma_1],\ j > 1. \end{aligned}$$

To bound $\Pr(B_i \cap B_j)$ we refer to the simple case $B_2 \cap B_3$ and then generalize the argument to account for arbitrary indices. But first we will require a calculation.

Remark 2.7. In $\mathscr{C} = \bigcup_{i \le 3} \mathscr{C}_i$ we assume $X = \bigcup_{i=1}^{3} A_i$ with $P(A_1) = p_1$, where $p_1 < ½$ and $P(A_j) = (1-p_1)/2$, $j = 2, 3$.

If we put $G_P(A_1) = x$, $G_P(A_2) = y$ we have the constraint $x + y + z = 0$ so that $z \equiv -x-y$. Now the joint distribution function of (x, y) is bivariate normal with density function $f(x,y)$ and covariance $\Sigma_{xy} = P(A_1 \cap A_2) - P(A_1)P(A_2)$. One easily calculates $\Sigma_{xy} = -p_1(1-p_1)/2$ for $x \ne y$, $\Sigma_{xx} = p_1(1-p_1)$, and $\Sigma_{yy} = (1-p_1)(1+p_1)/4$.

To bound $\Pr(B_2 \cap B_3)$ we will consider

$$\Pr[\sup_{t \in A_2} G_P(t) > M, \sup_{t \in A_3} G_P(t) > M].$$

We will make use of the conditional distributions to obtain a bound for this expression. Now from Lemma 2.4 we have

$$\Pr[\sup_{t\in A_2} G_P(t) > M \mid G_P(A_1) = x,\ G_P(A_2) = y]$$

$$= \begin{cases} \exp(-2(M-x)(M-x-y)/(t_2-t_1)) & \text{if } x+y<M \text{ and } x<M. \\ 1 & \text{if } x+y \ge M \text{ or } x \ge M. \end{cases}$$

The formula for A_3 can be obtained similarly. Thus,

$$\Pr[\sup_{t\in A_2} G_P(t) > M,\ \sup_{t\in A_3} G_P(t) > M] \tag{2.12}$$

$$= \iint \Pr[\sup_{t\in A_2} G_P(t) > M \mid G_P(A_1) = x,\ G_P(A_2) = y]$$

$$\cdot \Pr[\sup_{t\in A_3} G_P(t) > M \mid G_P(A_1) = x,\ G_P(A_3) = z]\ f(x,y)\,dx\,dy$$

which in accordance with Lemma 2.4 becomes

$$= \iint \Big(1_{\{x+y\ge M \text{ or } x\ge M\}}$$

$$+ 1_{\{x+y<M \text{ and } x<M\}} \exp(-2(M-x)(M-x-y)/(t_2-t_1))\Big)$$

$$\times \Big(1_{\{-y\ge M \text{ or } x\ge M\}}$$

$$+ 1_{\{-y<M \text{ and } x<M\}} \exp(-2(M-x)(M+y)/(t_3-t_2))\Big) f(x,y)\,dx\,dy.$$

To obtain an explicit formula for (2.12) we expand the product in the integrand and transform the integrals to standard bivariate normal form. It is a simple but tedious calculation (which will be omitted here) to then show that computation of the asymptotic bounds may be condensed and each exponential term is of the form $\exp(-(2+\delta_i)M^2)$, $\delta_i = \delta_i(p_1) > 0$ with $p_1 < ½$. Therefore,

$$\Pr[\sup_{t\in A_2} G_P(t) > M,\ \sup_{t\in A_3} G_P(t) > M]$$

$$= F_1 + F_2 + F_3 + F_4 + F_5 - F_6,$$

where

$$F_i = \exp(-(2+\delta_i)M^2) \quad i = 1, \ldots, 6$$

with
$$\delta_1 = (2p_1-1)^2[2p_1(1-p_1)]^{-1},$$

$$\delta_2 = 2(2-3p_1)^2[(1+9p_1)(1-p_1)]^{-1},$$

$$\delta_3 = 2(9p_1^2-8p_1+3)[(1+9p_1)(1-p_1)]^{-1},$$

$$\delta_4 = \begin{cases} [2p_1(1-p_1)]^{-1} - 2 & \text{when } p_1 \in (¼, ½) \\ 2(3-8p_1)/(1+8p_1) & \text{when } p_1 \in (0, ¼] \end{cases}$$

$$\delta_5 = 2(1 - p_1)/(1 + p_1)$$

$$\text{and } \delta_6 = \delta_5.$$

Let $\delta := \min(\delta_1, \delta_2, \ldots, \delta_6)$. then (2.12) $\leq O\exp(-(2 + \delta)M^2)$, $\delta > 0$. This inequality provides an upper bound for $\Pr(B_2 \cap B_3)$.

To compute a bound for $\Pr(B_i \cap B_j)$ we will use a new description of the intervals in $\mathscr{C}_k$. Let $\mathscr{C}_k'$ be the class obtained by reordering the subintervals of X as follows. W.l.o.g. assume k is odd. Reparametrize the sets $A_4, A_5, \ldots, A_k$, for k odd, so that $0 < t_1 < t_2 < t_4 < t_{2m-2} < t_{2m} < t_{k-1}$ and $0 < t_1 < t_3 < t_5 < t_{2m-1} < t_{2m+1} < t_k$. Then the sets $A_2, A_4, \ldots$ form one branch of the treelike ordering and the sets $A_3, A_5, \ldots$ form the other branch.

Insert Diagram 1

Thus the event $B_2 \cap B_3$ occurs on $\mathscr{C}_k$ and $\mathscr{C}_k'$ with equal probability.
Hence (2.13)

$$\Pr(B_2 \cap B_3) \leq \Pr((B_2 \cup \cdots \cup B_{k-1}) \cap (B_3 \cup \cdots \cup B_k))$$

for which this intersection probability can be written

$$\Pr(B_2 \cup \cdots \cup B_{k-1}) + \Pr(B_3 \cup \cdots \cup B_k)$$
$$-\Pr(B_2 \cup B_3 \cup \cdots \cup B_{k-1} \cup B_k).$$

To generalize our argument to include the case of pairwise intersections of the events B_i and B_j, $i \neq j$ we note the restrictions on i and j occur in $\binom{k-1}{2}$ ways. Therefore, $\Pr(\cup B_i) \geq (k-1)e^{-2M^2} - \binom{k-1}{2} O[\exp(-(2+\delta)M^2)]\ \delta > 0,$
or passing to the limit as $M \to \infty$

$$\Pr(\cup B_i) \sim (k-1)e^{-2M^2}$$

which completes the proof of this theorem. □

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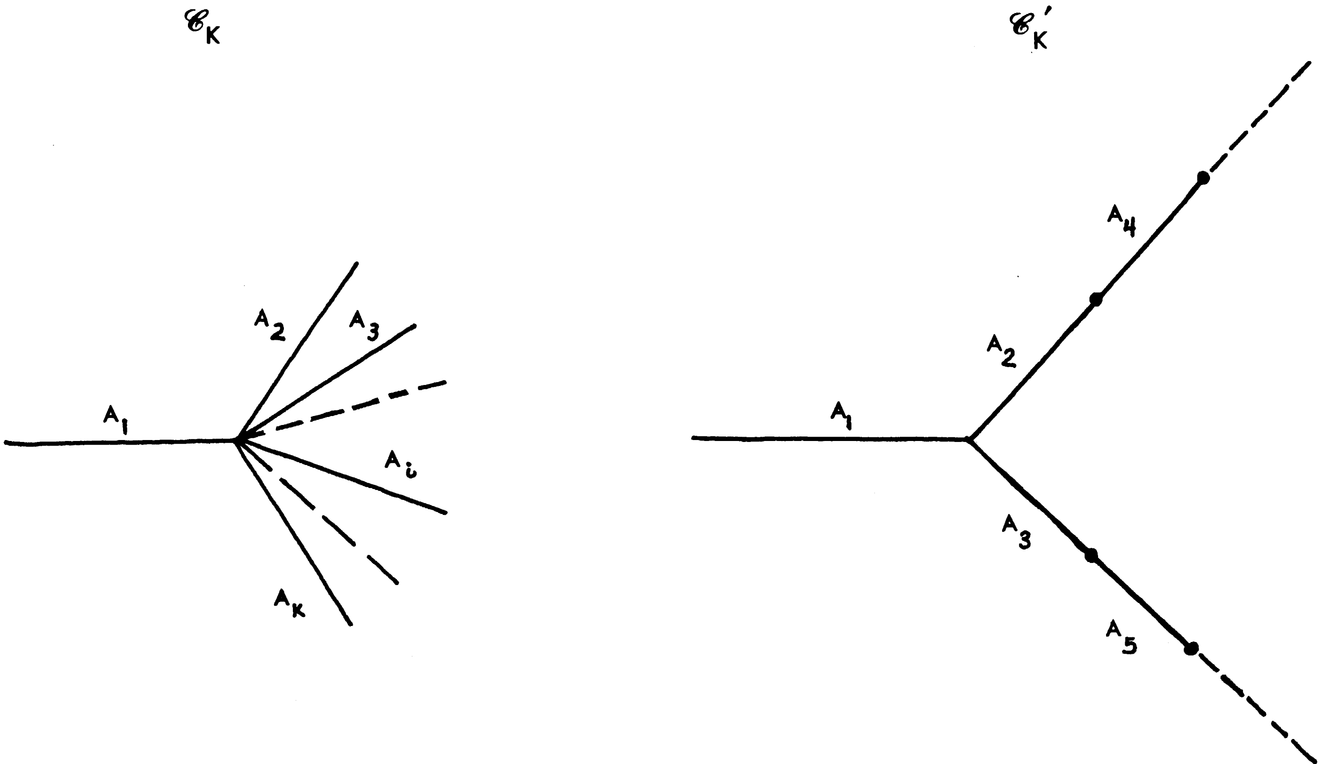
Diagram 1



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