An Asymptotic Expansion for the Mean of the Passage-Time Distribution of Integrated Brownian Motion.

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Abstract

The paper considers the first-passage time problem for an *integrated* Brownian motion process in the presence of two fixed boundaries located at $\pm b$. For a process starting at $X(0) = x$ with $V(0) = v$, where $V(t) = dX(t)/dt$ is Brownian motion, an asymptotic expansion (as $v \to \infty$) is developed for the mean of the passage time distribution. A truncated version of this expansion is proposed as an approximation to the true mean first passage time. In an extensive series of simulations the approximation is found to perform well even for small v.

Key words: Mean first-passage time, integrated Brownian motion, two barrier problem, asymptotic expansion.

AMS(MOS) subject classification: 60J65, 60J70.

1. Introduction

In many physical systems, chemical reactions and economic models (see e.g. Oppenheim et. al (1977) van Kampen (1981), Ross (1989)) a state variable $X(t)$ evolves according to the stochastic differential equation

$$\frac{dX(t)}{dt} = V(t) + F(X(t))$$

where $F(X(t))$ represents the total of deterministic forces acting on the state variable at time t and $V(t)$ is a Wiener process. Frequently, in these systems the question arises when the state variable first reaches an upper or lower threshold, exits from an interval, crosses one of two boundaries etc. The present paper makes a contribution to this problem context in the absence of deterministic forces and aims to approximate the expected time until thresholds are reached, boundaries are crossed, etc. More generally still, the problem under consideration has applications to any situation where the rate of change of a stochastic process (rather than the process itself) follows a Wiener process. For example, when modelling the price $X(t)$ of a certain stock it is sometimes assumed in econometrics that the rate of change of $X(t)$ equals the current inflation rate which in turn is assumed to behave as a Wiener process (Ross (1989)). The question when the stock price $X(t)$ first leaves a specified interval given that the inflation rate is $V(0) = v$ leads to the above first-passage problem. Another example is given by the evolution of the (one-dimensional) position process $X(t)$ of a particle

driven by white noise. Again, the expected time until exit from a prescribed interval may be approximated by the results of this paper.

First-Passage Time Problems (henceforth referred to as FPT problems) for these *integrated* Markov processes (of which $X(t)$ is an example) are unsolved in even the simplest cases (see e.g. Abrahams (1984)) Part of the complication is due to the fact that the integral of a Markov process is no longer Markovian. This implies, among other things, that the technology involving Kolmogorov backward and forward equations can no longer be used directly. However, the two-dimensional process

$$(dX(t)/dt,\ X(t))$$

is Markovian and this provides a starting point for analysis.

Unfortunately it turns out, as we will see, that the boundary and initial conditions provided by the respective contexts are often insufficient to constitute a well-posed problem in two dimensions. This impass can be circumvented by introducing a moving boundary with a specification and then letting the boundary approach infinity in a controlled fashion. We then use techniques for global asymptotic analysis on the resulting system hence obtaining asymptotic expansion for the mean first passage time. This is demonstrated in Sections 2 and 3. Section 4 reports the results of a series of simulations. The truncated asymptotic expansion for the mean turns out to be an extremely accurate approximation. For related work see Lefebure (1989), Lachal

(1990).

2. The Model

Consider the system

$$dX(t) = V(t)\,dt, \quad V(t) = 3w(t) + v$$

where $w(t)$ is a standard Wiener process. Let b be positive and let

$$\tau(b,v) = \inf\{t: X(t) = b\} \wedge \inf\{t: X(t) = -b\}$$

be the first time the process hits $+b$ or $-b$. We will approximate $E(\tau(b,v))$ for fixed b and v. Towards this end consider the bivariate Markov process

$$\{X(s), V(s), s \geq 0\} \quad X(0) = x, \quad V(0) = v$$

with two absorbing planes at $X = b$ and $X^* = -b$ and let $p(x_t, v_t, t, x, v)$ be the probability density associated with:

$X(0) = x$, $V(0) = v$, $X(t) = x_t$, $V(t) = v_t$ and $\{X(s), V(s)\}$ has not hit either of the absorbing planes in $[0, t)$.

The density p depends on b, of course, but still satisfies both the Kolmogorov forward and backward equations. Define, for positive a

$$P(a,t,x,v) = \int_{-a}^{a}\int_{-\infty}^{\infty} p(x_t, v_t, t, x, v)\,dv_t\,dx_t$$

so that for the density $f(t,x,v,b)$ of $\tau(b,v)$

$$f(t,x,v,b) = -\frac{\partial}{\partial t} P(b, t, x, v).$$

Write

$$\psi_{s,b}(x,v) = \int_0^\infty e^{-st} f(t,x,v,b)\,dt \tag{2.1}$$

and for $t \le \tau(b,v)$ introduce the random density $f(t', X(t), V(t), b)$ and the random Laplace transform $\psi_{s,b}(X(t), V(t))$. Also introduce the notation

$$\begin{aligned}
\partial_1 \psi_{s,b}(x,v) &= (\partial/\partial x)\,\psi_{s,b}(x,v) \\
\partial_2 \psi_{s,b}(x,v) &= (\partial/\partial v)\,\psi_{s,b}(x,v) \\
\partial_{22} \psi_{s,b}(x,v) &= (\partial^2/\partial v^2)\,\psi_{s,b}(x,v).
\end{aligned}$$

Then we have the following

Theorem 1: With the Laplace transform $\psi_{s,b}(\cdot,\cdot)$ of (2.1) the process $\{\Gamma(t) = \exp(-st)\,\psi_{s,b}, (X(t), V(t));\ 0 \le t \le \tau(k,v)\}$ for positive s has stochastic Itô differential

$$\begin{aligned}
d\Gamma(t) &= \exp(-st)\,\partial_2 \psi_{s,b}(X(t), V(t))\,\sigma\,dW(t) \\
&+ \exp(-st)\Big[\frac{\sigma^2}{2}\partial_{22}\psi_{s,b}(X(t), V(t)) + V(t)\,\partial_1 \psi_{s,b}(X(t), V(t)) \\
&+ s\,\psi_{s,b}(X(t), V(t))\Big]\,dt
\end{aligned}$$

where $W(t)$ is standard Brownian motion.

Proof: By Itô's formula or the following argument:

$$\begin{aligned}
\Delta\Gamma &= \Gamma(t+\Delta t) - \Gamma(t) \\
&= (\exp(-s(t+\Delta t) - \exp(-st))\,[\psi_{s,b}(X(t+\Delta t), V(t+\Delta t) \\
&+ (\exp(-s(t+\Delta t)) - \exp(-st))\,\psi_{s,b}(X(t), V(t)) \\
&+ \exp(-st)\,[\psi_{s,b}(X(t+\Delta t)), V(t+\Delta t)) - \psi_{s,b}(X(t), V(t)).
\end{aligned}$$

Retaining terms of order Δt only

$$\begin{aligned}\Delta\Gamma &= -s\exp(-st)\,\psi_{s,b}(X(t),V(t))\,\Delta t \\ &+ \exp(-st)\,[\,\partial_1\psi_{s,b}(X(t),V(t))\,V(t)\,\Delta t + \partial_2\psi_{s,b}(X(t),V(t))\,\partial V(t)\,] \\ &+ \frac{1}{2}\partial_{22}\psi_{s,b}(X(t),V(t))\,(\Delta V(t))^2 .\end{aligned}$$

Since $V(t) = \sigma w(t) + v$ we have

$$\begin{aligned}\Delta V(t) &= \sigma\Delta W(t) \\ (\Delta V(t))^2 &= \sigma^2(\Delta W(t))^2 = \sigma^2\Delta t\end{aligned}$$

and the claim follows by replacing infinitesimals by differentials.

Theorem 2: The process $\{\Gamma(t), 0 \le t \le \tau(b,v)\}$ as defined in Theorem 1 is a (local) martingale relative to the filtration generated by $\{\Gamma(s), s \le t\}$.

Proof: Theorem 2 follows easily from stopping time arguments. Now, combining Theorems 1 and 2 and using

$$E(\tau(b,v)) = (\partial/\partial s)\,\psi_{s,b}(x,v)\,|_{s=0}$$

and reparametrizing the space variable in such a way that z now denotes the distance between the starting position and the boundary on the right we obtain

$$\frac{\sigma^2}{2}(\partial^2/\partial v^2)\,E(\tau(z,v)) - v(\partial/\partial z)\,E(\tau(z,v)) = -1 \tag{2.3}$$

$$E(\tau(0,v)) = 0 \quad \text{for} \quad v > 0 \tag{2.4a}$$

$$E(\tau(2b,v)) = 0 \quad \text{for} \quad v < 0 \tag{2.4b}$$

if we can demonstrate that the expectation exists. However, this is so, since $X(t)$ for all t is normally distributed with mean $b - z + vt$ and variance $3^{-1}\sigma^2 t^3$ and

$$\begin{aligned}P(\tau(z,v) > t) &\le P(X(t) < x + z) - P(X(t) < -x - z) \\ &\le \min\{1, O(t^{-3/2})\}\end{aligned}$$

so that $P(\tau(z,v) > t)$ is integrable. The system (2.3) (2.4) is a nonhomogeneous parabolic partial differential equation. the handling of which is complicated due to its lack of sufficient boundary or initial information. Although (2.4) is obvious since

$$\lim_{z\to 0^+}\lim_{v\to v^*} E(\tau(z,v)) = E(\tau(0,v^*)) \;=\; 0 \quad \text{for all } v^* > 0,$$

it remains unclear how this limit behaves for $v^* < 0$. On the other hand, to obtain

$$\lim_{z\to z^*}\lim_{v\to v^*} E(\tau(z,v))$$

for a given v^* and all $z^* \in (0,2b)$ is as difficult as the original FPT problem itself. Therefore, both the boundary-value problem and the initial boundary-value problem are underspecified and it seems that we have arrived at an impasse. In the following section we demonstrate how to circumvent this difficulty and derive an asymptotic expansion for the mean FPT.

3. An asymptotic expansion for the mean FPT

Consider the perturbed process $(X^w(t), V(t))$ with

$$X^w(t) \;=\; \begin{cases} X(t) & \text{for } t \le t_0 \\ X(t_0) + V(t_0)(t_0 - t) & \text{for } t > t_0 \end{cases}$$

where t_0 is the stopping time

$$t_0 \;=\; \inf\{t\colon V(t) = w\} \;\wedge\; \inf\{t\colon V(t) = -w\}$$

for some fixed $w \ge |v|$. Also, define $\tau^w(z,v)$ and ψ^w, the hitting time and its Laplace transform, respectively, of $(X^w(t), V(t))$ $\{-b,+b\} \times \mathbf{R}$. Then, with probability one,

$\tau^w(z,v) \to \tau(z,v)$ as $w \to \infty$ and $E(\tau^w(z,v)) \to E(\tau(z,v))$. Hence, we attempt to solve

$$\frac{\sigma^2}{2}\frac{\partial^2 E(\tau^w(z,v))}{\partial v^2} - v\frac{\partial E(\tau^w(z,v))}{\partial z} = -1 \tag{3.1}$$

with

$$\begin{aligned} &E(\tau^w(0,v) = 0 \text{ for } v > 0 \\ &E(\tau^w(2b,v)) = 0 \text{ for } v < 0 \\ &E(\tau^w(z,w)) = zw^{-1} = E(\tau^w(2b-z,-w)) \text{ for } z < 2b \\ &E(\tau^w(z,-w)) = (2b-z)w^{-1} = E(2b-z,w) \text{ for } z < 2b \end{aligned} \tag{3.2}$$

in combination with the limit as $w \to \infty$ is taken. A possible approach to the system (3.1), (3.2) is the use of perturbation methods to transform the problem into a boundary-layer problem for the Laplace transform of $E(\tau^w(z,v))$. One then aims to approximate the system by a sequence of equations valid in the inner region and near the boundaries and combines the respective solutions by asymptotic matching techniques to obtain a globally valid approximation. Since we were unable to carry through this program in a fashion resulting in a mathematically simple global approximation, we decided on a different and simpler strategy.

Let

$$\hat{m}^w(c,v) = \int_0^\infty e^{-cz} E(\tau^w(z,v))\,dz$$

and

$$y = 2^{1/3} c^{1/3} \sigma^{-2/3} v$$

$$y_0 = 2^{1/3} c^{1/3} \sigma^{-2/3} w.$$

The homogeneous part of the transformed system

$$\frac{d^2 \hat{m}^{y_0}(c,y)}{dy^2} - y \hat{m}^{y_0}(c,y) = -2^{1/3} \sigma^{-2/3} c^{-5/3} \tag{3.3}$$

$$\hat{m}^{y_0}(c,y_0) = 2^{1/3} \sigma^{-2/3} c^{-5/3} y_0^{-1} \tag{3.4}$$

is a one-dimensional Schrödinger-equation which becomes amenable to techniques of global asymptotic analysis such as the method of dominant balance (for an irregular singular point at ∞).

The basic strategy is to first peel off the leading asymptotic behavior then, after having removed this, to determine the leading behavior of the remainder, and so on. To initiate this procedure set

$$\frac{d^2 \hat{m}^{\infty}(c,y)}{dy^2} \sim 0 \quad \text{as} \quad y \to \infty, \tag{3.5}$$

where here by the notation $f(x) \sim g(x)$ as $x \to x_0$ (''f is asymptotic to g as $x \to x_0$'') for functions f and g is meant that $\lim_{x \to x_0} f(x)/g(x) = 1$ if $g(x) \neq 0$) and $\lim_{x \to x_0} f(x) = 0$ if $g(x) \equiv 0$. If $g(x) = \sum_{n=0}^{\infty} a(n) x^{-n}$ is a power series then by $f(x) \sim g(x)$ as $x \to \infty$ is meant that $\lim_{x \to \infty} (f(x) - \sum_{n=0}^{N} a(n) x^{-n}) / x^{-N} = 0$ for every N. This yields

$$\hat{m}^{\infty}(c,y) \sim 2^{1/3} \sigma^{-2/3} c^{-5/3} y^{-1} \quad \text{as} \quad y \to \infty.$$

Corrections to this leading term are determined by setting

$$\hat{m}^{\infty}(c,y) = 2^{1/3} \sigma^{-2/3} c^{-5/3} (y^{-1} + \varepsilon(y)), \quad y \to \infty$$

where the correction term $\varepsilon(y)$ is of smaller order than y^{-1} and satisfies

$$\frac{d^2 \varepsilon(y)}{dy^2} + \frac{2}{y^3} = y\,\varepsilon(y).$$

Setting $\frac{d^2/dy^2)\,\varepsilon(y)}{dy^2} \sim 0$ (as $y \to \infty$) gives $\varepsilon(y) \sim 2y^{-4}$ and. continuing in this fashion the full asymptotic power series expansion:

$$\hat{m}^{\infty}(c,y) \sim 2^{1/3}\sigma^{-2/3}c^{-5/3}\sum_{n=0}^{\infty}\frac{(3n)!}{3^n n!}y^{-3n-1},$$

and hence

$$E(\tau(z,v)) \sim \sum_{n=0}^{\infty}\frac{(3n)!\,\sigma^{2n}z^{n+1}}{n!\,(n+1)!\,6^n v^{3n+1}}. \tag{3.6}$$

It is easily checked by differentiating termwise that $E(\tau(z,v))$ formally satisfies (2.3), (2.4a) and (2.4b) as $v \to -\infty$, although the sum in (3.6) does not converge for any nonzero value of $\sigma^2 z v^{-3}$. It is well-known that many problems in perturbation analysis and the theory of dominant balance lead to such divergent series. These series are still useful; under certain conditions formal solutions (such as divergent series) of differential equations are asymptotic expansions of actual solutions. In fact one can even go a step further: Typically, optimally truncated divergent series are very good approximations for these actual solutions (see Bender and Orszag (1978)). We chose to use the first 3 terms of the divergent series as our approximation

$$E^*(\tau(z,v)) = \frac{z}{v} + \frac{\sigma^2}{2}\frac{z^2}{v^4} + \frac{5\sigma^4}{3}\frac{z^3}{v^7}. \tag{3.7}$$

4. Simulations

It seems complicated to find an analytic bound for the error term introduced by the various approximations and asymptotic expansion arguments which led to (3.7). Therefore, we performed an extensive series of simulations, for different values of σ^2 and several distances z and initial velocities v, to compare the approximation (3.7) with simulated sample mean first-passage times and their sample standard deviations. These simulations indicate that the approximation performs well whenever $\sigma^2 z v^{-3} < 1/4$ (i.e. even for positive starting velocities which in view of the derivation is expected to be the last accurate case for the approximation). For the given values of σ^2 this is the case for all the listed values of z and v in Tables 1-4.

The basis for simulations is provided by

$$X(t + \Delta t) = X(t) + \int_t^{t+\Delta t} V(s)\,ds, \tag{4.1}$$

but of course, it is impossible to obtain complete (for all $s \geq 0$) realizations of the velocity process $V(s)$. Instead, we deduce the entire trajectory from the subset $V(k\Delta t)$, $k = 0,1,2,\ldots$. In view of this an assumption is necessary that governs the behavior of the velocity process between discrete time points $k\Delta t$. The only assumption which makes sense both physically and analytically is to require constant acceleration during $[k\Delta t, (k+1)\Delta t)$ for all k. This leads to position being a quadratic spline and hence necessitates a quadratic interpolation scheme to obtain the approximate first passage-time of the realization. Numerically, the effect of the constant acceleration

assumption is the approximation of the integral in (4.1) by the trapezoidal rule of quadrature.

If $X((k-1)\Delta t) = x_1 - b < 0$, $V((k-1)\Delta t) = v_1$ and $V(k\Delta t) = v_2$ then by quadratic interpolation first-passage occurs during an increment at time $((k-1)\Delta t + \tau_0)$ with

$$\tau_0 = (-v_1 + (v_1^2 + 2X_1(v_2 - v_1)/\Delta t)^{1/2}\Delta t)/(v_2 - v_1)$$

if either $(v_1 + v_2)\Delta t/2 \geq x_1$ or $v_1^2 + 2x_1(v_2 - v_1)/\Delta t \geq 0$ with $v_1 > 0 > v_2$.

The simulations were performed in the Statistical Laboratory at Queens University in Kingston, Canada. At time $t = 0$ we started 2000 realizations of the bivariate process $(X(t), V(t))$ with $X(0) = x = b - z$ and $V(0) = v$. The time increment Δt was chosen in such a way that always of the order of one thousand steps were needed for the particle to reach the boundary. This being a compromise between desired accuracy and computation cost. For each realization it was determined, via quadratic interpolation, when it crossed boundaries for $z = 1,2,\ldots$, and ensemble averages were taken.

For $\sigma = .1, .3, .5, .7$ the results of the simulations are reported in Tables 1,2,3,4 at the end of this section. The constant b was set equal to 50. The following notation is used (for convenience the dependence on z, v, σ^2 will not be explicitly indicated):

m_s: sample averages of simulated first-passage times for given z, v, σ^2.

$$A_1 = \frac{z}{v}, \quad A_2 = \frac{\sigma^2}{2}\frac{z^2}{v^4}, \quad A_3 = \frac{5\cdot\sigma^4}{3}\frac{z^3}{v^7}$$

$$M_1 = 10^3 \cdot \frac{m_s - A_1}{A_1}, \quad M_2 = 10^3 \cdot \frac{m_s - (A_1 + A_2)}{A_1}, \quad M_3 = 10^3 \cdot \frac{m_s - (A_1 + A_2 + A_3)}{A_1}$$

$SD(M_i)$: sample standard deviation of M_i, i = 1,2,3.

$$t_i = \frac{M_i}{SD(M_i)}, \quad i = 1,2,3.$$

In summary, the findings are as follows:

1. In 105 out of 224 cases (i.e. the total of different combinations of z, v, σ^2), $|t_1| \leq 2.00$ and in 21 cases $t_1 \leq 0.00$, indicating clearly that the 1st order approximation A_1 tends to underestimate the mean first-passage time. This is confirmed by the following summary statistics of t_1:

   Mean = 2.57, STDEV = 4.15, SEMEAN = 0.14

2. In 197 out of 224 cases, $|t_2| \leq 2.00$ and in 118 cases $t_2 \leq 0.00$. Hence, in 88% of the cases the 2nd order approximation is within 2 standard errors of the simulated mean. Also, the frequencies of underestimation and overestimation (relative to the simulated sample means) are about equal and the summary statistics of t_2 are:

   MEAN = -0.04, STDEV = 1.27, SEMEAN = 0.09

3. In 199 out of 224 cases, $|t_3| \leq 2.00$ and in 126 cases $t_3 \leq 0.00$. The 3rd order adjustment terms in the approximation and hence the difference between t_2 and t_3 are small, especially for large v and small z and σ^2. Hence the precision of the experiments (i.e., 2000 particles for given σ^2, z, v) is not sufficiently high to determine whether the 3rd order approximation improves over the 2nd order

approximation. The summary statistics of t_3 are:

MEAN = -0.22, STDEV = 1.32, SEMEAN = 0.09

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TABLE 1: M_i, t_i for $i=1,2,3$; $\sigma=.1$,

and various z, v

	v = 2						v = 3					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	-1.4	-7.61	-7.64	-0.30	-1.68	-1.69	5.3	3.42	3.42	2.12	1.38	1.3
2	6.1	-6.41	-6.51	0.95	-1.00	-1.02	11.4	7.70	7.69	3.24	2.19	2.1
4	9.4	-15.59	-16.01	1.03	-1.71	-1.75	21.3	13.90	13.87	4.13	2.69	2.6
6	14.3	-23.18	-24.12	1.28	-2.07	-2.16	27.1	15.97	15.89	4.24	2.50	2.4
8	26.0	-24.02	-25.69	1.99	-1.84	-1.97	33.1	18.26	18.12	4.49	2.48	2.4
12	54.9	-20.13	-23.88	3.33	-1.22	-1.45	42.4	20.19	19.86	4.68	2.23	2.1
16	83.8	-16.18	-22.85	4.33	-0.84	-1.18	50.0	20.33	19.75	4.77	1.94	1.8
20	110.0	-14.97	-25.38	5.00	-0.68	-1.15	56.1	19.09	18.18	4.78	1.63	1.5

	v = 4						v = 5					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	-1.0	-1.82	-1.82	-0.68	-1.20	-1.20	-0.8	-1.20	-1.20	-0.72	-1.07	-1.0
2	-0.6	-2.12	-2.12	-0.25	-0.95	-0.95	-1.2	-1.98	-1.98	-0.72	-1.22	-1.2
4	2.1	-1.07	-1.08	0.63	-0.33	-0.33	-0.5	-2.06	-2.06	-0.20	-0.90	-0.9
6	4.7	0.05	0.03	1.20	0.01	0.01	2.4	-0.03	-0.03	0.84	-0.01	-0.0
8	8.3	2.09	2.06	1.81	0.45	0.45	4.0	0.77	0.76	1.21	0.24	0.2
12	12.9	3.49	3.43	2.26	0.61	0.60	4.8	0.03	0.01	1.20	0.01	0.0
16	16.8	4.31	4.21	2.54	0.65	0.64	5.4	-0.96	-0.99	1.17	-0.21	-0.21
20	19.4	3.78	3.62	2.62	0.51	0.49	5.6	-2.40	-2.44	1.08	-0.46	-0.47

	v = 6						v = 7					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	0.6	0.33	0.33	0.70	0.41	0.41	1.0	0.82	0.82	1.55	1.32	1.32
2	0.7	0.22	0.22	0.56	0.18	0.18	1.1	0.78	0.78	1.14	0.83	0.83
4	0.1	-0.86	-0.86	0.04	-0.49	-0.49	0.9	0.30	0.30	0.64	0.22	0.22
6	0.0	-1.39	-1.39	0.00	-0.64	-0.64	0.8	-0.11	-0.12	0.45	-0.07	-0.07
8	-0.4	-2.23	-2.23	-0.15	-0.89	-0.89	1.1	-0.08	-0.08	0.55	-0.04	-0.04
12	-0.1	-2.91	-2.92	-0.04	-0.94	-0.95	1.5	-0.27	-0.27	0.61	-0.11	-0.11
16	1.0	-2.70	-2.71	0.28	-0.76	-0.76	1.7	-0.66	-0.66	0.59	-0.23	-0.23
20	2.2	-2.39	-2.41	0.56	-0.60	-0.60	2.1	-0.79	-0.80	0.68	-0.25	-0.25

	v = 8						v = 9					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	-0.2	-0.26	-0.26	-0.30	-0.48	-0.48	0.4	0.28	0.28	0.87	0.70	0.70
2	0.8	0.65	0.65	1.17	0.90	0.90	0.2	0.03	0.03	0.28	0.05	0.05
4	1.5	1.15	1.15	1.38	1.03	1.03	-0.0	-0.28	-0.28	-0.01	-0.31	-0.31
6	1.6	1.05	1.05	1.20	0.77	0.77	-0.1	-0.50	-0.50	-0.08	-0.45	-0.45
8	1.3	0.53	0.53	0.83	0.33	0.33	0.1	-0.40	-0.40	0.11	-0.31	-0.31
12	1.2	-0.01	-0.01	0.60	-0.00	-0.00	0.8	-0.02	-0.02	0.51	-0.02	-0.02
16	1.8	0.17	0.17	0.78	0.08	0.07	1.4	0.27	0.26	0.75	0.15	0.15
20	1.9	-0.03	-0.03	0.78	-0.01	-0.01	1.8	0.44	0.44	0.88	0.21	0.21

TABLE 2: M_i, t_i for i=1,2,3; σ=.3, and various z, v

	v = 2						v = 3					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	33.8	-22.48	-24.60	2.4	-1.58	-1.72	3.2	-13.44	-13.63	0.4	-1.84	-1.86
2	81.0	-31.52	-39.96	3.8	-1.49	-1.89	20.0	-13.38	-14.12	1.9	-1.27	-1.34
4	238.4	13.44	-20.31	6.3	0.36	-0.54	51.0	-15.68	-18.65	3.4	-1.04	-1.23
6	645.0	307.53	231.60	4.1	1.98	1.49	87.5	-12.50	-19.17	4.5	-0.64	-0.99
8	846.9	396.91	261.91	6.1	2.85	1.88	126.4	- 6.92	-18.78	5.4	-0.30	-0.80
12	1601.0	926.05	622.29	4.3	2.50	1.68	209.9	9.92	-16.75	6.8	0.32	-0.54
16	1945.2	1045.22	505.22	6.5	3.48	1.68	313.0	46.32	- 1.09	7.5	1.11	-0.03
20	3063.4	1938.43	1094.68	4.0	2.53	1.43	913.1	579.75	505.67	2.5	1.61	1.40

	v = 4						v = 5					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	11.0	4.01	3.98	2.2	0.81	0.81	7.9	4.35	4.34	2.3	1.26	1.25
2	16.5	2.44	2.31	2.3	0.35	0.33	15.8	8.63	8.59	3.3	1.78	1.77
4	26.5	- 1.60	- 2.13	2.7	-0.16	-0.22	28.2	13.85	13.71	4.1	1.99	1.97
6	33.2	- 9.00	-10.19	2.7	-0.74	-0.84	40.3	18.67	18.36	4.6	2.14	2.11
8	43.3	-12.92	-15.02	3.1	-0.91	-1.06	50.2	21.45	20.89	4.9	2.11	2.05
12	67.9	-16.46	-21.21	3.8	-0.93	-1.20	69.1	25.91	24.66	5.5	2.05	1.95
16	99.8	-12.66	-21.10	4.8	-0.61	-1.01	85.0	27.41	25.19	5.7	1.85	1.70
20	133.9	- 6.69	-19.87	5.6	-0.28	-0.83	102.3	30.29	26.84	6.2	1.82	1.61

	v = 6						v = 7					
z	M1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	1.6	- 0.50	- 0.51	0.6	-0.20	-0.20	- 0.7	- 2.02	- 2.02	-0.3	-0.99	-0.99
2	10.1	5.90	5.89	2.7	1.57	1.57	- 0.2	- 2.81	- 2.81	-0.1	-0.94	-0.95
4	16.4	8.07	8.02	3.1	1.53	1.52	2.5	- 2.79	-2.81	0.6	-0.66	-0.67
6	22.9	10.39	10.29	3.5	1.60	1.59	5.0	- 2.85	- 2.90	1.0	-0.55	-0.56
8	25.8	9.15	8.96	3.4	1.22	1.19	9.1	- 1.40	- 1.47	1.5	-0.23	-0.24
12	31.9	6.87	6.45	3.5	0.74	0.70	17.1	1.33	1.17	2.3	0.17	0.15
16	41.8	8.51	7.76	3.9	0.80	0.73	23.4	2.40	2.10	2.7	0.28	0.24
20	52.4	10.76	9.60	4.4	0.90	0.80	28.7	2.52	2.07	2.9	0.26	0.21

	v = 8						v = 9					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	0.0	- 0.88	- 0.88	0.0	-0.55	-0.55	2.9	2.25	2.25	2.0	1.60	1.60
2	0.1	- 1.68	- 1.68	0.0	-0.70	-0.70	2.0	0.78	0.78	1.0	0.39	0.39
4	- 1.3	- 4.85	- 4.86	-0.4	-1.43	-1.43	2.6	0.13	0.13	0.9	0.05	0.04
6	- 0.4	- 5.65	- 5.67	-0.1	-1.36	-1.37	4.3	0.64	0.63	1.2	0.18	0.18
8	0.9	- 6.11	- 6.14	0.2	-1.28	-1.28	5.0	0.09	0.08	1.2	0.02	0.02
12	3.4	- 7.13	- 7.21	0.6	-1.22	-1.24	7.1	- 0.29	- 0.32	1.4	-0.06	-0.06
16	6.4	- 7.66	- 7.79	1.0	-1.14	-1.16	9.3	- 0.59	- 0.65	1.6	-0.10	-0.11
20	10.2	- 7.37	- 7.58	1.4	-0.99	-1.01	12.2	- 0.10	- 0.21	1.8	-0.02	-0.03

TABLE 3: M_i, t_i for i=1,2,3; σ=.5,

and various z, v

	v = 4						v = 5					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t
1	21.2	1.71	1.45	2.61	0.21	0.18	4.0	- 6.00	- 6.07	0.72	-1.07	-1
2	47.3	8.24	7.22	4.01	0.70	0.61	10.5	- 9.55	- 9.82	1.31	-1.19	-1
4	85.2	7.05	2.96	4.92	0.41	0.17	27.5	-12.55	-13.62	2.35	-1.07	-1
6	130.2	13.01	3.86	5.96	0.60	0.18	49.9	-10.06	-12.46	3.41	-0.69	-(
8	173.3	16.08	- 0.20	6.56	0.61	-0.01	71.5	- 8.49	-12.76	4.19	-0.50	-(
12	269.9	35.53	- 1.09	7.57	1.00	-0.03	122.6	2.64	- 6.96	5.73	0.12	-(
16	409.9	97.42	32.32	7.96	1.89	0.63	185.8	25.82	8.75	7.16	0.99	(
20	5121.1	4730.44	4628.71	1.47	1.36	1.33	245.9	45.94	19.28	8.00	1.49	(

	v = 6						v = 7					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t
1	8.8	3.05	3.03	2.06	0.71	0.71	1.2	- 2.39	-2.40	0.36	-0.70	-(
2	14.9	3.33	3.24	2.41	0.54	0.52	4.3	- 2.96	-2.99	0.86	-0.59	-(
4	26.8	3.66	3.30	3.09	0.42	0.38	10.2	- 4.39	-4.53	1.44	-0.62	-(
6	40.8	6.07	5.26	3.78	0.56	0.49	14.9	- 6.93	-7.25	1.70	-0.79	-0
8	56.0	9.74	8.31	4.44	0.77	0.66	20.8	- 8.38	-8.94	2.03	-0.82	-0
12	87.2	17.78	14.57	5.47	1.12	0.91	37.0	- 6.77	-8.05	2.92	-0.54	-0
16	123.1	30.49	24.78	6.44	1.60	1.30	58.3	- 0.04	-2.31	3.91	-0.00	-0
20	159.3	43.55	34.62	7.17	1.96	1.56	76.9	3.97	0.43	4.54	0.23	0

	v = 8						v = 9					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t
1	3.0	0.60	0.59	1.13	0.22	0.22	- 0.0	- 1.73	- 1.73	-0.00	-0.78	-0.
2	6.2	1.36	1.34	1.55	0.34	0.33	2.4	- 1.05	- 1.06	0.72	-0.32	-0.
4	13.3	3.53	3.47	2.32	0.62	0.61	3.4	- 3.43	-3.46	0.73	-0.73	-0.
6	22.3	7.62	7.48	3.09	1.06	1.04	5.3	- 4.96	- 5.03	0.91	-0.85	-0.
8	30.2	10.70	10.44	3.59	1.27	1.24	8.9	- 4.83	- 4.95	1.30	-0.71	-0.
12	43.2	13.86	13.28	4.16	1.34	1.28	16.0	- 4.57	- 4.85	1.88	-0.53	-0.
16	56.9	17.80	16.79	4.71	1.47	1.39	22.4	- 5.00	- 5.50	2.26	-0.50	-0.
20	69.2	20.42	18.83	5.09	1.50	1.38	27.9	- 6.44	- 7.22	2.51	-0.58	-0.

TABLE 4: M_i, t_i for $i=1,2,3$; $\sigma=.7$,

and various z, v

	v = 4						v = 5					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	15.4	- 22.92	- 23.90	1.34	-2.00	-2.09	11.2	- 8.40	- 8.66	1.41	-1.05	-1.
2	52.8	- 23.76	- 27.61	3.12	-1.41	-1.64	20.3	- 18.93	- 19.95	1.75	-1.64	-1.
4	107.6	- 45.49	- 61.13	4.42	-1.87	-2.51	53.5	- 24.88	- 28.97	3.14	-1.46	-1.
6	158.6	- 71.13	-106.30	5.28	-2.37	-3.54	83.9	- 33.66	- 42.88	4.00	-1.61	-2.
8	213.6	- 92.68	-155.21	6.00	-2.61	-4.36	128.6	- 28.19	- 44.58	5.25	-1.15	-1.
12	300.4	-158.96	-299.64	6.87	-3.63	-6.85	217.8	- 17.42	- 54.30	7.02	-0.56	-1.
16	3811.9	3199.31	2949.21	1.10	0.92	0.85	511.5	197.94	132.38	3.45	1.33	0.
20	3195.5	2429.93	2039.14	1.15	0.87	0.74	580.9	198.87	96.43	4.71	1.58	0.

	v = 6						v = 7					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	9.0	- 2.32	- 2.41	1.46	-0.38	-0.39	- 2.9	-10.02	-10.06	-0.59	-2.07	-2.
2	13.6	- 9.11	- 9.45	1.56	-1.04	-1.08	0.1	-14.19	-14.33	0.01	-2.08	-2.
4	34.6	-10.75	-12.12	2.77	-0.86	-0.97	8.3	-20.30	-20.85	0.84	-2.07	-2.
6	61.9	- 6.12	- 9.20	4.02	-0.40	-0.60	16.8	-26.10	-27.33	1.38	-2.14	-2.
8	88.8	- 1.96	- 7.45	4.94	-0.11	-0.41	32.1	-25.03	-27.21	2.26	-1.76	-1.
12	138.4	2.28	-10.07	6.05	0.10	-0.44	58.7	-26.97	-31.87	3.31	-1.52	-1.
16	197.1	15.58	- 6.39	6.94	0.55	-0.22	90.4	-23.89	-32.60	4.31	-1.14	-1.
20	262.7	35.90	1.59	7.57	1.04	0.05	121.8	-21.10	-34.70	5.07	-0.88	-1.

	v = 8						v = 9					
z	M_1	M_2	M_3	t_1	t_2	t_3	M_1	M_2	M_3	t_1	t_2	t_3
1	- 1.0	- 5.75	- 5.76	-0.24	-1.46	-1.46	8.0	4.64	4.63	2.48	1.44	1.4
2	- 4.4	-13.97	-14.03	-0.80	-2.52	-2.53	10.4	3.66	3.63	2.15	0.76	0.7
4	- 7.0	-26.18	-26.42	-0.87	-3.25	-3.28	18.0	4.52	4.40	2.65	0.66	0.6
6	- 2.5	-31.19	-31.74	-0.25	-3.15	-3.21	28.5	8.36	8.08	3.40	1.00	0.9
8	4.4	-33.89	-34.87	0.38	-2.93	-3.02	39.3	12.40	11.92	4.03	1.27	1.2
12	23.9	-33.54	-35.74	1.66	-2.33	-2.49	55.4	15.03	13.95	4.55	1.23	1.1
16	43.1	-33.51	-37.42	2.57	-2.00	-2.24	70.0	16.27	14.35	4.95	1.15	1.0
20	64.0	-31.74	-37.84	3.40	-1.69	-2.01	80.8	13.56	10.55	5.10	0.86	0.6