On the convergence of a maximal correlation algorithm with alternating projections

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Abstract

In this paper a maximal correlation algorithm using alternating projections is investigated. This algorithm was suggested by Breiman and Friedman (1985) without a proof. We prove the convergence of a slightly modified version of the algorithm. The convergence is exponentially fast.

1. Introduction.

Let $S_0, S_1, \ldots, S_k$ be linear subspaces of a Hilbert space H. Breiman and Friedman (1985) considered the following problem:

(1.1) Find, if exist, $s_i^* \in S_i$, $i = 0, 1, \ldots, k$ such that $\|s_0^*\| = 1$ and $\|\sum_0^k s_i^*\| = c_0 \equiv \min\{\|\sum_0^k s_i\| : s_i \in S_i, i = 0, \ldots, k, \|s_0\| = 1\}$.

If s_i^* exist then s_0^* and $\sum_1^k s_k^*$ is the pair of elements belonging to S_0 and $S_1 + S_2 + \cdots + S_k$ respectively with maximum "correlation" (or, with minimal angle between them). Breiman and Friedman (1985) considered the situation where Y, $X_1, \ldots, X_k$ are random variables defined on the same probability space and one looks for function $s_0, \ldots, s_k$ that maximize the correlation between $s_0(Y)$ and $\Sigma s_i(X_i)$.

Breiman and Friedman (1985) suggested the following algorithm for an approximate solution of (1.1). Let P_i be the orthogonal projection on S_i, $i = 0, \ldots, k$ and let

$$T = (I - P_k)(I - P_{k-1}) \cdots (I - P_1). \tag{1.2}$$

Then define

$$s_0^{(n+1)} = -P_0 \sum_1^k s_i^{(n)} / \|P_0 \sum_1^k s_i^{(n)}\|$$

$$\sum_0^k s_i^{(n+1)} = T^{m(n)}(s_0^{(n+1)} + \sum_1^k s_i^{(n)}) \tag{1.3}$$

where m(n) is appropriate large number and $\Sigma_{i=0}^k s_i^{(1)}$ is arbitrary. It is well known that $T^m \to I - P$ where P is the projection on $S = S_1 + \ldots + S_k$.

Breiman and Friedman (1985) suggested without a proof a "single-loop" algorithm which is more symmetrical with respect to $P_0, \ldots P_k$. It is defined essentially as the above algorithm but with $m(n) \equiv 1$. They report that this single loop algorithm has a better performance.

We prove in this paper that this single loop algorithm converges exponentially fast if PP_0P is compact and A2 below is satisfied. Then we prove that the algorithm actually converges to $s_0^*, \ldots, s_k^*$ if the algorithm is modified as follows.

Let

$$T = (I - P_1)(I - P_2) \cdots (I - P_k)(I - P_{k-1}) \cdots (I - P_1) \tag{1.4}$$

and define

$$u_n = \sum_{j=1}^k s_j^{(n)}.$$

Then replace (1.3) by

$$u_{n+1} + s_0^{(n+1)} = [(1 - \alpha_n) I + \alpha_n T](u_n + s_0^{(n+1)})$$

where $\alpha_i \in (0, 1)$ is a predeterminant sequence of positive numbers (their actual values has no influence on the theoretical result).

2. Preliminaries.

Let $Q = PP_0P : S \to S$. We assume

A1: Q is a compact operator (e.g. $P_0 S \to S_0$ compact) and $\|Q\| > 0$

A2: $\inf\{\|\sum_{j=i}^{k} s_j\| : s_j \in S_j,\ j = i+1, \ldots, k,\ s_i \in S_i \cap (S_{i+1} + \cdots + S_k)^{\perp},\ \|s_i\| = 1\} > 0$, $i = 1,2, \ldots, k-1$.

Let $\sigma_0 \geq \sigma_1 \geq \cdots$ be the eigenvalues of Q (including multiplicities) and let $\sigma(Q)$ denotes the spectrum of Q. For any operator H, $N(H)$ denotes its null space.

Proposition 1.

i) $s_0^* = -P_0 \Sigma_{i=1}^k s_i^* / \|P_0 \Sigma_{i=1}^k s_i^*\|$, $\Sigma_{i=1}^k s_i^* = -P s_0^*$ and $Q \Sigma_{i=1}^k s_i^* = \|P_0 \Sigma_{i=1}^k s_i^*\| \sum_{i=1}^{k} s_i^*$.

ii) If $u \in S_1 + \cdots + S_k$, $Qu = \sigma u$ and $\|P_0 u\| = \sigma$. Then $\|u - P_0 u / \|P_0 u\|\|^2 = 1 - \sigma$. In particular, $Q \Sigma_{i=1}^k s_i^* = \sigma_0 \Sigma_{i=1}^k s_i^*$ and $c_0 = 1 - \sigma_0$.

iii) $(I - T)(I - P) = 0$ and $u_{i+1} = \{(1 - \alpha_i) I + \alpha_i T + \alpha_i \lambda_i (I - T) Q\} u_i$ $i = 1,2,\ldots$ where $\lambda_i = \|P_0 u_i\|^{-1}$, $i = 1,2,\ldots$

Proof.

i) Consider the problems:

(a) minimize $\|s_0 + \Sigma_{i=1}^k s_i^*\|$ subject to $s_0 \in S_0$ and $\|s_0\| = 1$

(b) minimize $\|s_0^* + u\|$ subject to $u \in S$

We get immediately that $s_0^* = -P_0 \Sigma_{i=1}^k s_i^* / \|P_0 \Sigma_{i=1}^k s_i^*\|$ and

(c)
$$\begin{aligned} \Sigma_{i=1}^k s_i^* &= -P s_0^* \\ &= P P_0 \Sigma_{i=1}^k s_i^* / \|P_0 \Sigma_{i=1}^k s_i^*\| \\ &= Q \Sigma_{i=1}^k s_i^* / \|P_0 \Sigma_{i=1}^k s_i^*\|. \end{aligned}$$

ii) Let u be any eigenvector of Q, $Qu = \sigma u$, normalized such that $\|P_0 u\| = \sigma$. Then

$$\begin{aligned} \langle u,u \rangle &= \langle u, \sigma^{-1} P P_0 u \rangle \\ &= \sigma^{-1} \langle u, P_0 u \rangle \\ &= \sigma^{-1} \langle P_0 u, P_0 u \rangle = \sigma. \end{aligned}$$

Therefore

$$\begin{aligned} c_0 &\geq \| u - P_0 u / \| P_0 u \| \|^2 \\ &= \| u \|^2 - 2\sigma^{-1} < u, P_0 u > + 1 \\ &= 1 - \sigma. \end{aligned}$$

Since $\Sigma_{i=1}^{k} s_i^*$ satisfies (c) we conclude that it corresponds to σ_0, $c_0 = 1 - \sigma_0$ and $\| u \|^2 = \sigma_0$.

iii) The first claim follows since $(I - P_i)(I - P) = I - P$. Hence

$$\begin{aligned} u_{i+1} &= (1 - \alpha_i) u_i + \alpha_i T u_i + \alpha_i \lambda_i (I - T) P_0 u_i \\ &= (1 - \alpha_i) u_i + \alpha_i \lambda_i (I - T) P P_0 P u_i. \quad \blacksquare \end{aligned}$$

3. Main result.

In this section we prove the following theorem.

Theorem: Suppose A1 and A2 hold. Let u_i $i = 2,3,\ldots$ be defined recursively as in Proposition 1(iii). Let $c_i = \| u_i - P_0 u_i / \| P_0 u_i \| \|^2$.

i) Suppose $0 < \alpha^* \le \alpha_i \le 1$ and that T is given either by (1.2) or by (1.4). Then $c_1, c_2, \ldots$ converges exponentially fast and $\sigma_\infty \equiv 1 - \lim_{i\to\infty} c_i \in \sigma(Q)$. $u_1, u_2, \ldots$ converges exponentially fast and $\lim_{i\to\infty} u_i \in N(\sigma_\infty I - Q)$. Both rates have uniform bounds which depend only on σ_∞. In particular the convergence is uniform on $\{u_1 : \| u_1 - P_0 u_1 / \| P_0 u_1 \| \|^2 \le \bar{c} < 1\}$.

ii) If further $0 < \alpha^* \le \alpha_i < 1$ and T is given by (1.4) then $\{u_1 : u_i \to N(\sigma_0 I - Q)\}$ is an open everywhere dense set. □

The theorem will be proved by the following lemmas. See in particular Lemmas 3(ii), 4 and 7.

We assume for simplicity that $\| P_0 u_1 \| > 1/2 \| u_1 \|^2$, which can always be achieved by normalization except in the trivial case where $u_1 = u_2 = \cdots$. Then $\| u_1 - P_0 u_1 / \| P_0 u_1 \| \|^2 = \| u_1 \|^2 - 2 \| P_0 u_1 \| + 1 < 1$. Note that this is the case in particular if $u_1 = (I - T)\, s_0 \ne 0$ for arbitrary $s_0 \in S_0$, $\| s_0 \| = 1$.

In the following K_1, K_2 and C_1, $C_2, \ldots$ are constants which do not depend on the particular sequence $u_1, u_2, \ldots$ or on i as long as $c_i < \bar{c} < 1$. In particular they hold uniformlly on $\{u_1 : c_1 \le \bar{c} < 1\}$. They may depend on $\bar{c}$.

In the remaining, all operators except P, $P_i : H \to H$ are considered as operating from **S** to **S**.

Lemma 1. $\| T \| < 1$.

Proof. The lemma will be proved by induction on the number of subspaces $S_1, \ldots, S_k$. For k = 1 the result is trivial ($\| T \| = 0$). Suppose the lemma holds for $k - 1$ subspaces. Assume T is given by (1.4). Let $T' = (I - P_2)(I - P_3) \cdots (I - P_K)(I - P_{K-1}) \cdots (I - P_2)$, thus $T = (I - P_1) T' (I - P_1)$. We prove now that $\| T \| < 1$. Assume otherwise, that

$$\| T \| = 1. \tag{3.1}$$

Since T is self adjoint operator, there are $w_1, w_2, \ldots$ such that $T w_i = \beta_i w_i$, $\| w_i \| = 1$ $i = 1,2,\ldots$ and $\beta_i \to 1$. Since $\| (I - P_i) \| \le 1$, $i = 1,2,\ldots,k$ we have: $\| (I - P_1) w_i \| \ge \| (I - P_1) T' (I - P_1) w_i \| = \beta_i$. Therefore:

$$\| P_1 w_i \|^2 = \| w_i \|^2 - \| (I - P_1) w_i \|^2 \le 1 - \beta_i^2 \quad i = 1,2,\ldots \tag{3.2}$$

In a similar way; $\|w_i\| \geq \|T'(I-P_1)w_i\| \geq \|(I-P_1)T'(I-P_1)w_i\| = \beta_i\|w_i\|$ and

$$\|P_1T'(I-P_1)w_i\|^2 = \|T'(I-P_1)w_i\|^2 - \|(I-P_1)T'(I-P_1)w_i\|^2 \leq 1-\beta_i^2, \qquad i = 1,2,... \tag{3.3}$$

Now, $(I-P_1)T'(I-P_1)w_i = \beta_i w_i$ hence

$$(I-T')w_i = (1-\beta_i)w_i - P_1T'(I-P_1)w_i - T'P_1w_i \qquad i = 1,2,... \tag{3.4}$$

Together (3.2) - (3.4) imply that

$$\|(I-T')w_i\| \leq 1-\beta_i + 2(1-\beta_i^2)^{1/2} \to 0 \tag{3.5}$$

Let P' be the (orthogonal) projection on $S_2 + \cdots + S_k$ operator. By proposition 1(iii), $\|(I-T')w_i\| = \|(I-T')P'w_i\|$ $i = 1,2,....$ Therefore (3.5) and the induction hypotheses imply that

$$\lim_{i\to\infty}\|P'w_i\| = 0. \tag{3.6}$$

Write $w_i = w_i' + w_i''$ where $w_i' \in S_1 \cap \{S_1 \cap (S_2 + \cdots + S_k)\}^{\perp}$ and $w_i'' \in S_2 + \cdots + S_k$. we conclude from (3.2) and (3.6) that

$$w_i' + P_1w_i'' \to 0 \text{ and } P'w_i' + w_i'' \to 0, \quad \text{hence} \tag{3.7}$$

$$P_1P'P_1w_i' - w_i' \to 0.$$

Clearly $w_i' \not\to 0$. (Otherwise $\lim w_i'' = -\lim P_1w_i' = 0$ contradicting $\|w_i' + w_i''\| = 1$). Now, $P_1P'P_1$ is a self adjoint and by (3.7) its spectral radius is 1. This contradicts assumption A2 by Proposition 1. Hence (3.1) was contradicted. Now, if T is given by (1.2) the same argument applies since $\|Tw\|^2 = w^TT^TTw$. □

Lemma 2.

i) $c_1 \geq c_2 \geq \cdots$

ii) $\|u_i - u_{i+1}\|^2 \leq (2k-1)\alpha_i(c_i - c_{i+1})$ $\quad i = 1,2,...$

Proof.

i) Fix any $i \geq 1$ let $v = P_0u_i/\|P_0u_i\|$ and define $w_0, w_1 \cdots$, by:

$$\begin{aligned} w_0 - v &= u_i - v, \\ w_{j+1} - v &= (I-P_j)(w_j - v) \qquad j = 1,\ldots,k-1, \\ w_{j+1} - v &= I - P_{2k-1-j}(w_j - v) \qquad j = k,\ldots,2k-2. \end{aligned} \tag{3.8}$$

Then

(3.9) $$u_{i+1} = (1 - \alpha_i) u_i + \alpha_i w_{2k-1}.$$

Now,

(3.10) $$\| w_{j+1} - v \|^2 = \| w_j - v \|^2 - \| w_{j+1} - w_j \|^2 \qquad j = 0, \ldots, 2k-2.$$

Hence:

(3.11) $$\| w_{2k-1} - v \|^2 = \| u_i - v \|^2 - \Sigma_{j=0}^{2k-2} \| w_{j+1} - w_j \|^2.$$

A convexity argument, (3.9) and (3.11) together imply that

(3.12) $$\begin{aligned} \| u_{i+1} - v \|^2 &\le (1 - \alpha_i) \| u_i - v \|^2 + \alpha_i \| w_{2k-1} - v \|^2 \\ &= \| u_i - v \|^2 - \alpha_i \Sigma_{j=0}^{2k-1} \| w_{j+1} - w_j \|^2. \end{aligned}$$

Therefore

(3.13) $$\begin{aligned} c_{i+1} &= \| u_{i+1} - \| P_0 u_{i+1} \|^{-1} P_0 u_{i+1} \|^2 \\ &\le \| u_{i+1} - v \|^2 \\ &\le c_i - \alpha_i \Sigma_{j=0}^{2k-1} \| w_{j+1} - w_j \|^2. \end{aligned}$$

Which proves part (i).

ii) The second part follows (3.13) and the Schwartz inequality since

(3.14) $$\begin{aligned} \| u_{i+1} - u_i \|^2 &= \alpha_i^2 \| w_{2k-1} - u_i \|^2 \\ &\le \alpha_i^2 (\Sigma_{j=0}^{2k-1} \| w_{j+1} - w_j \|)^2 \\ &\le \alpha_i^2 (2k-1) \Sigma_{j=0}^{2k-1} \| w_{j+1} - w_j \|^2 \\ &\le (2k-1) \alpha_i (c_i - c_{i+1}). \qquad \square \end{aligned}$$

Establishing Lemma 2, we are ready to prove a stronger convergence result.

Lemma 3.

i) $\| u_i - \lambda_i Q u_i \|^2 \le (2k-1)(1 - \| T \|)^{-2} \alpha_i^{-1} (c_i - c_{i+1}) \qquad i = 1,2,\ldots$ where $\lambda_i = \| P_0 u_i \|^{-1}$.

ii) $\lim_{i \to \infty} (1 - c_i) \equiv \sigma_\infty \in \sigma(Q)$

iii) $| \lambda_i^{-1} - \sigma_\infty | \le K_1 (c_i - 1 + \sigma_\infty)^{1/2} \qquad i = 1,2,\ldots$ where $K_1 = 1 + \{(2k-1) \alpha_i^*\}^{1/2} (I - \| T \|)^{-1}$.

iv) $\| u_i \| \ge 1 - c_i^{1/2}, \qquad i = 1,2,\ldots.$

v) $2/5 \le \lambda_i \le (1 - c_i^{1/2})^{-1}, \qquad i = 1,2,...$

Proof.

By proposition (iii) $u_{i+1} - u_i = \alpha_i (I - T)(\lambda_i Q u_i - u_i)$, $i = 1,2,...$, and part (i) follows Lemmas 1 and 2(ii). Write

$$(3.15) \qquad \begin{aligned} u_i &= \lambda_i Q u_i + w_i, \\ &= \lambda_i P P_0 u_i + w_i \qquad i = 1,2,... \end{aligned}$$

where $\| w_i \|^2 \le \dfrac{(2k-1)}{(1 - \| T \|)^2 \alpha_i} (c_i - c_{i+1})$. This proves part (i). Now

$$(3.16) \qquad \begin{aligned} 1 &= < \lambda_i P_0 u_i, \lambda_i P_0 u_i > \\ &= \lambda_i < u_i, \lambda_i P_0 u_i > \\ &= \lambda_i < u_i, \lambda_i P P_0 u_i > \\ &= \lambda_i < u_i, u_i - w_i > \qquad \text{by (3.15)} \\ &= \lambda_i \| u_i \|^2 - \lambda_i < u_i, w_i >, \qquad i = 1,2,... \end{aligned}$$

Therefore

$$(3.17) \qquad \begin{aligned} c_i &= \| u_i - \lambda_i P_0 u_i \|^2 \\ &= \| u_i \|^2 - 2 < u_i, \lambda_i P_0 u_i > + 1 \\ &= 1 - \lambda_i^{-1} - < u_i, w_i > \qquad i = 1,2,.... \end{aligned}$$

Since $\| u_i \| \le \| u_i - \lambda_i P_0 u_i \| + 1 \le 2$, (3.15) and (3.17) together imply that

$$(3.18) \qquad | c_i - 1 - \lambda_i^{-1} | \le 2 \{ \frac{2k-1}{(1 - \| T \|)^2 \alpha_i} (c_i - c_{i+1}) \}^{1/2}, \qquad i = 1,2,...$$

Therefore $\lambda_\infty^{-1} = \lim_{i \to \infty} \lambda_i^{-1} = 1 - \lim_{i \to \infty} c_i$. Now, Q is compact and $\| u_i \| < 2$ $i = 1,2,...$, hence the RHS of (3.15) has a converging subsequence, i(1), i(2),..., say. But (3.15) implies that $u_{i(1)}, u_{i(2)},...$ converges to u, say, such that $u - \lambda_\infty Q u = 0$. This proves part (ii). To prove the third part conclude from (3.18) and the monotonicity of $\{c_i\}$ that:

$$(3.19) \qquad \begin{aligned} | \lambda_i^{-1} - \sigma_\infty | &\le | c_i - 1 + \lambda_i^{-1} | + | 1 - c_i - \sigma_\infty | \\ &\le 2 \{ \frac{2k-1}{(1 - \| T \|)^2 \alpha_i} (c_i - 1 + \sigma_\infty) \}^{1/2} + c_i - 1 + \sigma_\infty. \end{aligned}$$

Part (iv) is proved by $\| u_i \| \ge \| \lambda_i P_0 u_i \| - \| u_i - \lambda_i P_0 u_i \| \equiv 1 - c_i^{1/2}$.

Finally, $c_i \equiv \| u_i - \| P_0 u_i \|^{-1} P_0 u_i \|^2 = \| u_i \|^2 - 2 \| P_0 u_i \| + 1$, hence

(3.20) $\lambda_i \equiv \|P_0 u_i\|^{-1} = 2(1 + \|u_i\|^2 - c_i)^{-1}, \quad i = 1,2,...$

Since $0 \le c_i \le 1$ and $(1 - c_i^{1/2})^{-2} \le \|u_i\|^2 \le 4$ the proof of the lemma is completed by (3.20). □

We need the following lemma to complete the proof of the first part of the Theorem.

Lemma 4.

$u_\infty = \lim_{i\to\infty} u_i$ exists and there are $\varepsilon > 0$, K_2 and $b \in (0,1)$ which depend on u_1 only through σ_∞, such that $c_m - 1 + \sigma_\infty < \varepsilon$ implies:

i) $c_i - 1 + \sigma_\infty \le (c_m - 1 + \sigma_\infty)b^{i-m}, \quad i \ge m$

ii) $|\lambda_i - \sigma_\infty| \le K_2(c_m - 1 + \sigma_\infty)b^{i-m}, \quad i \ge m$

iii) $\|u_i - u_\infty\| \le K_2(c_m - 1 + \sigma_\infty)b^{i-m}, \quad i \ge m.$

Proof. Let ξ_j, $j = 0,1,2,...$ satisfy $Q\xi_j = \sigma_j\xi_j$, $< \xi_j, \xi_l > = \delta_{jl}$. Let $J = \{j : \sigma_j = \sigma_\infty\}$ and $J_c = \{j : \sigma_j \ne \sigma_\infty\}$. Let $v_j = P_0\xi_j$. Note that $Pv_j = \sigma_j\xi_j$, $< v_j, \xi_l > = 0$ for $j \ne l$ and $< v_j, v_l > = < v_j, P_0\xi_l > = < v_j, \xi_l > = < Pv_j, \xi_l > = \sigma_j\delta_{jl}$. Since $\sigma_\infty \ge 1 - c_i > 0$ and Q is compact operator there is $\gamma > 0$ such that $2\gamma \le \min_{j\in J_c}|\sigma_j - \sigma_\infty|$. Suppose ε is small enough such that (3.21) and (3.25) below are satisfied when $c_i - 1 + \sigma_\infty < \varepsilon$.

First note that if ε is small enough then for $i \ge m$:

$$
\begin{aligned}
(3.21) \qquad \min_{j\in J_c}|1 - \lambda_i\sigma_j| &= \min_{j\in J_c}\lambda_i|\lambda_i^{-1} - \sigma_j| \\
&\ge \lambda_i(2\gamma - |\lambda_i^{-1} - \sigma_\infty|) \\
&\ge \frac{2\gamma - K_1(c_i - 1 + \sigma_\infty)^{1/2}}{\sigma_\infty + K_1(c_i - 1 + \sigma_\infty)^{1/2}} \quad \text{by Lemma 3(iii)} \\
&\ge \gamma.
\end{aligned}
$$

Now write $u_i = \Sigma\alpha_{ij}\xi_j$, $i \ge m$. Then

$$(3.22) \qquad \|u_i - \lambda_i Q u_i\|^2 = \|\sum_{j=0}^{\infty} \alpha_{ij}(1 - \lambda_i \sigma_j)\xi_j\|^2$$

$$= \sum_{j=0}^{\infty} \alpha_{ij}^2 (1 - \lambda_i \sigma_j^2) \qquad i \geq m.$$

Therefore, by Lemma 3(i)

$$(3.23) \qquad \sum_{j=0}^{\infty} \alpha_{ij}^2 (1 - \lambda_i \sigma_j)^2 \leq C_1 (c_i - c_{i+1}) \qquad i \geq m$$

where $C_1 = (2k - 1)(1 - \|T\|)^{-2} \alpha^{*-1}$. Combine (3.21) and (3.23) and get

$$(3.24) \qquad \sum_{j \in J_c} \alpha_{ij}^2 \leq \frac{C_1}{\gamma}(c_i - c_{i+1}) \qquad i \geq m.$$

Lemma 3(iv) and (3.24) imply that

$$(3.25) \qquad \sum_{j \in J} \alpha_{ij}^2 = \|u_i\|^2 - \sum_{j \in J_c} \alpha_{ij}^2$$

$$\geq 1 - c_i^{1/2} - \frac{C_1}{\gamma}(c_i - c_{i+1})$$

$$\geq 1/2(1 - c_m^{1/2}).$$

We conclude from (3.23) and (3.25) that

$$(3.26) \qquad (1 - \lambda_i \sigma_\infty)^2 \leq \frac{2C_1}{1 - c_m^{1/2}} \qquad i \geq m.$$

Our last preliminary result is:

$$(3.27) \qquad 1 = \lambda_i^2 \|P_0 u_i\|^2$$

$$= \lambda_i^2 \|\sum_{j=0}^{\infty} \alpha_{ij} v_j\|^2$$

$$= \lambda_i^2 \sum_{j=0}^{\infty} \alpha_{ij}^2 \sigma_j \qquad i \geq m.$$

Now,

$$(3.28) \qquad c_i = \|u_i - \lambda_i P_0 u_i\|^2$$

$$= \|\Sigma_{j=0}^{\infty} \alpha_{ij}(\xi_j - \lambda_i v_j)\|^2$$

$$= \Sigma_{j=0}^{\infty} \alpha_{ij}^2 (1 - 2\lambda_i < \xi_j, v_j > + \lambda_i^2 \|v_j\|^2)$$

$$= \Sigma_{j=0}^{\infty} \alpha_{ij}^2 (1 - 2\lambda_i \sigma_j + \lambda_i^2 \sigma_j)$$

$$= \Sigma_{j=0}^{\infty} \alpha_{ij}^2 (1 - \lambda_i \sigma_j)^2 + \lambda_i^2 \Sigma_{j=0}^{\infty} \alpha_{ij}^2 \sigma_j - \lambda_i^2 \Sigma_{j=0}^{\infty} \alpha_{ij}^2 \sigma_j^2 \qquad i \geq m.$$

The last term in the RHS of (3.28) can be rewritten as:

$$(3.29) \qquad \lambda_i^2 \sum_{j=0}^{\infty} \alpha_{ij}^2 \sigma_j^2 = \sigma_\infty \sum_{j\in J} \alpha_{ij}^2 \lambda_i^2 \sigma_j + \sum_{j\in J_c} \alpha_{ij}^2 \lambda_i^2 \sigma_j^2 \qquad i \geq m,$$

where we have used (3.27). Combine now (3.24), (3.26), (3.27) and (3.29) and get

$$(3.30) \qquad c_i = 1 - \sigma_\infty + \sigma_\infty \sum_{j\in J_c} \alpha_{ij}^2 \lambda_i^2 (\sigma_j - \sigma_j^2) + \sum_{j=0}^{\infty} \alpha_{ij}^2 (1 - \lambda_i \sigma_j)^2$$

$$\leq 1 - \sigma_\infty + 1/2\, \sigma_\infty \frac{C_1}{\gamma} \lambda_i^2 (c_i - c_{i+1}) + C_1 (c_i - c_{i+1}) \qquad i \geq m.$$

Lemma 3(v) and (3.30) imply that:

$$(3.31) \qquad c_i - 1 + \sigma_\infty \leq C_1 (1 + 1/2\, \sigma_\infty \frac{\lambda_i^2}{\gamma})(c_i - c_{i+1}) \leq C_2 (c_i - c_{i+1}) \qquad i \geq m$$

for some $C_2 > 0$.

Hence (note that $c_{i+1} \geq 1 - \sigma_\infty$):

$$(3.32) \qquad c_{i+1} - 1 + \sigma_\infty \leq \frac{C_2 - 1}{C_2} (c_i - 1 + \sigma_\infty)$$

$$\leq (c_m - 1 + \sigma_\infty) (\frac{C_2 - 1}{C_2})^{i-m} \qquad i \geq m$$

which proves Part (i). Part (ii) follows Part (i) and (3.26). Finally, Part (iii) is proved by Part (i) and Lemma 2(i). □

The first part of the theorem is now proved. We introduce now some new notation to simplify the proof of the second part. Let $x_i = (I - T)^{-1/2} u_i$ $i = 1,2,\ldots$ and $M = (I - T)^{1/2} Q (I - T)^{1/2}$ where $(I - T)^{1/2}$ is the self adjoint operator such that $(I - T)^{1/2} (I - T)^{1/2} = I - T$. Then it is easy to see that $x_{i+1} = (1 - \alpha_i) x_i + \alpha_i (T + \lambda_i M) x_i$ and $\lambda_i = < x_i, Mx_i >^{-1/2}$, $i \geq 1$. Define

$$(3.33) \qquad f_i(x) = (1 - \alpha_i) x + \alpha_i Tx + \alpha_i < x, Mx >^{-1/2} Mx, \quad x \in S,\ i \geq 1.$$

Clearly $x_{i+1} = f_i(x)$ $i \geq 1$.

In the next lemma we establish some properties of the operator $f_i(\cdot)$.

Lemma 5.

i) $f_i(\cdot)$ is monotone coerctive self adjoint operator with F-derivative at $x \in S$ given by

$$(3.34) \qquad \dot{f}_i(y; x) = (1 - \alpha_i) y + \alpha_i Ty + \alpha_i < X, Mx >^{-1/2} My$$

$$- \alpha_i < x, Mx >^{-1/2} < Mx, y > Mx, \qquad y \in S \qquad i \geq 1.$$

ii) f_i has a bounded inverse.

iii) If $x \in S$, $f_i(x) = x$ and $< x, Mx > < \sigma_0^2$, $(I - T)^{1/2} x \notin N(\sigma_0 I - Q)$ then $\| \dot{f}_i(\cdot; x) \| > 1$.

iv) For any $\varepsilon > 0$, and $x \in S$, $\dot{f}_i(\cdot; x)$ has a finite number of eigenvalues including multiplicities greater than $\alpha_i + (1 - \alpha_i) \| T \| + \varepsilon$.

Proof.

i) Immediate.

ii) Since M is self adjoint positive semidefinite operator, $< x, Mx > < y, My > - < x, My >^2 \geq 0$ (Schwartz inequality). T is self adjoint positive semi-definite operator. Hence

$$\begin{aligned} (3.35) \qquad < y, \dot{f}_i(y; x) > &= (1 - \alpha_i) \| y \|^2 + \alpha_i < y, Ty > \\ &\quad + \alpha_i < x, Mx >^{-3/2} (< x, Mx > < y, My > - < x, My >^2) \\ &\geq (1 - \alpha_i) \| Y \|^2. \end{aligned}$$

Therefore $\dot{f}_i(\cdot; x)$ is invertible (Joshi and Bose, (1985)).

iii) Let $\lambda = < x, Mx >^{-1/2}$ and $y = (I - T)^{-1/2} u_0$ where $u_0 \in N(\sigma_0 I - Q)$. Then

$$(3.36) \qquad y = (T + \sigma_0^{-1} M) y.$$

If $x = (T + \lambda M) x$, $x \in S$ with $\lambda > \sigma_0^{-1}$ then

$$\begin{aligned} (3.37) \qquad < y, x > &= < y, (T + \lambda M) x > \\ &= < (T + \lambda M) y, x > \\ &= < y + (\lambda - \sigma_0^{-1}) M y, x > \\ &= < y, x > + (\lambda - \sigma_0^{-1}) < My, x >. \end{aligned}$$

Hence $< y, Mx > = 0$ implying that $\dot{f}_i(y; x) = (1 - \alpha_i) y + \alpha_i Ty + \alpha_i \lambda My$ and

$$\begin{aligned} (3.38) \qquad < y, \dot{f}_i(y; x) > &= (1 - \alpha_i) \| Y \|^2 + \alpha_i < (T + \lambda M) y, y > \\ &= \| y \|^2 + \alpha_i (\lambda - \sigma_0^{-1}) < y, My >. \end{aligned}$$

Since $\| T \| < 1$, (3.36) implies that $< y, My > > 0$ hence $< y, \dot{f}_i(y; x) > \geq \| y \|^2$ which proved part (iii).

iv) Suppose $i \geq 1$, $x \in S$ and $\dot{f}_i(y_n; x) = \gamma_n y_n$, $n = 1,2,...$ where $y_n \in S$, $< y_n, y_m > = \delta_{nm}$ and $\gamma_n > 1 - \alpha_i (1 + \| T \|) + \varepsilon$.

Then

$$(3.39) \qquad (1-\alpha_i)y_n + \alpha_i(Ty_n + \lambda My_n - \lambda^3 < x, My_n > Mx) = \gamma_n y_n \qquad n \geq 1$$

where $\lambda = < x, Mx >^{-1/2}$. Now $\dot{f}_i$ is bounded and M is compact hence, after passing if necessary to a subsequence, $\gamma_n \to \gamma \geq \alpha_i + (1-\alpha_i)\|T\| + \varepsilon$ and $My_n \to y \in S$, say. Therefore

$$(3.39) \qquad y_n = [\gamma_n - \alpha_i 1 - (1-\alpha_i)T]^{-1}(\lambda My_n - \lambda^3 < x, My_n > Mx)$$

$$\underset{n\to\infty}{\to} [\gamma - \alpha_i - (1-\alpha_i)T)^{-1}(\lambda y - \lambda^3 < x, y > Mx)$$

contradicting the assumption $\|y_n - y_m\| = 2\delta_{nm}$. □

Let $\psi(x)$ denote the linear space spaned by the eigenvector of $\dot{f}_i(\cdot; x)$, $x \in S$ with eigenvalues greater than 1. (Note that $\psi(x)$ is independent of α_i). For any vector $e \neq 0$ and linear spaces ψ_1, ψ_2, $\measuredangle(e, \psi_1)$ is the arccos of the norm of the orthogonal projection of $e/\|e\|$ on ψ, ($\measuredangle(e, \psi_1) \in [0, \pi/2]$) and $\measuredangle(\psi_1, \psi_2) = \max\{\measuredangle(e, \psi_1), e \in \psi_2, \|e\| = 1\}$. Recall that by Lemma 4 $x_i \to x_\infty = (T + \lambda_\infty M)X_\infty$ and $u_i = (I-T)^{1/2}x_i \to u_\infty = \lambda_\infty Qu_\infty$.

Lemma 6. Suppose that $x_i \to x_\infty = (T + \sigma_\infty^{-1}M)x_\infty$ where $\sigma_\infty < \sigma_0$. Let

$$x_m' = x_m + e_m$$

$$x_{i+1} = f_i(x_i'), \quad e_{i+1} = x_{i+1}' - x_{i+1} \quad i > m.$$

Then there are $\varepsilon > 0$ and $\overline{\theta} > 0$ such that $u_i' = (I-T)^{1/2}x_i' \to u_\infty' = \lambda_\infty' Qu_\infty$, $\lambda_\infty' \neq \sigma_\infty^{-1}$ whenever $|c_m - 1 + \sigma_\infty| < \varepsilon$, $0 < \|e_m\| < \varepsilon$ and $\measuredangle\{e, \psi(x_\infty)\} < \overline{\theta}$.

Proof.

Suppose $c_m - 1 + \sigma_\infty < \varepsilon$, $0 < \|e_m\| < \varepsilon$, $\measuredangle(e, \psi(x_\infty) \leq \overline{\theta}$ and $x_i' \to N(I - T - \sigma_\infty^{-1}M)$. We will show that if $\overline{\theta} > 0$ and $\varepsilon > 0$ are small enough this is contradiction.

In the following $C_1, \ldots, C_s$ are appropriate constants which depends on u_1 only through σ_∞.

Let $c_i' = \|u_i' - \|P_0 u_i'\|^{-1} P_0 u_i'\|^2$, $i \geq m$. Then $|c_m' - c_m| \leq C_1\varepsilon$ for some $C_1 > 0$. Lemma 4(iii) implies therefore that

$$(3.40) \qquad \|u_i' - u_i\| \leq \|u_i - u_m\| + \|u_i' - u_m'\| + \|u_m - u_m'\|$$

$$\leq C_2\varepsilon \qquad i \geq m$$

for some constant C_2.

Hence

(3.41) $$\|e_i\| = \|(I-T)^{-1/2}(u_i' - u_i)\| \le C_3\varepsilon \qquad i \ge M.$$

Moreover, $x_i' \to x_\infty'$, say, again by Lemma 4(iii). Therefore:

(3.42) $$\begin{aligned} e_{i+1} &= f_i(x_i') - f_i(x_i) \\ &= \dot{f}_i(e_i; x_i^*) \quad x_i^* = x_i + \beta_i e_i, \ \beta_i \in [0,1] \\ &= \dot{f}_i(e_i; x_\infty^*) + \Delta_i \varepsilon_i \qquad i \ge m \end{aligned}$$

where $x_i^* \to x_\infty$ and $\Delta_m, \Delta_{m+1}, \ldots$ are self adjoint operator such that

(3.43) $$\|\Delta_i\| \le C_4 \varepsilon b^{i-m} \qquad i \ge m.$$

Let $\gamma > 0$ be such that $\dot{f}_i(\cdot; x_\infty)$ does not have any eigenvalue in $(1, 1+3\gamma)$. The existence of such γ is ensured by Lemma 5(iv). Now, Lemma 5(iii) ensures that $\dot{f}_i(\cdot, x_\infty)$ has p (including multiplicities) eigenvalues greater than $1 + 2\gamma (1 \le p < \infty)$. If $\|\dot{f}_i(\cdot; x_i^*) - \dot{f}_i(\cdot, x_\infty)\| < \gamma$ then $\dot{f}_i(\cdot, x_i^*)$ has exactly p eigenvalues greater than $(1+2\gamma)$ and none in $(1+\gamma, 1+2\gamma)$. Let $\theta_i = \measuredangle(e_i, \psi(x_i^*))$, $\delta_i = \measuredangle\{\psi(x_i^*), \psi(x_{i+1}^*)\}$ and $\xi_i = \measuredangle\{e_{i+1}, \psi(x_i^*)\}$ $i \ge M$. Then,

(3.44) $$\sin(\delta_i) \le \frac{\|\Delta_i - \Delta_{i+1}\|}{\gamma} \qquad i \ge M.$$

See Davis and Kahan (1970).

Now if $(\|\Delta_i\| + \|\Delta_{i+1}\|)/\gamma \le 1/2$ then (3.44) is equivalent to

(3.45) $$\begin{aligned} \delta_i &\le 2\sin\delta_i \\ &\le 2C_4\varepsilon b^{i-m} \qquad i \ge m \end{aligned}$$

(the last inequality follows (3.43)). Clearly $\xi_i \le \theta_i$, $i \ge m$. Hence

(3.46) $$\begin{aligned} \theta_{i+1} &\le \xi_i + \delta_i \\ &\le \theta_i + \delta_i \\ &\le \theta_m + \sum_{j=m}^{i} \delta_j \qquad i \ge m. \end{aligned}$$

So, (3.45) and (3.46) can be combined to give

(3.47) $$\theta_i \le \theta_m + 2C_4(1-b)^{-1}\varepsilon \qquad i \ge m.$$

If $\measuredangle(e_m, \psi(x_\infty)) < \overline{\theta}$ then

(3.48) $$\theta_m \le \overline{\theta} + \measuredangle\psi(x_\infty), \psi(x_\infty^*)\} \le \overline{\theta} + C_5\varepsilon.$$

Choose now $\overline{\theta}$ and ε such that

(3.49) $$(1 + 2\gamma)\cos\{\bar{\theta} + 2C_4\varepsilon(1 - b)^{-1} + C_5\varepsilon\} > 1.$$

Then

(3.50) $$\|e_{i+1}\| \geq (1 + 2\gamma)\|e_i\|\cos\theta_i \to \infty \quad \text{as} \quad i \to \infty$$

contradicting (3.41). □

Lemma 7.

i) Suppose $u_i \to u_\infty = \sigma_\infty^{-1} Q u_\infty$ and $\sigma_0 < \sigma_\infty$. Then for any open ball B_0 such that $u_1 \in B_0$ there is an open ball $B_1 \subseteq B_0$ such that the algorithm starting at any $u_1' \in B_1$ converges to $N(\sigma_0 I - q)$.

ii) If $u_i \to N(\sigma_0 I - Q)$ then there is an open ball B_2, $u_1 \in B_2$ and the algorithm starting at any point of B_2 converges to $N(\sigma_0 I - Q)$.

Proof.

Let $g_i(u) = (1 - \alpha_i)u + \alpha_i Tu + \alpha_i \|P_0 u\|^{-1}(I - T)Qu \quad i = 1,2,\ldots$ and let $g_{1,m} = g_{m-1} \circ g_{m-2} \circ \cdots \circ g_1$. Then $u_m = g_{1,m}(u_1)$. By Lemma 5 g_{1m} and its inverse have continuous bounded derivatives, hence both take open sets to open sets. But by Lemma 6, for m large enough, $g_{1m}(B_0)$ has an open subset such that for any point in the later the algorithm does not converge to $N(I - \sigma_\infty^{-1} Q)$. g_{1m}^{-1} maps this open set to an open set B_1' included in B_0. Since $\tilde{\sigma}: \sigma(Q) \cap \{\sigma : \sigma > 1 - \|u_1 - \|P_0 u_1\|^{-1} P_0 u_1\|^2\}$ is a finite set by taking B_1' small enough we ensure that the algorithm starting at points in B_1 converge to $\bigcup_{j=0}^{k} N(\sigma_i I - Q)$ for finite k. Repeating the above argument k times results in an open set $B_1 \subseteq B_0$ such that the algorithm starting at points of B_1 converges to $N(\sigma_0 I - Q)$.

ii) If $u_i \to N(\sigma_0 I - Q)$, then $c_i \to 1 - \sigma_0$ and for some m, $c_m < 1 - \sigma_I$ where $\sigma_0 = \sigma_1 = \cdots = \sigma_{I-1} > \sigma_I$. Now take B_2' open such that $\|u - \|P_0 u\|^{-1} P_0 u\|^2 < 1 - \sigma_I$ for all $u \in B_2'$ and $u_m \in B_2'$. Let $B_2 = g_{1m}^{-1} B_2'$. Then B_2 is open, $u_1 \in B_2$, and the algorithm, starting at points of B_2, converges to $N(\sigma_0 I - Q)$. □

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