STOCHASTIC BILLIARDS ON GENERAL TABLES

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ABSTRACT. We consider stochastic analogues of classical billiard systems. A particle moves at unit speed with constant direction in the interior of a bounded, d–dimensional region with continuously differentiable boundary. The boundary need not be connected; that is, the "table" may have interior "obstacles". When the particle strikes the boundary, a new direction is chosen uniformly at random from the directions that point back into the interior of the region and the motion continues. Such chains are closely related to those that appear in shake–and–bake simulation algorithms.

For the discrete time Markov chain that records the locations of successive hits on the boundary, we show that, uniformly in the starting point, there is exponentially fast total variation convergence to an invariant distribution. By analysing an associated non–linear, first–order PDE, we investigate which regions are such that this chain is reversible with respect to surface measure on the boundary. We also establish a result on uniform total variation Césaro convergence to equilibrium for the continuous time Markov process that tracks the position and direction of the particle.

A key ingredient in our proof is a result on the geometry of C^1 regions that can be described loosely as follows: associated with any bounded C^1 region is an integer N such that it is always possible to pass a message between any two locations in the region using a relay of exactly N locations with the property that every location in the relay is directly visible from its predecessor. Moreover, the locations of the intermediaries can be chosen from a fixed, finite subset of positions on the boundary of the region.

We also consider corresponding results for polygonal regions in the plane.

1. INTRODUCTION

Billiards are dynamical systems that model the motion of a particle in some d–dimensional region (the *table*) with conditions on the nature of "reflections" at the boundary. In the most classical case, the particle moves with unit speed and constant direction in the interior of the table and new directions are chosen at the boundary via the usual "angle of incidence equals angle of reflection" rule. Surveys of the extensive literature on such systems, their rôle as concrete examples of twist maps, and their connections with geometry, deterministic chaos, and asymptotics of partial differential equations can be found in [Sin91, KT91, CFS81, KH95].

We are interested in stochastic analogues with dynamics that can be described informally as follows. A particle moves with unit speed and constant direction in the interior of the table until it strikes the boundary. When the particle strikes the boundary a new direction for the particle is chosen uniformly at random from

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the hemisphere of directions that point into the interior of the region and the motion continues. We investigate here the equilibrium distribution of such stochastic billiard processes and the manner in which these processes converge to equilibrium.

The discrete time Markov chain that is obtained by observing the locations of successive hits of the stochastic billiard on the boundary is reminiscent of the chains that appear in so-called *shake–and–bake* algorithms for simulating distributions on the boundary of a convex polytope, detecting necessary constraints in convex optimization, and minimizing functions which attain their minimum on the boundary of a convex polytope (see, for example, [ST81, PS83, Rom91, BCM+91, Rom92, Rom98]). The shake–and–bake literature is concerned with convex regions, and the questions that arise in our work about the connection between the geometry of the table and *irreducibility* and *aperiodicity* of the chain are almost vacuous in the convex case.

Before describing our results, we need to be more explicit about our set-up and the construction of our process.

Our billiard table is a bounded, connected, open set $D \subset \mathbb{R}^d$, $d \geq 2$. Write $\bar{D}$ and ∂D for the closure and boundary of D. Our standing assumption will be that ∂D is C^1. That is, ∂D locally coincides with the graph of a continuously differentiable function. More precisely, for each $x \in \partial D$ there is an $\epsilon > 0$, a (linear) map $A \in \mathrm{Isom}(\mathbb{R}^d)$ (:= the group of isometries of $\mathbb{R}^d$), and $F \in C^1(\mathbb{R}^{d-1}, \mathbb{R})$ such that $Ax = 0$, $F(0) = 0$, and $A(D \cap B(x,\epsilon)) = \{u \in B(0,\epsilon) : u_d > F(u_1, \ldots u_{d-1})\}$ (where, as usual, $B(z,\delta)$ is the open ball $\{u \in \mathbb{R}^d : \|u - z\| < \delta\}$). Note that we are not assuming that ∂D is connected, so that our table might have *obstacles*.

As usual, write $S^{d-1} := \{v \in \mathbb{R}^d : \|v\| = 1\}$ for the unit sphere. By our assumption on D, the inward pointing normal vector $\nu_x \in S^{d-1}$ and tangent hyperplane $T_x := \{w \in \mathbb{R}^d : w \cdot \nu_x = 0\}$ are well–defined for all $x \in \partial D$.

For $x \in \partial D$ define the *folding map* at x, $\Phi_x : S^{d-1} \to S^{d-1}$, by $\Phi_x v = v + 2(v \cdot \nu_x)^- \nu_x$; that is, vectors in the same hemisphere of S^{d-1} as ν_x are left unchanged by Φ_x whereas vectors in the opposite hemisphere are reflected through the equatorial hyperplane T_x. Note that if W has the uniform distribution σ on S^{d-1} (that is, the σ is normalised Haar measure or, equivalently, normalised $(d-1)$–dimensional Hausdorff measure), then $\Phi_x W$ is distributed according to the normalised restriction of σ to $\{v \in S^{d-1} : v \cdot \nu_x > 0\}$.

Write E for the subset of $\bar{D} \times S^{d-1}$ consisting of pairs (x, v) such that $v \cdot \nu_x > 0$ if $x \in \partial D$. For $(x, v) \in E$ put

$$r(x,v) := \inf\{s > 0 : x + sv \in \partial D\} > 0 \tag{1.1}$$

and

$$\xi(x,v) := x + r(x,v)v. \tag{1.2}$$

In other words, $\xi(x, v)$ is the first point on ∂D that we hit when moving away from x in the direction v and $r(x, v)$ is the distance from x to this point.

Let $(W_k)_{k=1}^\infty$ be a collection of independent, σ–distributed, S^{d-1}–valued random variables. Fix $y = (x, v) \in E$ for the moment and define a càdlàg E-valued stochastic process $(Y_t)_{t \geq 0} = ((X_t, V_t))_{t \geq 0}$ and random times $0 =: S_0 < S_1 < \ldots$ inductively as follows. Set $Y(0) := y$. Suppose $0 =: S_0 < S_1 < \ldots < S_k$ and Y_t for

$t \in]0, S_k]$ have already been defined. Put

$$S_{k+1} := S_k + r(Y_{S_k}),$$
$$Y_t := (X_{S_k} + (t - S_k)V_{S_k}, V_{S_k}) \text{ for } t \in]S_k, S_{k+1}[,$$

and

$$Y_{S_{k+1}} := (\xi(Y_{S_k}), \Phi_{\xi(Y_{S_k})} W_{k+1}).$$

We will show below (see Remark 3.1) that $\lim_k S_k = \infty$ almost surely, so that the above prescription does indeed define Y_t for all $t \geq 0$. Write $\mathbb{P}^y$ for the law of $(Y_t)_{t \geq 0}$ on path space. It is clear that the collection of laws $(\mathbb{P}^y)_{y \in E}$ are those of a time–homogeneous strong Markov process. With a slight abuse of notation we will denote this process as $\mathbf{Y} = (Y_t, \mathbb{P}^y)$ and write $Y_t = (X_t, V_t)$. We think of X_t and V_t as, respectively, the position and velocity of our billiard ball at time t.

We remark in passing that $\mathbf{Y}$ is an example of a a *piecewise–deterministic Markov process* in the terminology of [Dav84] or a *jumping Markov process* in the terminology of [JS96], although $\mathbf{Y}$ is somewhat pathological in the context of that theory because it has predictable jumps.

It is clear that the locations of successive hits of $\mathbf{Y}$ on ∂D form a discrete–time Markov chain. More precisely, with a consistent re-use of notation put

$$S_0 := 0 \text{ and } S_{k+1} := \inf\{s > S_k : X_s \in \partial D\}, k \geq 0. \tag{1.3}$$

For $z \in \partial D$ write δ_z for the point mass at z and η_z for the push–forward of the Haar measure σ under the folding map Φ_z. Then under $\mathbb{P}^{\delta_z \otimes \eta_z}$ the process $(Z_k)_{k \geq 0} := (X_{S_k})_{k \geq 0}$ is a ∂D–valued, discrete–time Markov chain. Write $\mathbb{Q}^z$ for the law of $(Z_k)_{k \geq 0}$ under $\mathbb{P}^{\delta_z \otimes \eta_z}$.

Write μ for the $(d-1)$–dimensional Hausdorff measure on ∂D. By the assumptions on D, $0 < \mu(\partial D) < \infty$ and we normalise so that $\mu(\partial D) = 1$.

Theorem 1.1. *There is a probability measure ρ on ∂D and constants $a > 0$, $b > 1$ such that*

$$\sup_{z \in \partial D} \sup_C |\mathbb{Q}^z\{Z_k \in C\} - \rho(C)| \leq ab^{-k}, \quad k \geq 0,$$

where the inner supremum is over all Borel subsets of ∂D. The measures ρ and μ are mutually absolutely continuous.

Remark 1.2. If $D = B(0,1)$ and $\partial D = S^{d-1}$, then, by symmetry, $\rho = \sigma = \mu$. One might guess that $\rho = \mu$ for general D. This is, however, not the case. For $0 < r < 1$ consider the 2–dimensional annulus $D = \{x : r < \|x\| < 1\}$ with boundary $\partial D = rS^1 \cup S^1$. Write $\hat{\mathbf{Z}} = (\hat{Z}_k)_{k=0}^\infty$ for the $\{0,1\}$–valued process defined by $\hat{Z}_k = 0$ if $Z_k \in rS^1$ and $\hat{Z}_k = 1$ if $Z_k \in S^1$. By symmetry (more precisely, Dynkin's criterion for a functional transform of a Markov process to be Markov) the process $\hat{\mathbf{Z}}$ is a Markov chain. Some simple trigonometry shows that the transition matrix of $\hat{\mathbf{Z}}$ is

$$\begin{pmatrix} \hat{q}_{00} & \hat{q}_{01} \\ \hat{q}_{10} & \hat{q}_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{2 \arcsin r}{\pi} & 1 - \frac{2 \arcsin r}{\pi} \end{pmatrix}.$$

Therefore $\hat{\mathbf{Z}}$ has invariant distribution

$$(\rho_0, \rho_1) = \left(\frac{2 \arcsin r}{\pi + 2 \arcsin r}, \frac{\pi}{\pi + 2 \arcsin r} \right)$$

rather than $(\frac{r}{1+r}, \frac{1}{1+r})$, which would be the case if the invariant distribution of $\mathbf{Z}$ was given by normalised arc–length.

It is clear that if D is a ball, then $\mathbf{Z}$ is even reversible with respect to μ. We investigate in §6 whether reversibility of $\mathbf{Z}$ with respect to μ implies that D is a ball. This is indeed the case for $d = 2$, and under the added assumption of strict convexity of D we can show it is also true in higher dimensions. We conjecture that in all dimensions invariance of $\mathbf{Z}$ with respect to μ is equivalent to D being a ball. Along the same lines, we show in Remark 5.6 that normalised arc–length is **never** the invariant measure for the analogue of $\mathbf{Z}$ in planar regions with piecewise linear boundary.

Recall the definition of $\xi(x, v)$ from (1.2). Let π be the probability measure on E defined by

$$\pi(dx, dv) := \frac{\kappa}{|v \cdot \nu_{\xi(x,v)}|} \frac{d\rho}{d\mu}(\xi(x, v))\, \lambda(dx)\, \sigma(dv),$$

where λ is Lebesgue measure on D, σ is Haar measure on S^{d-1} as above, and κ is the correpsonding normalisation constant.

Theorem 1.3. *In the above notation,*

$$\lim_{t\to\infty} \sup_{y\in E} \sup_{C} \left| \frac{1}{t} \int_0^t \mathbb{P}^y\{Y_s \in C\}\, ds - \pi(C) \right| = 0,$$

where the inner supremum is over all Borel subsets of E.

Remark 1.4. The classical billiard flow preserves the product measure $\lambda \otimes \sigma$ on $D \times S^{d-1}$. In contrast, it is clear that for no choice of D is π a constant multiple of $\lambda \otimes \sigma$.

Theorem 1.3 will be deduced from Theorem 1.1 in §4 using shift–coupling ideas and a simple "renewal–reward" argument using the strong law of large numbers for positive Harris recurrent chains. The sort of *uniform, geometric ergodicity* exhibited in Theorem 1.1 is known to be equivalent to a number of other conditions on the chain in question (cf. Theorem 16.0.2 of [MT93]). All of these equivalent conditions say in one way or another that the chain is irreducible and aperiodic in a sufficiently strong sense. In order to establish such a condition in §3, we proceed via a result of independent interest on the geometry of C^1 regions which we now outline.

Notation 1.5. For $x', x'' \in \mathbb{R}^d$, write

$$[x', x''] := \{sx' + (1-s)x'' : 0 \le s \le 1\}$$

for the closed line segment joining x' and x'', and write

$$]x', x''[:= \{sx' + (1-s)x'' : 0 < s < 1\}$$

for the open line segment joining x' and x''.

Definition 1.6. Given $z', z'' \in \partial D$, write $z' \rightleftharpoons z''$ (and say that z' and z'' *see each other*) if $]z', z''[\subset D$, $(z'' - z') \cdot \nu_{z'} > 0$, and $(z' - z'') \cdot \nu_{z''} > 0$. Note that under the assumption $]z', z''[\subset D$ the condition $(z'' - z') \cdot \nu_{z'} > 0$ is implied by either of the equivalent conditions $(z'' - z') \cdot \nu_{z'} \ne 0$ or $(z'' - z') \ne T_{z'}$. That is, $z' \rightleftharpoons z''$ if the closed line segment joining z' and z'' does not encounter any other points of ∂D and makes a non-zero angle with the tangent hyperplanes to ∂D at z' and z''.

The proof of the following is given in §2.

Proposition 1.7. *There is an integer K and a finite set $\Delta \subset \partial D$ for which the following holds: for all $z', z'' \in \partial D$ there exist $z_0, \dots z_K$ with $z' = z_0$, $z'' = z_K$, $\{z_1, \dots, z_{K-1}\} \subseteq \Delta$, and $z_k \rightleftharpoons z_{k+1}$ for $0 \le k \le K-1$.*

Our results here suggest a number of possible extensions. For instance, it is natural to inquire whether similar behaviour is observed for the analogues of $\mathbf{Y}$ and $\mathbf{Z}$ in regions with boundaries that are only assumed to be Lipschitz: we take one very small step in this direction by showing in §5 that this is indeed the case for polygonal regions of the plane. Alternatively, one could seek to "interpolate" between the classical case and the fully stochastic case considered here and, for example, investigate systems in which there is random reflection in a restricted range of directions dictated by the direction of incidence.

2. Proof of Proposition 1.7

We lead up to the proof with a number of preliminary lemmas.

Lemma 2.1. *The set ∂D has finitely many connected components.*

Proof. It follows from the C^1 assumption that ∂D is locally connected. This coupled with the compactness of ∂D gives the result (cf. Problem 3 in §34 of [Sim63]). $\square$

Notation 2.2. For $z \in \partial D$ put $U_z := \{z' \in \partial D : z' \rightleftharpoons z\}$.

Lemma 2.3. *Each set U_z, $z \in \partial D$, is relatively open in ∂D and non–empty. Consequently, $\partial D = \bigcup_z U_z$.*

Proof. The openness of each U_z is clear from the C^1 assumption on ∂D. By construction, $z' \in U_z$ if and only if $z \in U_{z'}$, and so the claim $\partial D = \bigcup_z U_z$ will follow if we can show that each U_z is non–empty.

Fix $z \in \partial D$ and $v \in S^{d-1}$ with $v \cdot \nu_z > 0$. Put $y = \xi(z, v)$. If $(z-y) \cdot \nu_y > 0$ we are done. Suppose, therefore, that $(z-y) \cdot \nu_y = 0$.

Let $\epsilon > 0$, $A \in \mathrm{Isom}(\mathbb{R}^d)$, and $F \in C^1(\mathbb{R}^{d-1}, \mathbb{R})$ be such that $Ay = 0$, $F(0) = 0$, and $A(D \cap B(y, \epsilon)) = \{u \in B(0, \epsilon) : u_d > F(u_1, \dots u_{d-1})\}$. We can suppose without loss of generality that $Az \in \{u \in \mathbb{R}^d : u_1 < -\epsilon, u_2 = \dots = u_{d-1} = 0\}$. Write $c = \frac{\partial F}{\partial u_1}(0)$. Then we must have $Az = (-b, 0, \dots, 0, -bc)$ for some $b > \epsilon$.

Consider the function $f \in C^1(\mathbb{R}, \mathbb{R})$ defined by $f(s) := F(s, 0, \dots, 0)$. By construction, $f(0) = 0$ and $f'(0) = c$. Let $\delta > 0$ be such that $\{(s, f(s)) : |s| < \delta\} \subset B(0, \epsilon)$.

By definition of y it must be the case that $f(s) \neq cs$ for $s \in]-\delta, 0[$. By continuity it must then be that either

$$f(s) > cs \text{ for all } s \in]-\delta, 0[\tag{2.1}$$

or

$$f(s) < cs \text{ for all } s \in]-\delta, 0[. \tag{2.2}$$

Consider first the case (2.1). Write M for the set of $t \in]-\delta, 0[$ such that $f(t) - ct < f(s) - cs$ for all $s \in]-\delta, t[$. Note that $M \cap]-\eta, 0[\neq \emptyset$ for any $\eta > 0$. For $t \in M$ the line segment $](-b, -bc), (t, f(t))[$ does not intersect the set $\{(s, f(s)) : |s| < \delta\}$. Moreover, $f'(t) \le c < (f(t) - (-bc))/(t - (-b))$

Consequently, for $t \in M$ the line segment

$$](-b, 0, \ldots, 0, -bc), ((t, 0, \ldots, 0), F(t, 0, \ldots, 0))[$$

does not intersect the set $\{(u, F(u)) : \|u\| < \delta\}$ and the vector

$$((t, 0, \ldots, 0), F(t, 0, \ldots, 0)) - (-b, 0, \ldots, 0, -bc)$$

is not in the tangent plane to the function F at the point $(t, 0, \ldots, 0)$.

Put $y(t) := A^{-1}((t, 0, \ldots, 0), F(t, 0, \ldots, 0))$. Then $]z, y(t)[\subset D$ for all $t \in M$ sufficiently close to 0 because otherwise we could find a sequence $(t_k)_{k=1}^{\infty} \subset M$ converging to 0 such that $\xi(z, (y(t_k) - z)/\|y(t_k) - z\|) \in \partial D \cap A^{-1}\{x : x_1 \leq -\delta\}$, but then $(y(t_k))_{k=1}^{\infty}$ would have a subsequential limit converging to a point $y^* \in \partial D \cap A^{-1}\{x : x_1 \leq -\delta\}$ such that $(y^* - z)/\|y^* - z\| = v$ and $\|y^* - z\| < \|y - z\|$, contradicting the definition of y. By construction, $z - y(t) \notin T_{y(t)}$ for all $t \in M$, and hence $(z - y(t)) \cdot \nu_{y(t)} > 0$. Finally, $(y(t) - z) \cdot \nu_z > 0$ for all $t \in M$ sufficiently close to 0.

A similar argument handles (2.2). $\square$

Lemma 2.4. *For each connected component C of ∂D there is an integer G^C and a finite set $\Sigma^C \subset \partial D$ such that the following holds: for all $z', z'' \in C$ there exist $z_0, \ldots z_L$ with $L \leq G^C$, $z' = z_0$, $z'' = z_L$, $\{z_1, \ldots, z_{L-1}\} \subseteq \Sigma^C$, and $z_k \rightleftharpoons z_{k+1}$ for $0 \leq k \leq L - 1$.*

Proof. By the compactness of C and Lemma 2.3 there exist $x_1, \ldots, x_N \in \partial D$ such that $C = \bigcup_{i=1}^N V_{x_i}$, where we put $V_z := U_z \cap C$ for $z \in \partial D$. We may suppose that each V_{x_i} is non–empty. When $V_{x_i} \cap V_{x_j} \neq \emptyset$, let $y_{ij} = y_{ji}$ be an arbitrary point in the intersection. Define Σ^C to be the set consisting of the x_i and the y_{ij}.

By the connectedness of C, there exist $x_{i(1)}, \ldots, x_{i(I)}$ with $1 \leq i(1), \ldots, i(I) \leq N$ and $i(a) \neq i(b)$ for $a \neq b$ (hence $I \leq N$) such that $z' \in V_{x_{i(1)}}$, $V_{x_{i(a)}} \cap V_{x_{i(a+1)}} \neq \emptyset$ for $1 \leq a \leq I - 1$, and $z'' \in V_{x_{i(I)}}$. Put

$$\begin{aligned}
z_0 &:= z' \\
z_1 &:= x_{i(1)} \in U_{z'} = U_{z_0} \\
z_2 &:= y_{i(1),i(2)} \in U_{z_1} \\
z_3 &:= x_{i(2)} \in U_{z_2} \\
z_4 &:= y_{i(2),i(3)} \in U_{z_3} \\
&\cdots \\
z_{2I-1} &:= x_{i(I)} \in U_{z_{2I-2}} \\
z_{2I} &:= z'' \in V_{x_{i(I)}} \subseteq U_{z_{2I-1}}.
\end{aligned}$$

Thus $G^C = 2N$ works. $\square$

For the next lemma, recall from Lemma 2.1 that ∂D has finitely many connected components.

Lemma 2.5. *Let $C_1, \ldots, C_M$ denote the connected components of ∂D. There is a finite set $\Xi \subset \partial D$ such that the following holds: given $1 \leq i \neq j \leq M$ there are indices $1 \leq k_1, \ldots, k_H \leq M$ with $k_a \neq k_b$ for $a \neq b$ (hence $H \leq M$) such that $i = k_1$, $j = k_H$, and for $1 \leq a \leq H-1$ there exist $x_a \in C_{k_a} \cap \Xi$ and $y_{a+1} \in C_{k_{a+1}} \cap \Xi$ with $x_a \rightleftharpoons y_{a+1}$.*

Proof. For $1 \le i \ne j \le M$ let $z_{ij} \in C_i$ and $z_{ji} \in C_j$ be a (possibly non–unique) pair such that $\|z_{ij} - z_{ji}\| = \inf\{\|x-y\| : x \in C_i, y \in C_j\}$. Set $\Xi := \{z_{ij} : 1 \le i \ne j \le M\}$.

For $1 \le i, j \le M$ write $i \sim j$ if either $i = j$ or $i \ne j$ and there exists a sequence $1 \le k_1, \ldots, k_H \le M$ with the properties described in the statement. It is clear that $\sim$ is an equivalence relation, and the statement of the lemma asserts that this relation has only one equivalence class.

Suppose, to the contrary, that there are equivalence classes $\mathcal{E}_1, \ldots, \mathcal{E}_P$ with $P \ge 2$. By compactness we have

$$\inf_{1 \le I \le J \le P} \inf_{i \in \mathcal{E}_I, j \in \mathcal{E}_J} \inf_{x \in C_i, y \in C_j} \|x - y\| > 0,$$

and the infimum is attained for some pair $x_0 \in C_{i_0}$, $i_0 \in I_0$, and $y_0 \in C_{j_0}$, $j_0 \in J_0$, with $I_0 \ne J_0$. We may suppose that $x_0 = z_{i_0, j_0}$ and $y_0 = z_{j_0, i_0}$.

By construction, $]x_0, y_0[\cap \partial D = \emptyset$ and so $]x_0, y_0[\subset D$. It must also be the case that $(y_0 - x_0) \perp T_{x_0}$ because otherwise we could find $x_1 \in C_{I_0}$ arbitrarily close to x_0 such that $\|x_1 - y_0\| < \|x_0 - y_0\|$. Similarly, $(x_0 - y_0) \perp T_{y_0}$. Thus $x_0 \rightleftharpoons y_0$ and we obtain the contradiction $i_0 \sim j_0$. $\square$

Lemma 2.6. *There exist distinct points $\hat{x}, \hat{y}, \hat{z} \in \partial D$ such that $\hat{x} \rightleftharpoons \hat{y}$, $\hat{x} \rightleftharpoons \hat{z}$, and $\hat{y} \rightleftharpoons \hat{z}$.*

Proof. Fix $w \in D$ and let $\hat{x} \in \partial D$ be such that $\|\hat{x} - w\| = \sup\{\|x' - w\| : x' \in D\}$, so that $\bar{D} \subseteq \bar{B}(w, \|\hat{x} - w\|)$. Consequently, $\nu_{\hat{x}} = (w - \hat{x})/\|w - \hat{x}\|$ and $(x' - \hat{x}) \cdot \nu_{\hat{x}} > 0$ for $x' \in \bar{D} \backslash \{\hat{x}\}$.

Let $\epsilon > 0$, $A \in \mathrm{Isom}(\mathbb{R}^d)$, and $F \in C^1(\mathbb{R}^{d-1}, \mathbb{R})$ be such that $A\hat{x} = 0$, $F(0) = 0$, and $A(D \cap B(\hat{x}, \epsilon)) = \{u \in B(0, \epsilon) : u_d > F(u_1, \ldots u_{d-1})\}$.

Define $f \in C^1(\mathbb{R}, \mathbb{R})$ by $f(t) := F(t, 0, \ldots, 0)$. By construction, $f(0) = 0$ and $f(t) > tf'(0)$ for all $t \ne 0$ sufficiently small. Put

$$\zeta(b) := \begin{cases} \inf\{t > 0 : f(t) = tb\}, & \text{if } b > f'(0), \\ \sup\{t < 0 : f(t) = tb\}, & \text{if } b < f'(0). \end{cases}$$

Note that $\zeta(b) \ne 0$ and $\lim_{b \to f'(0)} \zeta(b) = 0$. By the argument in the proof of Lemma 2.3, for any $b > f'(0)$ with $\zeta(b) < \infty$ we can find s arbitrarilty close to $\zeta(b)$ such that $f(t) < tf(s)/s$ for all $t \in]0, s[$ and $f'(s) > f(s)/s > f'(0)$. Similarly, for any $b < f'(0)$ with $\zeta(b) > -\infty$ we can find s arbitrarilty close to $\zeta(b)$ such that $f(t) < tf(s)/s$ for all $t \in]s, 0[$ and $f'(s) < f(s)/s < f'(0)$. Combining these observations, we see that for any $\delta > 0$ we can find $-\delta < \sigma < 0 < \tau < \delta$ such that

$$\begin{aligned} f(t) &< t\frac{f(\sigma)}{\sigma}, \quad t \in]\sigma, 0[, \\ f'(\sigma) &< \frac{f(\sigma)}{\sigma} < f'(0), \\ f(t) &< t\frac{f(\tau)}{\tau}, \quad t \in]0, \tau[, \\ f'(0) &< \frac{f(\tau)}{\tau} < f'(\tau). \end{aligned}$$

It is not hard to conclude from these inequalities that

$$f'(\sigma) < \frac{f(\sigma)}{\sigma} < \frac{f(\tau) - f(\sigma)}{\tau - \sigma} < \frac{f(\tau)}{\tau} < f'(\tau),$$

and

$$f(t) < \frac{\tau - t}{\tau - \sigma} f(\sigma) + \frac{t - \sigma}{\tau - \sigma} f(\tau), \quad t \in]\sigma, \tau[,$$

so that

$$](\sigma, f(\sigma)), (\tau, f(\tau))[\subset \{u \in \mathbb{R}^2 : u_2 > f(u_1)\}.$$

We can thus take

$$\hat{y} := A^{-1}((\sigma, 0, \ldots, 0), F(\sigma, 0, \ldots, 0))$$

and

$$\hat{z} := A^{-1}((\tau, 0, \ldots, 0), F(\tau, 0, \ldots, 0))$$

provided δ is sufficiently small. □

Completion of the proof of Proposition 1.7. Given what we have already shown, the argument follows a pattern familiar from the elementary theory of discrete time Markov chains, but we include the details for completeness.

In the notation of Lemmas 2.4, 2.5 and 2.6, set $\Delta := (\bigcup_C \Sigma^C) \cup \Xi \cup \{\hat{x}, \hat{y}, \hat{z}\}$. Given $y', y'' \in \partial D$, say that it is possible to see y'' from y' in I steps if there exist $y_0, \ldots y_I \in \partial D$ with $y' = y_0$, $y'' = y_I$, $\{y_1, \ldots, y_{I-1}\} \subseteq \Delta$, and $y_i \rightleftharpoons y_{i+1}$ for $0 \le i \le I - 1$. Write $I(y', y'')$ for the minimal number of steps necessary (with $I(y', y'') = \infty$ if no such sequence exists). Combining Lemmas 2.4 and 2.5, we see that there is an integer R such $I(y', y'') \le R$ for all $y', y'' \in \partial D$.

Given $z', z'' \in \partial D$, we can, by passing through $\hat{x}$, see z'' from z' in $I(z', \hat{x}) + I(\hat{x}, z'')$ steps. From Lemma 2.6 we conclude that we can pad out the sequence of states visited in seeing z'' from z' via $\hat{x}$ to include "redundant" subsequences of consecutive visits of the form $\ldots, \hat{x}, \hat{y}, \hat{x}, \ldots$ or $\ldots, \hat{x}, \hat{y}, \hat{z}, \hat{x}, \ldots$. We can thereby see z'' from z' in $I(z', \hat{x}) + I(\hat{x}, z'') + k$ steps for any $k \ge 2$. It is therefore always possible to see z'' from z' in exactly $2R + 2$ steps. □

3. Proof of Theorem 1.1

Write Q for the one–step transition kernel of $\mathbf{Z}$. Note for a suitable normalisation constant α that

$$\begin{aligned} Q(z', dz'') &= \alpha \mathbf{1}\{z' \rightleftharpoons z''\}\, \nu_{z''} \cdot \frac{(z' - z'')}{\|z' - z''\|^d} \mu(dz'') \\ &:= q(z', z'') \mu(dz''). \end{aligned} \tag{3.1}$$

Let K be as in Proposition 1.7. It suffices by Theorem 16.0.2 of [MT93] to show for some strictly positive Borel function $\gamma : \partial D \to \mathbb{R}$ that

$$\inf_{z' \in \partial D} Q^K(z', B) \ge \int_B \gamma(z'')\, \mu(dz'') \tag{3.2}$$

for all Borel sets $B \subseteq \partial D$.

Now

$$Q^K(z', dz'') = q^K(z', z'') \mu(dz'')$$

where

$$q^K(z', z'') = \int_{(\partial D)^{K-1}} q(z', z_1) q(z_1, z_2) \ldots q(z_{K-1}, z'')\, \mu(dz_1) \ldots \mu(dz_{K-1}).$$

By Proposition 1.7 and the C^1 assumption, the set $\{(z_1, \ldots, z_{K-1}) \in (\partial D)^{K-1} : z' \rightleftharpoons z_1 \rightleftharpoons z_2 \rightleftharpoons \cdots \rightleftharpoons z_{K-1} \rightleftharpoons z''\}$ is non–empty and (relatively) open for all $z', z'' \in \partial D$. Therefore $q^K(z', z'') > 0$ for all $z', z'' \in \partial D$. Now $q(\cdot, z)$ is clearly lower–semicontinuous for all $z \in \partial D$, and so, by Fatou's lemma, $q^K(\cdot, z)$ is also lower–semicontinuous for all $z \in \partial D$. Because a lower–semicontinuous function achieves its infimum, we must have $\gamma(z'') := \inf_{z' \in \partial D} q^K(z', z'') > 0$ for all $z'' \in \partial D$, and (3.2) holds as required. □

Remark 3.1. The conclusion $Q^K(z', dz'') \geq \gamma(z'')\mu(dz'')$ with $\gamma > 0$ established in the above proof certainly shows that $\mathbf{Z}$ is, in the language of §6 of [MT93], a μ–irreducible T–chain (T stands for *topological*). The chain $\mathbf{Z}$ is certainly Harris recurrent (for example, by Theorem 9.0.2 of [MT93]) and so for all $z \in \partial D$ and Borel $C \subseteq \partial D$ with $\mu(C) > 0$ we have $Q^z\{Z_k \in C \text{ i.o.}\} = 1$. Now recall the stopping times $(S_k)_{k=0}^\infty$ defined in (1.3). Because $\|X_s - X_t\| \leq |s-t|$ for $s, t \in [0, \sup_k S_k[$, it follows that on the event $\{\sup_k S_k < \infty\}$ we have that $\lim_k X_{S_k}$ exists $\mathbb{P}^y$–a.s. for all $y \in E$. However, $\mathbb{P}^y\{\lim_k X_{S_k} \text{ exists}\} = \mathbb{Q}^{\xi(y)}\{\lim_k Z_k \text{ exists}\} = 0$ by what we have just observed about Harris recurrence. Consequently, Y_t is well–defined for all $t \geq 0$, as claimed in the Introduction.

4. Proof of Theorem 1.3

It suffices to show that π is an invariant probability measure for $\mathbf{Y}$ and that

$$\lim_{t\to\infty} \sup_{y',y''\in E} \sup_C \left| \frac{1}{t}\int_0^t \mathbb{P}^{y'}\{Y_s \in C\}\, ds - \frac{1}{t}\int_0^t \mathbb{P}^{y''}\{Y_s \in C\}\, ds \right| = 0. \tag{4.1}$$

Consider first the invariance of π. Recall the sequence of stopping times $(S_k)_{k=0}^\infty$ defined in (1.3). Note that $(\tilde{Z}_k)_{k=1}^\infty := ((Z_k, Z_{k+1}))_{k=1}^\infty = ((X_{S_k}, X_{S_{k+1}}))_{k=1}^\infty$ has the distribution of a positive Harris recurrent Markov chain on $E \times E$ with transition kernel $\tilde{Q}((z_1', z_2'), d(z_1'', z_2'')) := \delta_{z_2'}(dz_1'')Q(z_2', dz_2'')$ and invariant probability measure $\tilde{\rho}(d(z_1, z_2)) := \rho(dz_1)Q(z_1, dz_2)$.

Recalling the definition of $r(\cdot,\cdot)$ from (1.1), put

$$\Theta_j := \frac{X_{S_{j+1}} - X_{S_j}}{\|X_{S_{j+1}} - X_{S_j}\|} = V_{S_j}$$

and

$$R_j := S_{j+1} - S_j = r(X_{S_j}, \Theta_j).$$

If f is a bounded, Borel function on E, then, by the strong law of large numbers for positive Harris recurrent chains (cf. Theorem 17.0.1 of [MT93]),

$$\begin{aligned}
&\lim_k \frac{1}{k-1}\int_{S_1}^{S_k} f(Y_s)\, ds = \lim_k \frac{1}{k-1}\sum_{j=1}^{k-1}\int_0^{R_j} f(X_{S_j} + s\Theta_j)\, ds \\
&= \beta \int_{\partial D}\int_{S^{d-1}}\int_0^\infty f(z+sv, v)\mathbf{1}\{z \rightleftharpoons z+sv\}\, ds\, \sigma(dv)\, \rho(dz), \quad \mathbb{P}^y\text{-a.s. for all } y \in E
\end{aligned}$$

for a suitable constant β. Therefore, by taking $f \equiv 1$ and dividing,

$$\begin{aligned}
&\lim_k \frac{\int_{S_1}^{S_k} f(Y_s)\,ds}{S_k - S_1} \\
&= \beta' \int_{\partial D} \int_{S^{d-1}} \int_0^\infty f(z+sv, v)\mathbf{1}\{z \rightleftharpoons z+sv\}\, ds\, \sigma(dv)\, \rho(dz) \qquad (4.2)\\
&:= \int_E f(x,v)\,\bar{\pi}(d(x,v)), \quad \mathbb{P}^y\text{-a.s. for all } y \in E,
\end{aligned}$$

for some other constant β'. Because each $S_{j+1} - S_j$ is at most the diameter of ∂D, we can interpolate and take expectations to conclude that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}^y[f(Y_s)]\,ds = \int_E f(x,v)\,\bar{\pi}(d(x,v)), \quad \text{for all } y \in E.$$

Thus $\bar{\pi}$ is an invariant probability measure for $\mathbf{Y}$.

We next show that $\bar{\pi} = \pi$. Transforming from polar co-ordinates around z to Cartesian co-ordinates for the inner two integrals in equation (4.2), we see that

$$\int_E f(x,v)\,\bar{\pi}(d(x,v)) = \beta'' \int_{\partial D}\int_D f\left(u, \frac{u-z}{\|u-z\|}\right) \frac{\mathbf{1}\{z \rightleftharpoons u\}}{\|u-z\|^{d-1}}\,\lambda(du)\,\rho(dz)$$

for some constant β''. Defining yet another set of co-ordinates by $x = u$ and $v = (u-z)/\|u-z\|$, so that $z = \xi(x,v)$ when $z \rightleftharpoons u$, shows that $\bar{\pi} = \pi$.

It remains to show that (4.1) holds. From the proof of Theorem 16.2.4 in [MT93], we see that the condition (3.2) implies we can successfully couple versions of $\mathbf{Z}$ exponentially quickly starting from any two initial distributions. More precisely, given two probability measures ζ' and ζ'' on ∂D it is possible to build on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ two processes $(Z'_k)_{k\ge 0}$ and $(Z''_k)_{k\ge 0}$ and a random time τ such that $(Z'_k)_{k\ge 0}$ has distribution $\mathbb{Q}^{\zeta'}$, $(Z''_k)_{k\ge 0}$ has distribution $\mathbb{Q}^{\zeta''}$, $\mathbb{P}\{\tau > n\} \le c^n$ for some universal constant $0 \le c < 1$ that does not depend on ζ' and ζ'', and $Z'_{\tau+k} = Z''_{\tau+k}$ for $k \ge 0$.

Given how $\mathbf{Y}$ can be recovered from $\mathbf{Z}$ from time S_1 onwards and the remark we have already made that $S_{k+1} - S_k$ is at most the diameter of ∂D for all k, we see that we can successfully shift–couple versions of $\mathbf{Y}$ starting from any two initial positions. More precisely, given $y', y'' \in E$ it is possible to build on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ two processes $(Y'_t)_{t\ge 0}$ and $(Y''_t)_{t\ge 0}$ and random times τ', τ'' such that $(Y'_t)_{t\ge 0}$ has distribution $\mathbb{P}^{y'}$, $(Y''_t)_{t\ge 0}$ has distribution $\mathbb{P}^{y''}$, $\mathbb{P}\{\tau' > t\} \vee \mathbb{P}\{\tau' > t\} \le c^* e^{-c^{**}t}$ for some universal constants $c^*, c^{**} > 0$ that do not depend on y' and y'', and $Z'_{\tau'+t} = Z''_{\tau''+t}$ for $t \ge 0$. By the shift–coupling inequality (see, for example, §3 of [Tho94] or Proposition 5 of [RR96]) we have

$$\begin{aligned}
&\sup_C \left| \frac{1}{t}\int_0^t \mathbb{P}^{y'}\{Y_s \in C\}\,ds - \frac{1}{t}\int_0^t \mathbb{P}^{y''}\{Y_s \in C\}\,ds \right| \\
&= \left| \frac{1}{t}\int_0^t \mathbb{P}\{Y'_s \in C\}\,ds - \frac{1}{t}\int_0^t \mathbb{P}\{Y''_s \in C\}\,ds \right| \\
&\le \frac{1}{t}\left(\int_0^\infty \mathbb{P}\{\tau' > s\}\,ds + \int_0^\infty \mathbb{P}\{\tau'' > s\}\,ds \right) \\
&\le \frac{2c^*}{c^{**}t},
\end{aligned}$$

as required. □

5. Polygonal tables

In this section we sketch how analogues of Theorems 1.1 and 1.3 can be obtained in the case where the table is a polygonal subset of $\mathbb{R}^2$.

Suppose now that $D \subset \mathbb{R}^2$ is a bounded, connected, open, subset of $\mathbb{R}^2$. Suppose further that D is polygonal. That is, the boundary ∂D of D can be written as the union of a finite collection $\mathcal{S}$ of closed line segments (the *sides* of D) such that any two distinct sides are either disjoint or intersect in a common endpoint, and any endpoint of a side belongs to precisely one other side. It will be convenient to adopt the convention that $\mathcal{S}$ is chosen minimally, so that the union of any two sides in $\mathcal{S}$ is not a line segment (equivalently, if two sides intersect in a point, then the angle between the sides is not π). With this convention, $\mathcal{S}$ is uniquely defined. Write $\mathcal{C} \subset \partial D$ for the finite collection of endpoints of sides (the *corners* or *vertices* of D). Note that we are again not precluding the possibility that ∂D is disconnected. Of course, the number of connected components of ∂D is necessarily finite.

We can define analogues of our billiard processes $\mathbf{Y}$ and $\mathbf{Z}$. The rule for reflection on the boundary is that a reflection angle is chosen uniformly from the directions that point back into the interior of the table. These directions form an open semicircle at points of ∂D other than the corners. We leave the simple task of formalising this prescription to the reader.

Definition 5.1. For $z' \in \partial D$ (re–)define the set $U_{z'}$ as $\{z'' \in \partial D :]z', z''[\subset D\}$. Let $U_{z'}^o$ denote the (relative) interior of $U_{z'}$. Note that $U_{z'} \backslash U_{z'}^o \subset \mathcal{C}$. Moreover, if $z'' \in U_{z'}^o$ for some $z' \in \partial D \backslash \mathcal{C}$, then $z' \in U_{z''}^o$.

The following result is the analogue of Lemma 2.3.

Lemma 5.2. *In the above notation,* $\partial D = \bigcup_{z \in \partial D \backslash \mathcal{C}} U_z^o$.

Proof. Fix $z'' \in \partial D$. Let $v \in S^1$ be a direction that is not parallel to any of the (at most 2) sides that contain z'', points from z'' into D, and does not point to a corner (that is, $z'' + sv \in \mathbb{R}^2 \backslash \bar{D}$ for all $s < 0$ sufficiently close to 0, $z'' + sv \in D$ for all $s > 0$ sufficiently close to 0, and $\mathcal{C} \cap \{z'' + sv : s > 0\} = \emptyset$). Put $z' := \xi(z'', v) \notin \mathcal{C}$. Then $z'' \in U_{z'}^o$. □

The following result is the analogue of Lemma 2.4 and can be proved the same way, with Lemma 5.2 playing the rôle of Lemma 2.3.

Lemma 5.3. *For each connected component C of ∂D there is an integer G^C and a finite set $\Sigma^C \subset \partial D \backslash \mathcal{C}$ such that the following holds: for all $z', z'' \in C$ there exist $z_0, \ldots z_L$ with $L \le G^C$, $z' = z_0$, $z'' = z_L$, $\{z_1, \ldots, z_{L-1}\} \subseteq \Sigma^C$, and $z_{k+1} \in U_{z_k}^o$ for $0 \le k \le L-1$.*

The following result is the analogue of Lemma 2.6

Lemma 5.4. *There exist distinct points $\hat{x}, \hat{y}, \hat{z} \in \partial D \backslash \mathcal{C}$ such that $\hat{y}, \hat{z} \in U_{\hat{x}}^o$, $\hat{x}, \hat{z} \in U_{\hat{y}}^o$, and $\hat{x}, \hat{y} \in U_{\hat{z}}^o$.*

Proof. There must exist at least one corner $w \in \mathcal{C}$ such that the interior angle at w is less than π. Let $v \in S^1$ be a direction that points from w into D and doesn't point towards any other corner (that is, $w + sv \in D$ for all $s > 0$ sufficiently close to 0 and $\mathcal{C} \cap \{w + sv : s > 0\} = \emptyset$). Put $\hat{x} := \xi(w, v) \in \partial D \backslash \mathcal{C}$. It is not hard to see that if we choose $\hat{y}$ and $\hat{z}$ sufficiently close to w with one each from the (relative interiors of) the two sides meeting at w, then $\hat{x}, \hat{y}, \hat{z}$ will have the desired properties. □

Using Lemmas 5.3 and 5.4 in place of Lemmas 2.4 and 2.6 and the obvious analogue of Lemma 2.5, the statement and proof of which we omit, the following counterpart of Proposition 1.7 can be proved along the same lines. The proofs of the obvious counterparts of Theorems 1.1 and 1.3 are then straightforward. We leave the details to the reader.

Proposition 5.5. *There is an integer K and a finite set $\Delta \subset \partial D \backslash \mathcal{C}$ for which the following holds: for all $z', z'' \in \partial D$ there exist $z_0, \dots z_K$ with $z' = z_0$, $z'' = z_K$, $\{z_1, \dots, z_{K-1}\} \subseteq \Delta$, and $z_{k+1} \in U^o_{z_k}$ for $0 \le k \le K-1$.*

Remark 5.6. In the same spirit as Remark 1.2, we note that the invariant distribution for the analogue of $\mathbf{Z}$ in this setting is never normalised arc–length μ. To see this, first note that there must be a corner of D with interior angle ψ strictly less than π. By applying a suitable isometry of $\mathbb{R}^2$, we may suppose that the corner is at $(0,0)$ and the 2 corresponding sides lie in the half-lines $\{(a\cos\psi, a\sin\psi) : a \ge 0\}$ and $\{(b,0) : b \ge 0\}$. The one–step transition kernel Q is of the form $Q(z', dz'') = q(z', z'')\mu(dz'')$ where, for some $\delta > 0$, we have

$$q((a\cos\psi, a\sin\psi),(b,0)) = \alpha \frac{a\sin\psi}{(b - a\cos\psi)^2 + a^2\sin^2\psi}, \quad 0 < a, b < \delta,$$

for a suitable normalisation constant α (cf. (3.1)). Thus,

$$\int_{\partial D} q(z,(b,0))\,\mu(dz) \ge \alpha \int_0^\delta \frac{a\sin\psi}{(b - a\cos\psi)^2 + a^2\sin^2\psi}\,da, \quad 0 < b < \delta,$$

and the quantity on the left–hand side cannot be equal to 1 for μ-a.e. $b \in]0,\delta[$ because the integral on the right–hand side blows up to ∞ as b converges to 0.

6. Reversibility

As we observed in Remark 1.2, in the case of $D = B(0,1)$ and $\partial D = S^{d-1}$ not only do we have that $\rho = \mu = \sigma$, but also that $\mathbf{Z}$ is reversible with respect to this probability measure. In this section we investigate whether this property is unique to balls. We first consider planar regions.

Proposition 6.1. *For $d = 2$, the Markov chain $\mathbf{Z}$ is reversible with respect to μ if and only if D is a disc.*

Proof. First note from (3.1) that reversibility with respect to μ implies that

$$\nu_{z'} \cdot (z'' - z') = \nu_{z''} \cdot (z' - z''), \quad z', z'' \in \partial D,\ z' \rightleftharpoons z''. \tag{6.1}$$

Fix $z \in \partial D$ and suppose without loss of generality that $z = 0$ and $\nu_z = (0,1)$. The function $\tilde{\xi} :]0,\pi[\to \mathbb{R}^2$ defined by $\tilde{\xi}(\theta) := \xi(0, (\cos\theta, \sin\theta))$ is continuous, because at any discontinuity θ_0 of $\tilde{\xi}$ a little thought shows we must have $\nu_{\tilde{\xi}(\theta_0)} \cdot \tilde{\xi}(\theta_0) = 0$, contradicting (6.1). It follows that $\tilde{\xi}$ is actually continuously differentiable. If we put $\tilde{r}(\theta) := r(0, (\cos\theta, \sin\theta)) = ||\tilde{\xi}(\theta)||$, then $\tilde{r}$ is continuously differentiable and, from (6.1), satisfies the ordinary differential equation $\tilde{r}'(\theta) = \tilde{r}(\theta)\cot\theta$. It follows that $\tilde{r}(\theta) = 2c\sin\theta$ for some constant $c > 0$, so that ∂D is a circle with centre $(0,c)$ and radius c. □

If we move to dimensions greater than 2, then the problem seems to become somewhat more delicate. We remark that if one drops the condition that D is bounded, then regions such as cylinders also have the local property expressed by

(6.1); that is, that any line makes equal angles with the normals at the two points where it meets the boundary of the region. It is natural to try to mimic the proof of Proposition 6.1 and work in spherical coordinates around some fixed point on the boundary. This approach leads to a non–linear, first–order PDE that we are unable to analyse completely. We do, however, have the following result. Recall that $\bar{D}$ is *strictly convex* if it is convex and the tangent hyperplane at each point on the boundary meets the boundary only at that point.

Proposition 6.2. *Suppose that $\bar{D}$ is strictly convex and ∂D is C^2. Then the Markov chain $\mathbf{Z}$ is reversible with respect to μ if and only if D is a ball.*

Proof. Note that if S' and S'' are two spheres in $\mathbb{R}^d$ (that is, two isometric images of S^{d-1}) and U' (resp. U'') is relatively open in S' (resp. S''), then $U' \cap U''$ is relatively open in both S' and S'' if and only if $S' = S''$. From this observation it suffices to show that any point in ∂D has a neighbourhood that is contained in some sphere.

Fix $z^* \in \partial D$. Let $\epsilon > 0$, $A \in \mathrm{Isom}(\mathbb{R}^d)$, and $F \in C^2(\mathbb{R}^{d-1}, \mathbb{R})$ be such that $Az^* = 0$, $F(0) = 0$, and $A(D \cap B(z^*, \epsilon)) = \{u \in B(0, \epsilon) : u_d < F(u_1, \ldots u_{d-1})\}$. (Note that here we are thinking of D in the neighbourhood of z^* as looking like the region under, rather than over, the graph of a C^2 function — this seems more natural for the proof). We can assume that the function F is strictly concave and that $\nabla F(0) = 0$.

For x sufficiently close to 0 the choice of unit normal to the graph of F at $(x, F(x))$ given by $(\nabla F(x), -1)/(1 + \|\nabla F(x)\|^2)^{1/2}$ corresponds under the mapping A to the inward pointing unit normal to ∂D at $A^{-1}(x, F(x))$. In particular, the unit normal to the graph of F at $(0, F(0))$ corresponding to the inward unit normal at z^* is $(0, \ldots, 0, -1)$. Thus, $A\xi(z^*, \nu_{z^*}) = (0, \ldots, 0, b)$ for some $b < 0$. Moreover, it follows from (6.1) that $\nu_{\xi(z^*, \nu_{z^*})} = -\nu_{z^*}$. Therefore, if we apply (6.1) with $z'' = \xi(z^*, \nu_{z^*})$ and z' varying in some neighbourhood of z^* and map forward with A, then we see that the function $G := F - b$ solves the PDE

$$x \cdot \nabla G(x) + G(x) \left[(1 + \|\nabla G(x)\|^2)^{1/2} - 1\right] = 0 \tag{6.2}$$

in some neighbourhood U of 0. That is,

$$\Psi(\nabla G(x), G(x), x) = 0,$$

where

$$\Psi(p, z, x) = x \cdot p + z \left[(1 + \|p\|^2)^{1/2} - 1\right].$$

We will solve (6.2) using the method of characteristic ODEs (see, for example, §3.2 of [Eva98]) as follows. Consider the system of ODEs

$$\begin{aligned} \dot{p}(s) &= -\nabla_x \Psi(p(s), z(s), x(s)) - \nabla_z \Psi(p(s), z(s), x(s)) p(s) \\ &= -(1 + \|p(s)\|^2)^{1/2} p(s) \end{aligned} \tag{6.3}$$

$$\begin{aligned} \dot{z}(s) &= \nabla_p \Psi(p(s), z(s), x(s)) \cdot p(s) \\ &= x(s) \cdot p(s) + \frac{\|p(s)\|^2}{(1 + \|p\|^2)^{1/2}} z(s) \end{aligned} \tag{6.4}$$

$$\begin{aligned} \dot{x}(s) &= \nabla_p \Psi(p(s), z(s), x(s)) \\ &= x(s) + z(s) \frac{p(s)}{(1 + \|p\|^2)^{1/2}}. \end{aligned} \tag{6.5}$$

Suppose that G solves (6.2) and $x(\cdot)$ solves (6.5) with $p(\cdot) = \nabla G(x(\cdot))$ and $z(\cdot) = G(x(\cdot))$. Then the fundamental theorem of characteristic ODE says that $p(\cdot)$ solves (6.3) and $z(\cdot)$ solves (6.4) for those s such that $x(s)$ is in the neighbourhood U of 0 where (6.2) holds. We will assume that $x(0) \in U \backslash \{0\}$ so that, by strict convexity, $p(0) = \nabla G(x(0)) \neq 0$.

It follows from (6.3) that $\|p(s)\|$ is decreasing in s and

$$p(s) = \operatorname{cosech}(s+c)\bar{p}, \tag{6.6}$$

with $\bar{p} = \nabla G(x(0))/\|\nabla G(x(0))\|$ and $\operatorname{cosech}(c) = \|\nabla G(x(0))\|$.

We next claim that if $x(0)$ is in a sufficiently small neighbourhood of 0, then $x(s) \in U$ for all $s \geq 0$. To see this, let $\epsilon' > 0$ be such that $\bar{B}(0, \epsilon') \subset U$. Set $m := \inf\{\|\nabla G(x)\| : \|x\| = \epsilon'\}$. By strict convexity and the C^2 assumption, $m > 0$. Now let $0 < \epsilon'' < \epsilon'$ be such that $\sup\{\|\nabla G(x)\| : \|x\| < \epsilon''\} < m$. Because $\|p(s)\|$ is decreasing in s, it follows that if $x(0) \in B(0, \epsilon'')$, then $x(s) \in B(0, \epsilon') \subset U$ for all $s \geq 0$. Suppose from now on that $x(0) \in B(0, \epsilon'')$.

Note that

$$\lim_{s \to \infty} p(s) = 0. \tag{6.7}$$

By strong convexity, $\|\nabla G(x)\| > 0$ for $x \neq 0$, whereas $\nabla G(0) = 0$, and hence

$$\lim_{s \to \infty} x(s) = 0. \tag{6.8}$$

Substitute (6.6) into (6.4) and (6.5) to get

$$\dot{z}(s) = \operatorname{cosech}(s+c)x(s) \cdot \bar{p} + \operatorname{cosech}(s+c)\operatorname{sech}(s+c)z(s) \tag{6.9}$$

$$\dot{x}(s) = x(s) + \operatorname{sech}(s+c)z(s)\bar{p}. \tag{6.10}$$

Write $w(s) := x(s) - (x(s) \cdot \bar{p})\bar{p}$ for the projection of $x(s)$ onto the subspace orthogonal to $\bar{p}$. We see from (6.10) that

$$\dot{w}(s) = w(s),$$

and so $w(s) = [x(0) - (x(0) \cdot \bar{p})\bar{p}]\exp(s)$. This, however, contradicts (6.8) unless $x(0) = (x(0) \cdot \bar{p})\bar{p}$, which implies that

$$x(s) = y(s)\bar{p}, \quad s \geq 0,$$

where

$$y(s) := x(s) \cdot \bar{p}.$$

From (6.9) and (6.10) we get

$$\dot{z}(s) = \operatorname{cosech}(s+c)y(s) + \operatorname{cosech}(s+c)\operatorname{sech}(s+c)z(s) \tag{6.11}$$

$$\dot{y}(s) = y(s) + \operatorname{sech}(s+c)z(s). \tag{6.12}$$

Note that $G(0) = -b > 0$, and so we may suppose that $G(x) > 0$ for all $x \in U$. Thus, $r(s) := y(s)/z(s)$ is well–defined for all $s \geq 0$. It follows from (6.11) and (6.12) that $r(\cdot)$ solves the generalised Riccati equation

$$\begin{aligned} \dot{r}(s) = {} & \operatorname{sech}(s+c) + (1 - \operatorname{cosech}(s+c)\operatorname{sech}(s+c))\, r(s) \\ & - \operatorname{cosech}(s+c)r(s)^2. \end{aligned} \tag{6.13}$$

A solution of (6.13) is

$$r(s) = -\exp(-(s+c)). \tag{6.14}$$

Using the standard recipe for constructing general solutions from this particular one (cf. App. A, Table 14.1.vi of [IK77]), other solutions are of the form

$$r(s) = -\exp(-(s+c)) + \frac{\exp(-3(s+c)) - \exp(s+c)}{\exp(-2(s+c)) - 2(s+c) + C} \tag{6.15}$$

for some constant C. It follows from (6.8) that $\lim_{s\to\infty} r(s) = 0$, and so r must be given by the solution (6.14).

Substituting into (6.12) gives

$$\dot{y}(s) = (1 - \exp(s+c)\,\mathrm{sech}(s+c))\,y(s),$$

so that

$$y(s) = k\frac{\exp(s+c)}{1+\exp(2(s+c))}$$

for some constant k. Thus

$$x(s) = k\frac{\exp(s+c)}{1+\exp(2(s+c))}\bar{p} \tag{6.16}$$

and

$$z(s) = -k\frac{\exp(2(s+c))}{1+\exp(2(s+c))}, \tag{6.17}$$

from which we deduce that

$$\|x(s)\|^2 + \left(z(s) + \frac{k}{2}\right)^2 = \frac{k^2}{4}, \quad s \geq 0.$$

By (6.17) we have

$$-k = \lim_{s\to\infty} z(s) = G(0) = -b$$

Therefore,

$$\|x\|^2 + \left(G(x) + \frac{b}{2}\right)^2 = \frac{b^2}{4}, \quad x \in B(0, \epsilon''),$$

or, equivalently,

$$\|x\|^2 + \left(F(x) - \frac{b}{2}\right)^2 = \frac{b^2}{4}, \quad x \in B(0, \epsilon''),$$

as required. □

Remark 6.3. Balls are certainly not the only case where $\mathbf{Z}$ is reversible with respect to the invariant probability measure ρ. It is not hard to see that the annulus example of Remark 1.2 and its higher dimensional analogues are reversible. Another example is given by the ellipsoid $D := \{x \in \mathbb{R}^d : a_1x_1^2 + \cdots + a_dx_d^2 < 1\}$ for $a_1, \ldots, a_d > 0$. Here

$$\nu_z = -\frac{(a_1z_1, \cdots, a_dz_d)}{\|(a_1z_1, \cdots, a_dz_d)\|}$$

for $z \in \partial D = \{x \in \mathbb{R}^d : a_1x_1^2 + \cdots + a_dx_d^2 = 1\}$, and so

$$\|(a_1z_1', \ldots, a_dz_d')\|\,\nu_{z'} \cdot (z'' - z') = \|(a_1z_1'', \ldots, a_dz_d'')\|\,\nu_{z''} \cdot (z' - z'')$$

for $z', z'' \in \partial D$. Thus $\mathbf{Z}$ is reversible with respect to the invariant probability measure

$$\rho(dz) := K \, \|(a_1 z_1, \ldots, a_d z_d)\| \; \mu(dz)$$

for a suitable normalisation constant K.

Remark 6.4. Consider the Markov chain $\breve{\mathbf{Z}}$ on ∂D that has transition kernel $\breve{Q}$ given by

$$\begin{aligned}\breve{Q}(z', dz'') := & \frac{\nu_{z'} \cdot (z'' - z')}{\nu_{z'} \cdot (z'' - z') + \nu_{z''} \cdot (z' - z'')} Q(z', dz'') \\ & + \left[\int \frac{\nu_z \cdot (z' - z)}{\nu_{z'} \cdot (z - z') + \nu_z \cdot (z' - z)} Q(z', dz)\right] \delta_{z'}(dz'').\end{aligned}$$

That is, $\breve{\mathbf{Z}}$ attempts to make steps using the same mechanism as $\mathbf{Z}$, but a step from z' to z'' is rejected with conditional probability $[\nu_{z''} \cdot (z' - z'')]/[\nu_{z'} \cdot (z'' - z') + \nu_{z''} \cdot (z' - z'')]$. It is clear from (3.1) that $\breve{\mathbf{Z}}$ is reversible with respect to μ. The proof of Theorem 1.1 can be easily modified to show that $\breve{\mathbf{Z}}$ is also uniformly, geometrically ergodic. The chain $\breve{\mathbf{Z}}$ is the counterpart in our setting of the chain called *original shake–and–bake* in [BCM+91].

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