Infinitely divisible laws associated with hyperbolic functions

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Abstract

The infinitely divisible distributions on $\mathbb{R}^+$ of random variables C_t, S_t and T_t with Laplace transforms

$$\left(\frac{1}{\cosh\sqrt{2\lambda}}\right)^t, \quad \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^t, \text{ and } \left(\frac{\tanh\sqrt{2\lambda}}{\sqrt{2\lambda}}\right)^t$$

respectively are characterized for various $t > 0$ in a number of different ways: by simple relations between their moments and cumulants, by corresponding relations between the distributions and their Lévy measures, by recursions for their Mellin transforms, and by differential equations satisfied by their Laplace transforms. Some of these results are interpreted probabilistically via known appearances of these distributions for $t = 1$ or 2 in the description of the laws of various functionals of Brownian motion and Bessel processes, such as the heights and lengths of excursions of a one-dimensional Brownian motion. The distributions of C_1 and S_2 are also known to appear in the Mellin representations of two important functions in analytic number theory, the Riemann zeta function and the Dirichlet L-function associated with the quadratic character modulo 4. Related families of infinitely divisible laws, including the gamma, logistic and generalized hyperbolic secant distributions, are derived from S_t and C_t by operations such as Brownian subordination, exponential tilting, and weak limits, and characterized in various ways.

Keywords: Riemann zeta function, Mellin transform, characterization of distributions, Brownian motion, Bessel process, Lévy process, gamma process, Meixner process

AMS subject classifications. 11M06, 60J65, 60E07

Contents

1 Introduction

This paper is concerned with the infinitely divisible distributions generated by some particular processes with stationary independent increments (*Lévy processes* [6, 56]) associated with the hyperbolic functions cosh, sinh and tanh. In particular, we are interested in the laws of the processes $\hat{C}, C, \hat{S}, S, \hat{T}$ and T characterized by the following formulae: for $t \geq 0$ and $\theta \in \mathbb{R}$

$$E[\exp(i\theta\hat{C}_t)] = E\left[\exp\left(-\tfrac{1}{2}\theta^2 C_t\right)\right] = \left(\frac{1}{\cosh\theta}\right)^t \tag{1}$$

$$E[\exp(i\theta\hat{S}_t)] = E\left[\exp\left(-\tfrac{1}{2}\theta^2 S_t\right)\right] = \left(\frac{\theta}{\sinh\theta}\right)^t \tag{2}$$

$$E[\exp(i\theta\hat{T}_t)] = E\left[\exp\left(-\tfrac{1}{2}\theta^2 T_t\right)\right] = \left(\frac{\tanh\theta}{\theta}\right)^t \tag{3}$$

according to the mnemonic C for cosh, S for sinh and T for tanh. These formulae show how the processes $\hat{C}$, $\hat{S}$ and $\hat{T}$ can be constructed from C,S and T by Brownian subordination: for $X = C, S$ or T

$$\hat{X}_t = \beta_{X_t} \tag{4}$$

where $\beta := (\beta_u, u \geq 0)$ is a *standard Brownian motion*, that is the Lévy process such that β_u is has Gaussian distribution with $E(\beta_u) = 0$ and $E(\beta_u^2) = u$, and β is assumed independent of the increasing Lévy process (*subordinator*) X. Both $\hat{C}$ and $\hat{S}$ belong to the class of *generalized z-processes* [30], whose definition in recalled in Section 4. The distributions of $\hat{C}_1$, $\hat{S}_1$ and $\hat{T}_1$ arise in connection with Lévy's stochastic area formula [40] and in the study of the Hilbert transform of the local time of a symmetric Lévy process [9, 25]. As we discuss in Section 6.5, the distributions of $\hat{S}_1$ and $\hat{S}_2$ arise also in a completely different context, which is the work of Aldous [2] on asymptotics of the random assignment problem.

The laws of C_t and S_t arise naturally in many contexts, especially in the study of Brownian motion and Bessel processes [70, §18.6]. For instance, the distribution of C_1 is that of the hitting time of ± 1 by the one-dimensional Brownian motion β. The distribution of S_1 is that of the hitting time of the unit sphere by a Brownian motion in $\mathbb{R}^3$ started at the origin [17], while $(\pi/2)\sqrt{S_2}$ has the same distribution as the maximum of a standard Brownian excursion [15, 9]. This distribution also appears as an asymptotic distribution in the study of conditioned random walks and random trees [60, 1]. The distributions of C_t and S_t for $t = 1, 2$ are also of significance in analytic number theory, due to the Mellin representations of the entire function

$$\xi(s) := \tfrac{1}{2}s(s-1)\left(\frac{1}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \qquad \text{where } \zeta(s) := \sum_{n=1}^{\infty} n^{-s} \qquad (\Re s > 1) \tag{5}$$

is Riemann's zeta function, and the entire function

$$\xi_4(s) := \left(\frac{4}{\pi}\right)^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L_{\chi_4}(s) \qquad \text{where } L_{\chi_4}(s) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s} \qquad (\Re s > 0) \tag{6}$$

is the Dirichlet series associated with the quadratic character modulo 4. The functions $2\xi(2s)$ and $\xi_4(2s+1)$ appear as the Mellin transforms of $\frac{\pi}{2}S_2$ and $\frac{\pi}{2}C_1$ respectively, and the Mellin transforms of S_1, C_2, T_1 and T_2 are also simply related to ξ. These results are presented in Table 1, where $\theta \in \mathbb{R}$, $s \in \mathbb{C}$, (with $\Re s > -t/2$ for $\hat{T}_t$), and $n = 1, 2, \ldots$. The discussion around (8) and (9) below recalls the classical definitions of the numbers A_m and B_{2n} appearing in Table 1.

Table 1. The Mellin transforms of C_1, C_2, S_1, S_2, T_1 and T_2.

X	$E\left(e^{-\frac{1}{2}\theta^2 X}\right)$	$E\left[\left(\frac{\pi}{2}X\right)^s\right]$	$\frac{(2n)!}{2^n n!}E(X^n) = E(\hat{X}^{2n})$
C_1	$\dfrac{1}{\cosh\theta}$	$\xi_4(2s+1)$	A_{2n}
C_2	$\left(\dfrac{1}{\cosh\theta}\right)^2$	$\dfrac{(4^{s+1}-1)}{(s+1)}\dfrac{2}{\pi}\xi(2s+2))$	A_{2n+1}
S_1	$\dfrac{\theta}{\sinh\theta}$	$\dfrac{(2^{1-2s}-1)}{(1-2s)}2\xi(2s)$	$(2^{2n}-2)\|B_{2n}\|$
S_2	$\left(\dfrac{\theta}{\sinh\theta}\right)^2$	$2\xi(2s)$	$(2n-1)2^{2n}\|B_{2n}\|$
T_1	$\left(\dfrac{\tanh\theta}{\theta}\right)$	$\dfrac{(4^{s+1}-1)}{(2s+1)(s+1)}\dfrac{2}{\pi}\xi(2s+2))$	$\frac{A_{2n+1}}{2n+1}$
T_2	$\left(\dfrac{\tanh\theta}{\theta}\right)^2$	$\dfrac{(4^{s+2}-1)}{(s+1)(s+2)}\dfrac{2}{\pi^2}\xi(2s+4))$	$\frac{A_{2n+3}}{2n+2}$

See [8] for a recent review of these and other properties of the laws of C_t and S_t with emphasis on the special cases when $t = 1$ or 2. The formulae in the table for T_t are derived in Section 6.4 of this paper. As shown in [9] and [8, §3.3], the classical functional equations $\xi(s) = \xi(1-s)$ and $\xi_4(s) = \xi_4(1-s)$ translate into symmetry properties of the laws of S_2 and C_1, and there is a reciprocal relation between the laws of C_2 and S_1. The formulae for positive integer moments in the table are read by comparison of the expansions

$$E\left(e^{-\frac{1}{2}\theta^2 X}\right) = \sum_{n=0}^{\infty} E(X^n)\frac{\left(-\frac{1}{2}\theta^2\right)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n E(\hat{X}^{2n})\frac{\theta^{2n}}{(2n)!} \qquad (|\theta| < \varepsilon), \tag{7}$$

for some $\varepsilon > 0$, with classical expansions of the hyperbolic functions (see [28, p. 35], or (83) and (96) below). The formulae for moments of C_t and T_t for $t = 1$ or 2 involve the numbers A_m of *alternating permutations* of $\{1, 2, \ldots, m\}$, that is permutations $(a_1, \ldots, a_m)$ with $a_1 > a_2 < a_3 > \cdots$. The A_{2n} are called *Euler* or *secant numbers*, and the A_{2n+1} are called *tangent numbers*, due to the expansions [19, p. 259]

$$\frac{1}{\cos\theta} = \sum_{n=0}^{\infty} A_{2n}\frac{\theta^{2n}}{(2n)!}; \qquad \tan\theta = \sum_{n=0}^{\infty} A_{2n+1}\frac{\theta^{2n+1}}{(2n+1)!}. \tag{8}$$

The formulae for moments of S_1 and S_2 involve the rational *Bernoulli numbers* B_{2n}, due to the expansion

$$\theta\coth\theta - 1 = \sum_{n=1}^{\infty} B_{2n}\frac{(2\theta)^{2n}}{(2n)!} \tag{9}$$

and the elementary identity $1/\sinh\theta = \coth(\theta/2) - \coth\theta$. Because $\tan\theta = \cot\theta - 2\cot 2\theta$ the tangent and Bernoulli numbers are related by

$$A_{2n+1} = (-1)^n \frac{(2^{2n+2}-1)2^{2n+1}}{n+1} B_{2n+2}. \tag{10}$$

The first few A_m and B_{2n} are shown in Table 2. See [21] for a bibliography of the Bernoulli numbers and their applications.

Table 2. Secant, tangent and Bernoulli numbers.

n	1	2	3	4	5	6
A_{2n}	1	5	61	1385	50521	2702765
A_{2n+1}	2	16	272	7936	353792	22368256
B_{2n}	$\frac{1}{6}$	$\frac{-1}{30}$	$\frac{1}{42}$	$\frac{-1}{30}$	$\frac{5}{66}$	$\frac{-691}{2730}$

Implicit in Table 1 is Euler's famous evaluation [65] of $\zeta(2n)$, and so is the companion evaluation of $L_{\chi_4}(2n-1)$ given in [22, 1.14 (14)]: for $n = 1, 2, \ldots$

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}| \text{ and } L_{\chi_4}(2n-1) = \frac{\pi^{2n-1}}{2^{2n}(2n-2)!}A_{2n-2}. \tag{11}$$

Our interest in these results led us to investigate the Mellin transforms of S_t, C_t and T_t for arbitrary $t > 0$, and to provide some characterizations of the various infinitely divisible laws involved. These characterizations are related to various special features of these laws: special recurrences satisfied by their moments and Mellin transforms, simple relations between their moments and cumulants, corresponding relations between the laws and their Lévy measures, and differential equations satisfied by their Laplace or Fourier transforms. These analytic results are related to various representations of the laws in terms of Brownian motion and Bessel processes, in particular the heights and lengths of excursions of a one-dimensional Brownian motion.

The rest of this paper is organized as follows. Section 2 recalls some background for the discussion of Lévy processes. Section 3 presents some special recurrences satisfied by the moments and Mellin transforms of the laws of S_t and C_t. We also show in this section how various processes involved can be characterized by such moment recurrences. In Section 4, we briefly review some properties of the gamma process, and the construction of both S and C as weighted sums of independent gamma processes. Section 6 presents a number of characterizations of the infinitely divisible laws under study. Some of these characterizations were presented without proof in [8, Proposition 2]. Section 7 presents several constructions of functionals X of a Brownian motion or Bessel process such that X has the distribution of either S_t or C_t for some $t > 0$. There is some overlap between that section and Section 4 of [8]. There we reviewed the large number of different Brownian and Bessel functionals whose laws are related to S_t and C_t. Here we focus attention on constructions where the structure of the underlying stochastic process brings out interesting properties of the distributions of S_2 and C_2, in particular several of those properties involved in the characterizations of Section 6.

2 Preliminaries

For a Lévy process (X_t), with $E(X_t^2) < \infty$ for some (and hence all) $t > 0$, the characteristic function of X_t admits the well known *Kolmogorov representation* [10, §28]

$$E[e^{i\theta X_t}] = \exp[t\Psi(\theta)] \text{ with } \Psi(\theta) = i\theta c + \int (e^{i\theta x} - 1 - i\theta x)x^{-2}K(dx) \qquad (12)$$

Here $c \in \mathbb{R}$, the integrand is interpreted as $-\theta^2/2$ at $x = 0$, and $K = K_X$ is the finite *Kolmogorov measure* associated with (X_t),

$$K_X(dx) = \sigma^2\delta_0(dx) + x^2\Lambda_X(dx) \qquad (13)$$

with σ^2 the variance parameter of the Brownian component of (X_t), with δ_0 a unit mass at 0, and with Λ_X the usual *Lévy measure* of (X_t). Assuming that the exponent Ψ in (12) can be expanded as

$$\Psi(\theta) = \sum_{m=1}^{\infty} \kappa_m \frac{(i\theta)^m}{m!} \qquad (|\theta| < \varepsilon) \qquad (14)$$

for some $\varepsilon > 0$, it follows from (12) that

$$\kappa_1 = c; \quad \kappa_2 = \sigma^2 + \int x^2 \Lambda_X(dx); \quad \kappa_n = \int x^n \Lambda_X(dx) \text{ for } n \geq 3. \tag{15}$$

According to the definition of *cumulants* of a random variable, recalled later in (91) the nth cumulant of X_t is $\kappa_n t$. In particular, if $c = \int x \Lambda_X(dx)$ and $\sigma^2 = 0$, as when $X = S, C, \hat{S}$ or $\hat{C}$, the cumulants κ_n of X_1 are just the moments of the Lévy measure Λ_X.

For a Lévy process (X_t) with all moments finite, it is a well known consequence of the Kolmogorov representation (12) that the sequence of functions $t \to E(X_t^n)$ is a sequence of *polynomials of binomial type* [20]. That is to say, $E(X_t^n)$ is a polynomial in t of degree at most n, and

$$E(X_{t+u}^n) = \sum_{k=0}^{n} \binom{n}{k} E(X_t^k) E(X_u^{n-k}). \tag{16}$$

The coefficients of these *moment polynomials* are determined combinatorially by the κ_n via the consequence of (12) and (14) that for $0 \leq k \leq n$

$$k![t^k]\, E(X_t^n) = n![\eta^n] \left(\sum_{m=1}^{\infty} \frac{\kappa_m}{m!} \eta^m \right)^k \tag{17}$$

where $[y^p]f(y)$ denotes the coefficient of y^p in the expansion of $f(y)$ in powers of y. Equivalently, starting from $E(X_t^0) = 1$, these polynomials are determined by the following recursion due to Thiele [31, p. 144, (4.2)] (see also [44, p.74, Th. 2], [20, Th. 2.3.6]): for $n = 1, 2, \ldots$

$$E(X_t^n) = t \sum_{i=0}^{n-1} \binom{n-1}{i} E(X_t^i) \kappa_{n-i}. \tag{18}$$

Table 3 displays the first few moment polynomials for five of the Lévy processes considered in this paper: the standard gamma process Γ defined by

$$P(\Gamma_t \in dx) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x}\, dx \qquad (t > 0, x > 0), \tag{19}$$

Brownian motion β, and the processes C, S and T.

Table 3. Some moment polynomials

X	$E(X_t)$	$E(X_t^2)$	$E(X_t^3)$	$E(X_t^4)$
Γ	t	$t(t+1)$	$t(t+1)(t+2)$	$t(t+1)(t+2)(t+3)$
β	0	t	0	$3t^2$
C	t	$\frac{t(2+3t)}{3}$	$\frac{t(16+30t+15t^2)}{15}$	$\frac{t(272+588t+420t^2+105t^3)}{105}$
S	$\frac{t}{3}$	$\frac{t(2+5t)}{45}$	$\frac{t(16+42t+35t^2)}{945}$	$\frac{t(144+404t+420t^2+175t^3)}{14175}$
T	$\frac{2t}{3}$	$\frac{4t(7+5t)}{45}$	$\frac{8t(124+147t+35t^2)}{945}$	$\frac{16t(2286+3509t+1470t^2+175t^3)}{14175}$

The moment polynomials of Γ and β illustrate some basic formulae, which we recall here for ease of later reference. First of all,

$$E(\Gamma_t^s) = \frac{\Gamma(t+s)}{\Gamma(t)} \qquad (\Re s > -t), \tag{20}$$

which reduces to $t(t+1)\cdots(t+n-1)$ for $s = n$ a positive integer. The identity in distribution $\beta_t^2 \stackrel{d}{=} 2t\Gamma_{1/2}$ and (20) give

$$E(|\beta_t|^{2s}) = (2t)^s \frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{1}{2})} = 2\left(\frac{t}{2}\right)^s \frac{\Gamma(2s)}{\Gamma(s)} \qquad (\Re s > -\tfrac{1}{2}), \tag{21}$$

where the second equality is the gamma duplication formula. In particular,

$$E(\beta_1^{2n}) = \frac{(2n)!}{2^n n!} \qquad (n = 0, 1, 2, \ldots). \tag{22}$$

For $X = C$, S or T we do not know of any explicit formula or combinatorial interpretation for the nth moment polynomial for general n. Table 5 in Section 4 shows that in these cases the cumulants κ_n, which appear in the descriptions (17) and (18) of $E(X_t^n)$, turn out to involve the Bernoulli numbers B_{2n}, which are themselves recursively defined. The recursive description of the moment polynomials via (17) or (18) is consequently rather cumbersome. It is therefore remarkable that for $X = C, S$ and T there are simple recurrences for the moments and Mellin transforms of X_t, which make no reference to the Bernoulli numbers. These recurrences are the subject of the next section.

3 Some special recurrences

Theorem 1 *(i) The process C is the unique Lévy process satisfying the following moment recurrence for $t \geq 0$ and $s = 1, 2, \ldots$:*

$$(t^2 + t)E[C_{t+2}^s] = t^2 E[C_t^s] + (2s+1)E[C_t^{s+1}]. \tag{23}$$

Moreover, the recurrence continues to hold for all $t \geq 0$ and $s \in \mathbb{C}$, with $E(C_t^s)$ an entire function of s for each t.

(ii) The process S is the unique Lévy process satisfying the following moment recurrence for $t \geq 0$ and $s = 1, 2, \ldots$:

$$(t^2 + t)E[S_{t+2}^s] = (t - 2s)(t - 2s + 1)E[S_t^s] + 2st^2 E[S_t^{s-1}]. \tag{24}$$

Moreover, the recurrence continues to hold for all $t \geq 0$ and $s \in \mathbb{C}$, with $E(S_t^s)$ an entire function of s for each t.

(iii) The process T is the unique Lévy process satisfying the following moment recurrence for $t \geq 1$ and $s = 1, 2, \ldots$:

$$(2s + t)E[T_t^s] = tE[T_{t-1}^s] + 2stE[T_{t+1}^{s-1}]. \tag{25}$$

Moreover, the recurrence continues to hold for all $t \geq 1$ and $s \in \mathbb{C}$ with $\Re s > (1-t)/2$.

Here and in similar assertions below, *unique* means of course *unique in law*. The fact that $E(C_t^s)$ and $E(S_t^s)$ are entire functions of s for each t is easily seen. The random variables C_t and S_t have all positive moments finite, because they have moment generating functions which converge in a neighbourhood of 0, and they have all negative moments finite by application to $X = C_t$ or $X = S_t$ of the following general formula: for X a non-negative random variable with $\varphi_X(\lambda) := E(e^{-\lambda X})$,

$$E[X^{-p}] = \frac{2^{1-p}}{\Gamma(p)} \int_0^\infty \theta^{2p-1} \varphi_X(\tfrac{1}{2}\theta^2) d\theta \qquad (p > 0). \tag{26}$$

This holds for a positive constant X by definition of $\Gamma(p)$, hence for every positive random variable X by Fubini's theorem. Similarly, consideration of (26) shows that $E[T_t^s] < \infty$ for real s iff $s > -t/2$. See also [69] and Yor [70, Exercise 11.1] for other applications of (26) to S_t. Another application of (26) shows that $E(T_t^s) < \infty$ iff $s > -t/2$. For $\hat{X}_t = \beta_{X_t}$ as in (4), where X may be C, S or T, Brownian scaling gives $\hat{X}_t \stackrel{d}{=} \beta_1 \sqrt{X_t}$, hence

$$E(|\hat{X}_t|^{2s}) = E(|\beta_1|^{2s})E(X_t^s) \qquad (\Re s > -\tfrac{1}{2}). \tag{27}$$

Using (21) and $\Gamma(x+1) = x\Gamma(x)$ we see that

$$E[|\beta_1|^{2(s+1)}] = (2s+1)E[|\beta_1|^{2s}] \quad (\Re s > -\tfrac{1}{2}). \tag{28}$$

The recurrences in Theorem 1 are equivalent via (27) and (28) to the following recurrences for $\hat{C}$, $\hat{S}$ and $\hat{T}$: for all $t > 0$ and $\Re(s) > -\frac{1}{2}$

$$(t^2+t)E[|\hat{C}_{t+2}|^{2s}] = t^2 E[|\hat{C}_t|^{2s}] + E[|\hat{C}_t|^{2s+2}], \tag{29}$$

$$(t^2+t)E[|\hat{S}_{t+2}|^{2s}] = (t-2s)(t-2s+1)E[|\hat{S}_t|^{2s}] + 2s(2s-1)t^2 E[|\hat{S}_t|^{2s-2}], \tag{30}$$

and for all $t \geq 1$ and s with $\Re(2s) > -1$ and $\Re(2s) > 1-t$

$$(2s+t)E[|\hat{T}_t|^{2s}] = tE[|\hat{T}_{t-1}|^{2s}] + 2s(2s-1)tE[|\hat{T}_{t+1}|^{2s-2}]. \tag{31}$$

The following Lemma presents some recurrences for probability density functions. The formulae (29) and (30) for $s > 0$ are obtained by multiplying both sides of (33) and (35) by $|x|^{2s}$ and integrating, using integration by parts, which presents no difficulty since the functions $\phi_t(x)$ and $\psi_t(x)$ are Fourier transforms of functions in the Schwartz space, hence also members of the Schwartz space. This establishes the recurrences of the theorem for all real $s > 0$, hence all $s \in \mathbb{C}$ by analytic continuation. A similar argument allows (31) to be derived from (37). More care is required to justify the integrations by parts, but this can be done using the fact that $E(|\hat{T}_t|^{2s}) < \infty$ for all $s > 0$, which is used in the proof for large s. Theorem 1 follows, apart from the uniqueness claims, which we establish in Section 3.1.

Lemma 2 *The density*

$$\psi_t(x) := \frac{P(\hat{C}_t \in dx)}{dx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{1}{\cosh y}\right)^t e^{iyx}\,dy \tag{32}$$

satisfies the recurrence

$$t(t+1)\psi_{t+2}(x) = (t^2+x^2)\psi_t(x) \tag{33}$$

while

$$\phi_t(x) := \frac{P(\hat{S}_t \in dx)}{dx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{y}{\sinh y}\right)^t e^{iyx}\,dy \tag{34}$$

satisfies the recurrence

$$t(t+1)\phi_{t+2}(x) = (x^2+t^2)\phi_t''(x) + (2t+4)x\phi_t'(x) + (1+t)(2+t)\phi_t(x) \tag{35}$$

and

$$\eta_t(x) := \frac{P(\hat{T}_t \in dx)}{dx} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\tanh y}{y}\right)^t e^{iyx}dy \tag{36}$$

satisfies the recurrence

$$-x\eta_t'(x) + (t-1)\eta_t(x) = t\eta_{t-1}(x) + t\eta_{t+1}''(x) \tag{37}$$

for $x \neq 0$ *and* $t \geq 1$.

Remarks. Formula (33) for $t = 1, 2, \ldots$ was given by Morris [44, (5.3)]. As noted there, this recursion and known formulae for $\psi_t(x)$ for $t = 1$ or 2 (displayed in Table 6) show that $\psi_t(x)$ is a polynomial of degree t divided by $\cosh(\pi x/2)$ if t is an odd integer, and divided by $\sinh(\pi x/2)$ if t is even. It is known that the classical integral representations of the beta function

$$\int_{-\infty}^{\infty}\frac{ce^{-qcy}dy}{(1+e^{-cy})^{p+q}} = \int_0^1 u^{p-1}(1-u)^{q-1}du =: B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{38}$$

where $c > 0, \Re p > 0, \Re q > 0$, yield the formula [23, 1.9.5], [32]

$$\psi_t(x) = \frac{2^{t-2}}{\pi}B\left(\frac{t+ix}{2}, \frac{t-ix}{2}\right) = \frac{2^{t-2}}{\pi\Gamma(t)}\left|\Gamma\left(\frac{t+ix}{2}\right)\right|^2. \tag{39}$$

As shown in [9] and [8] the Laplace transforms of C_t and S_t can be inverted to give series formulae for the corresponding densities for a general $t > 0$.

Proof. The recurrence (33) follows from (39) using $\Gamma(x+1) = x\Gamma(x)$. In the case of ϕ_t we do not know of any explicit formula like (39) for general $t > 0$. So we proceed by the following method, which can also be used to derive the recurrence for ψ_t without appeal to (39). By differentiating (34) with respect to x, then integrating by parts, we obtain

$$x\left(\phi_t'(x) + \frac{t+1}{x}\phi_t(x)\right) = t\int_{-\infty}^{\infty}\cosh y\left(\frac{y}{\sinh y}\right)^{t+1}e^{iyx}dy. \tag{40}$$

Differentiating again with respect to x, and integrating by parts again, leads to (35). Lastly, by standard formulae for Fourier transforms, the recurrence (37) is equivalent to the fact that $g_t(\theta) := \tanh\theta/\theta$ solves the following differential equation:

$$\frac{d}{d\theta}(\theta g_t(\theta)) + (t-1)g_t(\theta) = tg_{t-1}(\theta) - t\theta^2 g_{t+1}(\theta). \tag{41}$$

$\square$

Examples. To illustrate (35), from the known results recalled in Table 6, we have $\phi_1(x) = (\pi/4)/\cosh^2(\pi x/2)$, so deduce from (35) that

$$\phi_3(x) = \frac{\pi[6 - 2\pi^2(1 + x^2) - 6\pi x \sinh(\pi x) + (6 + \pi^2(1 + x^2))\cosh(\pi x)]}{16\cosh^4\left(\frac{\pi x}{2}\right)}$$

and $\phi_4(x)$ can be derived similarly from $\phi_2(x)$, but the result is quite messy. We note in passing that the density $\phi_n(x)$ of $\hat{S}_n$ appears for $n = 1, 2, \ldots$ in the formula of Gaveau [27] for the fundamental solution of the heat equation on the Heisenberg group of real dimension $2n + 1$. As shown by Gaveau, this is closely related to the appearance of the distribution of $\hat{S}_1$ in connection with Lévy's stochastic area formula [40].

From the Mellin transform $E[(\frac{\pi}{2}S_2)^s] = 2\xi(2s)$ in Table 1 we deduce with (24) that

$$E[(\tfrac{\pi}{2}S_4)^s] = \frac{2}{3}[(s-1)(2s-3)\xi(2s) + 2\pi s\xi(2s-2)] \qquad (s \in \mathbb{C}). \tag{42}$$

Using $\xi(s) = \xi(1-s)$ and the duplication formula for the gamma function, formula (42) can also be deduced by analytic continuation of the series for $E(S_4^{-m})$ for $m > 0$ derived from (26) in [70, Exercise 11.1] (which should be corrected by replacing 2^{3m-2} by 2^{3m}). In particular, (42) implies

$$\left(\frac{\theta}{\sinh\theta}\right)^4 = 1 + \sum_{n=1}^{\infty} \frac{2^{2n}}{3(2n)!}(2n-1)(2n-3)(nB_{2n-2} - (n-1)B_{2n})\,\theta^{2n} \tag{43}$$

where the series converges for $|\theta| < \pi$, and the coefficient of θ^{2n} is $(-1)^n E(S_4^n)/(2^n n!)$. Formula (43) can also be checked using (96) and (52) below.

As a generalization of (42), we deduce using (24) that for $n = 1, 2, \ldots$

$$E[(\tfrac{\pi}{2}S_{2n})^s] = \sum_{j=0}^{n-1} b_{n,j}(s)\xi(2s-2j) \qquad (s \in \mathbb{C}) \tag{44}$$

where the $b_{n,j}(s)$ for $0 \le j \le n-1$ are polynomials in s with real coefficients, of degree at most $2(n-1)$, which are determined by $b_{1,0}(s) = 2$ and the following recurrence: for $n = 1, 2, \ldots$

$$2n(2n+1)b_{n+1,j}(s) = (2n-2s)(2n+1-2s)b_{n,j}(s)1(j < n) + (2n)^2\pi b_{n,j}(s)1(j > 0).$$

By combining Theorem 1 and the results of Table 1, similar descriptions can be given for $E[(\frac{\pi}{2}C_{2n})^s]$ and $E[(\frac{\pi}{2}S_{2n-1})^s]$ involving ξ, and for $E[(\frac{\pi}{2}C_{2n-1})^s]$ involving ξ_4.

3.1 Uniqueness in Theorem 1

By consideration of moment generating functions, to establish the uniqueness claims in Theorem 1 it is enough to show that the recursions in Theorem 1 determine the positive integer moments of C_t, S_t and T_t for all $t > 0$. We complete the proof of Theorem 1 by establishing the following corollary, which presents the desired conclusion in more combinatorial language.

Corollary 3 *Each one of the following three recursions* (45), (46) *and* (47)*, with* $p_0(t) = 1$*, defines a unique sequence of polynomials* $p_n(t), n = 0, 1, 2, \ldots$ *of binomial type:*

$$(t + t^2)p_n(t + 2) = t^2 p_n(t) + (2n + 1)p_{n+1}(t); \tag{45}$$

$$(t + t^2)p_n(t + 2) = (t - 2n)(t - 2n + 1)p_n(t) + 2nt^2 p_{n-1}(t); \tag{46}$$

$$(2n + t)p_n(t) = tp_n(t - 1) + 2ntp_{n-1}(t + 1). \tag{47}$$

The corresponding generating functions

$$G_t(\theta) := \sum_{n=0}^{\infty} p_n(t) \frac{(\frac{1}{2}\theta^2)^n}{n!} \qquad (|\theta| < \pi/2). \tag{48}$$

are $(1/\cos\theta)^t$ *for* (45)*,* $(\theta/\sin\theta)^t$ *for* (46)*, and* $(\theta^{-1}\tan\theta)^t$ *for* (47)*.*

Proof. That the polynomials defined by the generating functions satisfy the recursions follows from the result established in the previous section that the moments of the associated Lévy processes satisfy these recursions. Or see the remarks below. For (45) the uniqueness is obvious. To deal with uniqueness for (46), we consider this recurrence with

$$(t - 2n)(t - 2n + 1) = 2n(2n - 1) + (1 - 4n)t + t^2 \tag{49}$$

replaced by $\alpha_n + \beta_n t + t^2$, and argue that the solution will be unique provided $\alpha_n \neq 0$ and

$$\beta_n \notin \{3, 5, \ldots, 2n + 1\} \tag{50}$$

for all n, as is the case in (49) with $\beta_n = 1 - 4n$. Suppose that $p_n(t)$ solves the recurrence. Take $t = 0$ and use $\alpha_n \neq 0$ to see that $p_n(0) = 0$ for all n. For $n = 1$ the recurrence amounts to

$$\alpha_1 = 2 \text{ and } p_1(t) = 2t/(3 - \beta_1).$$

So any solution of the recurrence must be of the form $p_n(t) = \sum_{j=1}^{n} a_{n,j} t^j$ for some array of coefficients $(a_{n,j})$. Assume inductively that suitable coefficients $a_{n-1,j}$ exist for some

$n \geq 2$. The recurrence amounts to a system of $n+3$ coefficient identities obtained by equating coefficients of t^k for $0 \leq k \leq n+2$. These coefficient identities are trivial for $k=0$ and $k=n+2$, leaving a system of $n+1$ linear equations in n unkowns $a_{n,j}, 1 \leq j \leq n$. The identity of coefficients of t^{n+1} reduces easily to $a_{n,n}(2n+1-\beta_n) = \delta_n a_{n-1,n-1}$ which determines $a_{n,n}$ by (50). For $2 \leq k \leq n$ it is easily checked that the identity of coefficients of t^k involves only $a_{n,j}$ for $j \geq k-1$ and $a_{n-1,k-2}$, and that the coefficient of $a_{n,k-1}$ in this identity is $2(k-1)+1-\beta_n \neq 0$ by (50). Thus for each $2 \leq k \leq n$ the coefficient $a_{n,k-1}$ can be expressed in terms of the $a_{n,j}$ for $j \geq k$ and $a_{n-1,k-2}$. This completes the inductive proof of uniqueness for (46). A similar argument establishes uniqueness for (47). □

Remarks. For the recurrence (46), with $(t-2n)(t-2n+1)$ replaced by $\alpha_n + \beta_n t + t^2$, the identity of coefficients of t reads

$$p_n(2) = \alpha_n p_n'(0) \text{ or } \sum_{j=1}^{n} a_{n,j} 2^j = \alpha_n a_{n,1}. \tag{51}$$

In general, this identity provides a constraint on α_n and β_n which is necessary for the generalized recurrence to admit a solution. That (51) holds for the $p_n(t)$ generated by $G_t(\theta) = (\theta / \sin\theta)^t$ with $\alpha_n = 2n(2n-1)$ can be checked using formula (17) with $k=1$ and the expressions for the moments and cumulants of $\hat{S}_2$ displayed in Table 6. We show later in Theorem 8 how this identity (51) provides simple characterizations of the laws of S_2 and $\hat{S}_2$.

The recursions can also be checked by showing that the corresponding generating function $G_t(\theta)$ satisfies a suitable differential equation. For instance, by routine manipulations, the recursion (46) is equivalent to the differential equation

$$(t+t^2)G_{t+2}(\theta) = (t+t^2+t^2\theta^2)G_t(\theta) - 2\theta t G_t'(\theta) + \theta^2 G_t''(\theta) \tag{52}$$

where the primes denote differentiation with respect to θ. But if $G_t(\theta) = (G(\theta))^t$, then after dividing both sides by $(G(\theta))^t$, the equation (52) reduces to an equality of coefficients of t and an equality of coefficients of t^2, which read respectively

$$-1 + G^2 + \theta^2 \left(\frac{G'}{G}\right)^2 - \theta \frac{G''}{G} = 0 \tag{53}$$

and

$$-1 - \theta^2 + G^2 + 2\theta \frac{G'}{G} - \theta^2 \frac{G''}{G} = 0. \tag{54}$$

For $G(\theta) = \theta / \sin\theta$ we have

$$\frac{G'}{G} = \frac{1}{\theta} - \cot\theta \text{ and } \frac{G''}{G} = \frac{-2\cot\theta}{\theta} + \cot^2\theta + \frac{1}{\sin^2\theta} \tag{55}$$

which imply (53) and (54), hence (52). The corresponding differential equation for $G_t(\theta) = (\theta^{-1}\tan\theta)^t$ appeared in (41), expressed in terms of $g_t(\theta) := G_t(i\theta)$. The recursion (47) for this case is a generalization of the recursion

$$T(n+1,k) = T(n,k-1) + k(k+1)T(n,k+1)$$

found by Comtet [19, p. 259] for the array of positive integers $T(n,k)$ defined by $(\tan\theta)^k/k! = \sum_{n\geq k} T(n,k)\theta^n/n!$. The differential equation for $G_t(\theta) = (1/\cos\theta)^t$ appears below, again in terms of $g_t(\theta) := G_t(i\theta)$, in the argument leading to (69). In this case, the polynomials $p_n(t)$ evaluated for t a positive integer are related to the numbers $E(n,k)$ defined by $(1/\cosh\theta)^k = \sum_n E(n,k)\theta^n/n!$. These *Euler numbers of order* k were studied by Carlitz [14].

3.2 Some special moments

For $X = C_t$ we find that (26) for $p = 1/2$ reduces using (38) to the simple formula

$$E[C_t^{-1/2}] = \frac{\Gamma(t/2)}{\sqrt{2}\Gamma((t+1)/2))}. \tag{56}$$

As a check, the recursion (23) for $s = -\frac{1}{2}$ simplifies to $(t+1)E[C_{t+2}^{-1/2}] = tE[C_t^{-1/2}]$, which is also implied by (56) and $\Gamma(x+1) = x\Gamma(x)$. The following proposition presents some explicit formulae for $E[S_t^s]$ in particular cases which correspond to a simplification in the recursion (23) for this function of s and t.

Proposition 4 *For all* $t > 0$

$$E[S_t^{(t-1)/2}] = \frac{2^{(t-1)/2}\Gamma(t/2)}{\sqrt{\pi}} \tag{57}$$

and

$$E[S_t^{(t-2)/2}] = \sqrt{\pi}2^{(t-4)/2}\frac{(\Gamma(t/2))^2}{\Gamma((t+1)/2)}. \tag{58}$$

Remarks. Comparing formulae (56), (57) and (58), we observe the remarkable identity

$$2E[S_t^{(t-2)/2}] = \pi E[C_t^{-1/2}]E[S_t^{(t-1)/2}] \tag{59}$$

for all $t > 0$, but we have no good explanation for this. We also note that the expectations in (56), (57) and (58) are closely related to the moments of $|\beta_1|$ and $\sqrt{A}$, where A has the arc sine law on $[0,1]$. Specifically the expectations in (56), (57) and (58) are equal to $(\pi/2)^{1/2}E[(\sqrt{A})^{t-1}]$, $E[|\beta_1|^{t-1}]$ and $(\pi/2)^{3/2}E[(|\beta_1|\sqrt{A})^{t-1}]$ respectively, where β_1 and A are assumed independent. But we do not see any good explanation of these coincidences either. As a check, the case $t = 2$ of (57) can also be read from Table 1 using $\xi(1) = 1/2$.
Proof. Observe from (24) that

$$(1+t)E[S_{t+2}^s] = 2stE[S_t^{s-1}] \text{ if } t = 2s-1 \text{ or } 2s. \tag{60}$$

Use these recurrences on one side, and $\Gamma(x+1) = x\Gamma(x)$ on the other side, to see that it suffices to verify (57) and (58) for $0 < t \leq 2$. Formula (57) for $t \in (0,1)$ is established by use of (26) with $p = (1-t)/2$, so $2p + t - 1 = 0$ and the right hand side of (26) reduces to a beta integral. The case $t = 1$ is trivial, and the formula is obtained for $t \in (2,3]$, by the recurrence argument. The case $t \in (1,2]$ is filled in by analytic continuation, using the following variant of (26) to show that $E[S_t^{(t-1)/2}]$ is an analytic function of t for $\Re t \in (1,3)$: for any non-negative random variable X and $0 < p < 1$

$$E(X^p) = \frac{p2^{1+p}}{\Gamma(1-p)}\int_0^\infty \frac{d\theta}{\theta^{2p+1}}(1 - \varphi_X(\tfrac{1}{2}\theta^2)). \tag{61}$$

This formula, which appears in [61, p. 325], is easily verified using Fubini's theorem,

In the case of (58), for $0 < t < 2$ we can apply (26) with $p = (2-t)/2$, so $2p+t-1 = 1$. The integral in (26) for $X = S_t$ can then be evaluated using the result of differentiation of (38) with respect to p, which is [28, p. 538, 4.253.1]

$$\int_0^1 u^{p-1}(1-u)^{q-1}\log u\, du = B(p,q)[\psi(p) - \psi(p+q)] \tag{62}$$

for $\Re p > 0, \Re q > 0$, where $\psi(x) := \Gamma'(x)/\Gamma(x)$ is the digamma function. For $p = t/2$ and $q = 1-t$ we find using the reflection formulae for ψ and Γ that

$$\psi(t/2) - \psi(1-t/2) = \pi\cot\pi t/2 = \pi\frac{\Gamma(t/2)\Gamma(1-t/2)}{\Gamma((1-t)/2)\Gamma((1+t)/2)}$$

and the right hand expression in (58) is finally obtained after simplification using the gamma duplication formula. $\square$

Comparison of the formulae (57) and (58) with those obtained from (61) yields the following two evaluations of integrals involving $(1/\sinh\theta)^t$:

$$\int_0^\infty d\theta \left(\frac{1}{\theta^t} - \frac{1}{(\sinh\theta)^t}\right) = \frac{\Gamma((3-t)/2)\Gamma(t/2)}{\sqrt{\pi}(t-1)} \qquad (1 < t < 3) \tag{63}$$

and

$$\int_0^\infty \theta d\theta \left(\frac{1}{\theta^t} - \frac{1}{(\sinh\theta)^t}\right) = \frac{\sqrt{\pi}\Gamma((4-t)/2)(\Gamma(t/2))^2}{2(t-2)\Gamma((t+1)/2)} \qquad (2 < t < 4). \tag{64}$$

As a check, (63) for $t = 2$ can be obtained by Fourier inversion of (88), using the expression for $\rho_{\hat{S}}$ in Table 5. We also confirmed these evaluations for various t by numerical integration using *Mathematica*. But we do not know how to prove them analytically without the Fourier argument involved in the recursion (35), which yielded (24) and (60). For X a positive random variable with $E(X^n) < \infty$ for some $n = 0, 1, 2, \ldots$, and $n < s < n+1$ there is the formula [61, (14)]

$$E(X^s) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{d\lambda}{\lambda^{s+1}} \left(\varphi_X(\lambda) - \sum_{k=0}^n \frac{(-\lambda)^k}{k!} E(X^k)\right). \tag{65}$$

With $\varphi_X(\frac{1}{2}\theta^2)$ replaced $(1/\cosh\theta)^t$ or $(\theta/\sinh\theta)^t$, these formulae (26), (61) and (65) give expressions for $E(C_t^s)$ and $E(S_t^s)$ for all real s except $s = 1, 2, 3, \ldots$, when the moments can be obtained from the moment polynomials discussed in Section 3. Comparison of (65) with (57) and (58) then gives two sequences of integral identities.

3.3 Variants of Theorem 1

We start by writing (29) in the functional form

$$(t^2 + t)E[f(\hat{C}_{t+2})] = t^2 E[f(\hat{C}_t)] + E[\hat{C}_t^2 f(\hat{C}_t)] \tag{66}$$

where f is an arbitrary bounded Borel function. This follows from (29) first for symmetric f by uniqueness of Mellin transforms, then for general f using

$$E[f(\hat{C}_t)] = E[f(-\hat{C}_t)] = E[\tilde{f}(|\hat{C}_t|)] \text{ where } \tilde{f}(x) := (f(x) + f(-x))/2.$$

Consider now the *Meixner process* $M^{(a)}$, that is the Lévy process whose marginal laws are derived by exponential tilting from those of $\hat{C}$, according to the formula

$$E[f(M_t^{(a)})] = (\cos a)^t \, E[f(\hat{C}_t) \exp(a\hat{C}_t)] \qquad (t \geq 0, -\pi/2 < a < \pi/2). \tag{67}$$

The functional recursion (66) for $\hat{C}$ generalizes immediately to show that $X = M^{(a)}$ satisfies the functional recursion (68) presented in the following theorem. There is a similar variant for $M^{(a)}$ of the moment recursion (29) for $\hat{C}$. These observations lead us to the following characterizations

Theorem 5 *A Lévy process X satisfies the functional recursion*

$$c(t^2+t)E[f(X_{t+2})] = t^2 E[f(X_t)] + E[X_t^2 f(X_t)] \tag{68}$$

for all bounded Borel f and all $t \geq 0$, for some constant c, if and only if X is a Meixner process $M^{(a)}$ for some $a \in (-\pi/2, \pi/2)$; then $c = 1/\cos^2 a \geq 1$ and (68) holds for all Borel f such that the expectations involved are well defined and finite.

Before the proof, we note the following immediate corollary of this theorem and the discussion leading to (66):

Corollary 6 *The process $X = \hat{C}$ is the unique Lévy process such that either*

(i) the moment recursion (29) holds all $s = 0, 1, 2, \ldots$ and the distribution of X_1 is symmetric about 0, or

(ii) the functional recursion (68) holds with $c = 1$ for all bounded Borel f.

Proof of Theorem 5. That $X = M^{(a)}$ satisfies (68) has already been established via the moment recursion (23) for $\hat{C}$. (This can also be seen by reversing the steps in the following proof of the converse, which parallels the analysis around (52).) Suppose that X satisfies (68). By consideration of (68) for f constant, it is obvious that $E(X_1^2) < \infty$ and $c = 1 + (E(X_1))^2$. Thus $c \geq 1$ and X_1 has characteristic function g with two continuous derivatives g' and g''. Now take $f(x) = e^{i\theta x}$ in (68) to obtain the following identity of functions of θ:

$$c(t^2+t)g^{t+2} = t^2 g^t - (g^t)'' = t^2 g^t - (t^2 - t)g^{t-2}(g')^2 - t g^{t-1} g''$$

where all differentiations are with respect to θ, and for instance $g^t(\theta) = (g(\theta))^t$. Cancelling the common factor of g^t and equating coefficients of t^2 and t, this amounts to the pair of equalities

$$\left(\frac{g'}{g}\right)' = -cg^2 = \left(\frac{g'}{g}\right)^2 - 1 \tag{69}$$

The argument is completed by the following elementary result: *the unique solution g of the differential equation*

$$\left(\frac{g'}{g}\right)' = \left(\frac{g'}{g}\right)^2 - 1 \text{ with } g(0) = 1 \text{ and } g'(0) = i\tan\phi \text{ for } \phi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \tag{70}$$

is $g(\theta) = (\cos\phi)/\cosh(\theta + i\phi)$. $\square$

For $\hat{S}$ instead of $\hat{C}$ the following functional recursion can be derived from (24). For f with two continuous derivatives, and $t \geq 0$, let

$$L_t f(x) := \tfrac{1}{2}(x^2 + t^2) f''(x) - t x f'(x).$$

Then for all $t \geq 0$

$$t(t+1)E[f(\hat{S}_{t+2})] = t(t+1)E[f(\hat{S}_t)] + 2E[L_t f(\hat{S}_t)]. \tag{71}$$

As a check, for $f(x) = e^{\theta x}$ this relation reduces to the previous equation (52). There is also a variant for the family of processes derived from $\hat{S}$ by exponential tilting. Presumably this could be used to characterize these processes, by a uniqueness argument for the appropriate variation of the differential equation (52). Similar remarks can be made for $\hat{T}$ instead of $\hat{S}$.

To give another application of these recurrences, for any Lévy process X subject to appropriate moment conditions, the formula

$$P_n(y,t) := E[(y + X_t)^n] = \sum_{k=0}^{n} \binom{n}{k} E[X_t^k] y^{n-k}$$

defines a polynomial in two variables y and t. Using $E(X_u^n | X_t) = P_n(X_t, u-t)$ for $0 \leq t \leq u$, it is easily shown that for each $u \in \mathbb{R}$, in particular for $u = 0$, the process

$$(P_n(X_t, u-t), t \geq 0)$$

is a martingale. In other terms, $P_n(y, -t)$ is a *space-time harmonic function* for X. The formulae (68) and (71) yield recurrences for these space-time harmonic polynomials $P_n(y,-t)$ in the particular cases when X is a Meixner process, or when $X = S$. These space-time harmonic polynomials are not the same as those considered by Schoutens and Teugels [58], because for fixed t the $P_n(y,-t)$ are not orthogonal with respect to $P(X_t \in dy)$. In particular, for X a Meixner process the polynomials $P_n(y,-t)$ are not the Meixner polynomials, and their expression in terms of Meixner polynomials appears to involve complicated connection coefficients. Thus there does not seem to be any easy way to relate the recurrence for the $P_n(y,t)$ deduced from (68), which involves evaluations with t replaced by $t+2$, to the classical two-term recurrence for the Meixner polynomials [3, p. 348], in which t is fixed.

4 Connections with the gamma process

The following proposition presents some elementary characterizations of the gamma process (Γ_t) in the same vein as the characterizations of (C_t), (S_t) and related processes provided elsewhere in this paper. Recall that the distribution of Γ_t can be characterized by the density (19), by the moments (20), or by the Laplace transform

$$E(\exp(-\lambda\Gamma_t)) = \left(\frac{1}{1+\lambda}\right)^t. \tag{72}$$

Proposition 7 *The gamma process $(\Gamma_t, t \geq 0)$ is the unique subordinator with any one of the following four properties:*
(i) for all $m = 0, 1, \ldots$

$$tE[\Gamma_{t+1}^m] = E[\Gamma_t^{m+1}]; \tag{73}$$

(ii) for all bounded Borel f

$$tE[f(\Gamma_{t+1})] = E[f(\Gamma_t)\Gamma_t]; \tag{74}$$

(iii) $\varphi(\lambda) := E[e^{-\lambda\Gamma_1}]$ solves the differential equation

$$\frac{\varphi'(\lambda)}{\varphi(\lambda)} = -\varphi(\lambda); \tag{75}$$

(iv) the Lévy measure Λ_Γ of (Γ_t) is such that

$$P(\Gamma_1 \in dx) = x\Lambda_\Gamma(dx). \tag{76}$$

Proof. From (74) we deduce

$$t\varphi^{t+1}(\lambda) = -\varphi'(\lambda)\varphi^{t-1}(\lambda)t$$

and hence the differential equation (75), whose unique solution with $\varphi(0) = 1$ is obviously $\varphi(\lambda) = 1/(1+\lambda)$. It is elementary and well known that Γ satisfies (76), with both sides equal to $e^{-x}dx$, and (75) is the Laplace transform equivalent of (76). □

Following [16], [5] and [8], let $(\Gamma_{n,t}, t \geq 0)$ be a sequence of independent gamma processes, and consider for $\alpha > 0$ the subordinator $(\Sigma_{\alpha,t}, t \geq 0)$ which is the following weighted sum of these processes

$$\Sigma_{\alpha,t} := \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{\Gamma_{n,t}}{(\alpha+n)^2}, t \geq 0. \tag{77}$$

The weights are chosen so by (72)

$$E\left[e^{-\frac{1}{2}\theta^2\Sigma_{\alpha,t}}\right] = \prod_{n=0}^{\infty}\left(1+\frac{\theta^2}{\pi^2(\alpha+n)^2}\right)^{-t} = \begin{cases} (1/\cosh\theta)^t & \text{if } \alpha=\frac{1}{2} \\ (\theta/\sinh\theta)^t & \text{if } \alpha=1 \end{cases} \tag{78}$$

where the second equality expresses Euler's infinite products for $\cosh\theta$ and $\sinh\theta$. Thus

$$(C_t) \stackrel{d}{=} (\Sigma_{\frac{1}{2},t}) \text{ and } (S_t) \stackrel{d}{=} (\Sigma_{1,t}). \tag{79}$$

Consider now the subordinated process $(\beta_{\Sigma_{\alpha,t}}, t \geq 0)$ derived from Brownian motion β and the subordinator $(\Sigma_{\alpha,t}, t \geq 0)$ as in (77). As shown in [5],

$$\beta_{\Sigma_{\alpha,1}} \stackrel{d}{=} \pi^{-1}\log(\Gamma_\alpha/\Gamma'_\alpha) \tag{80}$$

where Γ_α and Γ'_α are independent, with $\Gamma'_\alpha \stackrel{d}{=} \Gamma_\alpha$. By (79), formula (80) for $\alpha = \frac{1}{2}$ and $\alpha = 1$ describes the distributions of $\hat{C}_1$ and $\hat{S}_1$ respectively. Thus $\hat{C}$ and $\hat{S}$ are instances of Lévy processes X such that for some $a > 0, b > 0$ and $c \in \mathbb{R}$ there is the equality in distribution $(X_a - c)/b \stackrel{d}{=} \log(\Gamma_\alpha/\Gamma'_\beta)$ for some $\alpha, \beta > 0$ where Γ_α and Γ'_β are independent with $\Gamma'_\beta \stackrel{d}{=} \Gamma_\beta$. These are the *generalized z-processes* studied by Grigelionis [30]. The distribution of $\log(\Gamma_\alpha/\Gamma'_\beta)$, known as a *z-distribution*, has found applications in the theory of statistics [5], and in the study of Bessel processes [50].

5 Lévy measures

For a Lévy process X whose Lévy measure Λ_X has a density, let $\rho_X(x) := \Lambda_X(dx)/dx$ be this *Lévy density*. Directly from (76) and (77), the subordinator Σ_α has Lévy density at $x > 0$ given by

$$\rho_{\Sigma_\alpha}(x) = \frac{1}{x}\sum_{n=0}^{\infty} e^{-\pi^2(\alpha+n)^2 x/2} = \begin{cases} \rho_C(x) & \text{if } \alpha=\frac{1}{2} \\ \rho_S(x) & \text{if } \alpha=1 \end{cases} \tag{81}$$

Integrating term by term gives

$$\int_0^\infty x^s \rho_{\Sigma_\alpha}(x)dx = \left(\frac{2}{\pi^2}\right)^s \Gamma(s)\sum_{n=0}^{\infty}\frac{1}{(\alpha+n)^{2s}} \qquad (\Re s > \tfrac{1}{2}) \tag{82}$$

which for $\alpha = \frac{1}{2}$ or 1 involves Riemann's zeta function. On the other hand, from (2) and (12) we can compute for $0 \neq |\theta| < \pi$

$$\int_0^\infty \theta x e^{-\frac{1}{2}\theta^2 x} \rho_S(x) dx = -\frac{d}{d\theta}\left(\log\left(\frac{\theta}{\sinh\theta}\right)\right) = \coth\theta - \frac{1}{\theta}. \tag{83}$$

By expanding the leftmost expression of (83) in powers of θ, and comparing with (82) and the expansion (9) of $\theta\coth\theta - 1$, we deduce the descriptions of the Lévy measure of S given in the following table, along with Euler's formula for $\zeta(2n)$ displayed in (11). In this table, $x > 0, \Re s > \frac{1}{2}$, and $n = 1, 2, \ldots$.

Table 4: The Lévy densities of C,S and T

X	$\rho_X(x) = \frac{\Lambda_X(dx)}{dx}$	$\int_0^\infty x^s \rho_X(x) dx$	$\kappa_n(X_1) = \int_0^\infty x^n \rho_X(x) dx$
S	$x^{-1}\sum_{n=1}^{\infty} e^{-\pi^2 n^2 x/2}$	$\frac{2^s}{\pi^{2s}}\Gamma(s)\zeta(2s)$	$2^{3n-1}\frac{(n-1)!}{(2n)!}\lvert B_{2n}\rvert$
C	$x^{-1}\sum_{n=1}^{\infty} e^{-\pi^2 (n-\frac{1}{2})^2 x/2}$	$(4^s - 1)\frac{2^s}{\pi^{2s}}\Gamma(s)\zeta(2s)$	$(4^n - 1)2^{3n-1}\frac{(n-1)!}{(2n)!}\lvert B_{2n}\rvert$
T	$\rho_C(x) - \rho_S(x)$	$(4^s - 2)\frac{2^s}{\pi^{2s}}\Gamma(s)\zeta(2s)$	$(4^n - 2)2^{3n-1}\frac{(n-1)!}{(2n)!}\lvert B_{2n}\rvert$

The formulae for moments of the Lévy measure of C follow immediately from those for S and the formula

$$\frac{1}{4}\rho_S\left(\frac{x}{4}\right) = \rho_S(x) + \rho_C(x) \tag{84}$$

which is easily checked using the series for $\rho_S(x)$ and $\rho_C(x)$. Put another way, (84) amounts to

$$4S_t \stackrel{d}{=} S_t + C_t \tag{85}$$

where S_t and C_t are assumed independent. By the Laplace transforms (78), this is just a probabilistic expression of the duplication identity $\sinh 2\theta = 2\sinh\theta\cosh\theta$. Similarly, the formula

$$\rho_C(x) = \rho_S(x) + \rho_T(x)$$

corresponds to the identity in distribution

$$C_t \stackrel{d}{=} S_t + T_t \tag{86}$$

where S_t and T_t are independent. By the Laplace transforms (78), this is a probabilistic expression of the identity $1/\cosh\theta = (\theta/\sinh\theta)(\tanh\theta)/\theta$. Knight [38] discovered the decomposition (86) of C_1 using a representation of S_1, T_1 and C_1 in terms of Brownian motion, which we recall in Section 7.

From Table 4 we deduce the formulae presented in the next table, where $x \in \mathbb{R}, \Re s > \frac{1}{2}$ and $n = 1, 2, \ldots$:

Table 5: The Lévy densities of $\hat{C}$,$\hat{S}$ and $\hat{T}$

$\hat{X}$	$\rho_{\hat{X}}(x) := \frac{\Lambda_{\hat{X}}(dx)}{dx}$	$\int_{-\infty}^{\infty} \lvert x\rvert^{2s}\rho_{\hat{X}}(x)dx$	$\kappa_{2n}(\hat{X}_1) = \int_{-\infty}^{\infty} x^{2n}\rho_{\hat{X}}(x)dx$
$\hat{S}$	$\dfrac{\coth\left(\frac{\pi\lvert x\rvert}{2}\right) - 1}{2\lvert x\rvert}$	$\dfrac{2\Gamma(2s)}{\pi^{2s}}\zeta(2s)$	$\dfrac{2^{2n-1}}{n}\lvert B_{2n}\rvert$
$\hat{C}$	$\dfrac{1}{2x\sinh(\pi x/2)}$	$(4^s-1)\dfrac{2\Gamma(2s)}{\pi^{2s}}\zeta(2s)$	$(4^n-1)\dfrac{4^n}{2n}\lvert B_{2n}\rvert$
$\hat{T}$	$\rho_{\hat{C}}(x) - \rho_{\hat{S}}(x)$	$(4^s-2)\dfrac{2\Gamma(2s)}{\pi^{2s}}\zeta(2s)$	$(4^n-2)\dfrac{4^n}{2n}\lvert B_{2n}\rvert$

The formulae for Mellin transforms and integer moments of $\Lambda_{\hat{X}}$ are read from those of X in the previous table using the following fact: if X is a subordinator without drift, then by [56, (30.8)]

$$\int_{-\infty}^{\infty} |x|^{2s}\Lambda_{\hat{T}}(dx) = E[|\beta_1|^{2s}]\int_0^{\infty} x^s\Lambda_T(dx) \qquad (s > -\tfrac{1}{2}). \tag{87}$$

where $E[|\beta_1|^{2s}]$ is given by (27). The formulae for $\rho_{\hat{S}}(x)$ and $\rho_{\hat{C}}(x)$ can also be checked by inverting the following Fourier transforms, obtained from the Kolmogorov representation (12):

$$\int_{-\infty}^{\infty} x^2 e^{i\theta x}\rho_{\hat{S}}(x)dx = -\frac{d^2}{d\theta^2}\left(\log\left(\frac{\theta}{\sinh\theta}\right)\right) = \frac{1}{\theta^2} - \frac{1}{\sinh^2\theta}, \tag{88}$$

$$\int_{-\infty}^{\infty} x^2 e^{i\theta x}\rho_{\hat{C}}(x)dx = -\frac{d^2}{d\theta^2}\left(\log\left(\frac{1}{\cosh\theta}\right)\right) = \frac{1}{\cosh^2\theta}. \tag{89}$$

See [18, p. 261], [30] for variations and applications of (88). By an obvious variation of (85), there is the identity $2\hat{S}_t \stackrel{d}{=} \hat{S}_t + \hat{C}_t$, so

$$\rho_{\hat{C}}(x) = \frac{1}{2}\rho_{\hat{S}}\left(\frac{x}{2}\right) - \rho_{\hat{S}}(x). \tag{90}$$

This allows each of the formulae in the table for $\rho_{\hat{C}}(x)$ and $\rho_{\hat{S}}(x)$ to be deduced from the other using the elementary identity $\coth z/2 - \coth z = 1/\sinh z$. Similar remarks apply to the formulae for $\hat{T}$, using (86).

6 Characterizations

Recall that for a random variable X with $E[|X|^m] < \infty$ the *cumulants* $\kappa_n(X)$ for $1 \leq n \leq m$ are defined by the formula

$$\log E[e^{i\theta X}] = \sum_{n=1}^{m} \kappa_n(X) \frac{(i\theta)^n}{n!} + o(\theta^m) \quad \text{as } \theta \to 0 \text{ with } \theta \in \mathbb{R}. \tag{91}$$

The following table collects together formulae for the characteristic functions, probability densities, even moments and even cumulants of $\hat{C}_t$, $\hat{S}_t$ and $\hat{T}_t$ for $t = 1$ or 2. Except perhaps for $\hat{T}_2$, these formulae are all known. As indicated in the table, for $\hat{X} := \beta_X$, as in (4), for $n = 1, 2, \ldots$ the nth moment or cumulant of X is obtained from the $2n$th moment or cumulant of $\hat{X}$ by simply dividing by $E(\beta_1^{2n}) = (2n)!/(2^n n!)$. For moments this is just (27), and the companion result for cumulants is easily verified. Thus the moments and cumulants of C_t, S_t and T_t for $t = 1$ or 2 can also be read from the table. The cumulants in particular are already determined by Table 5 and the sentence following (15). There is no such simple recipe for recovering the density of X from that of $\hat{X} := \beta_X$, because the elementary formula

$$P(\hat{X} \in dx) = dx \int_0^\infty \frac{P(X \in dt)}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \qquad (x \in \mathbb{R}) \tag{92}$$

shows that recovering the density of X from that of $\hat{X}$ amounts to inverting a Laplace transform. As indicated in [8, Table 1], the densities of C_t and S_t for $t = 1, 2$ are known to be given by infinite series related to derivatives of Jacobi's theta function. But we shall not make use of these formulae here. The distributions of T_1 and T_2 are described in Section 6.4.

Table 6. Features of the laws of $\hat{C}_1$, $\hat{C}_2$, $\hat{S}_1$, $\hat{S}_2$, $\hat{T}_1$ and $\hat{T}_2$.				
$\hat{X}$	$E(e^{i\theta\hat{X}})$	$P(\hat{X} \in dx)/dx$	$E(\hat{X}^{2n}) = \frac{(2n)!}{2^n n!}E(X^n)$	$\kappa_{2n}(\hat{X}) = \frac{(2n)!}{2^n n!}\kappa_n(X)$
$\hat{C}_1$	$\dfrac{1}{\cosh\theta}$	$\dfrac{1}{2\cosh\left(\frac{\pi}{2}x\right)}$	A_{2n}	A_{2n-1}
$\hat{C}_2$	$\left(\dfrac{1}{\cosh\theta}\right)^2$	$\dfrac{x}{2\sinh\left(\frac{\pi}{2}x\right)}$	A_{2n+1}	$2A_{2n-1}$
$\hat{S}_1$	$\dfrac{\theta}{\sinh\theta}$	$\dfrac{\pi}{4\cosh^2\left(\frac{\pi}{2}x\right)}$	$(4^n-2)\|B_{2n}\|$	$\frac{4^n}{2n}\|B_{2n}\|$
$\hat{S}_2$	$\left(\dfrac{\theta}{\sinh\theta}\right)^2$	$\dfrac{\frac{\pi}{2}\left(\frac{\pi}{2}x\coth\left(\frac{\pi}{2}x\right)-1\right)}{\sinh^2\left(\frac{\pi}{2}x\right)}$	$(2n-1)4^n\|B_{2n}\|$	$\frac{4^n}{n}\|B_{2n}\|$
$\hat{T}_1$	$\dfrac{\tanh\theta}{\theta}$	$\dfrac{1}{\pi}\log\coth\left(\dfrac{\pi}{4}\|x\|\right)$	$\frac{A_{2n+1}}{2n+1}$	$\frac{(4^n-2)4^n}{2n}\|B_{2n}\|$
$\hat{T}_2$	$\left(\dfrac{\tanh\theta}{\theta}\right)^2$	$\displaystyle\int_{\|x\|}^{\infty}\frac{y(y-\|x\|)dy}{2\sinh(\pi y/2)}$	$\frac{A_{2n+3}}{2n+2}$	$\frac{(4^n-2)4^n}{n}\|B_{2n}\|$

The formulae for the densities of $\hat{C}_1$, $\hat{C}_2$, $\hat{S}_1$ and $\hat{T}_1$ are well known Fourier transforms [40], [23, 1.9], [32], [9]. The Fourier transform expressed by the density of $\hat{S}_2$ was found in [9]. The density η_2 of $\hat{T}_2$ is derived from the density η_1 of $\hat{T}_1$ using (37) for $t=1$, which reduces to

$$\eta_2''(x) = -x\eta_1'(x) = P(\hat{C}_2 \in dx)/dx \tag{93}$$

where the second equality is read from the formulae in the table for the densities of $\hat{T}_1$ and $\hat{C}_2$. Thus

$$\eta_2''(x) = E[(\hat{C}_2 - |x|)^+]. \tag{94}$$

In particular

$$\eta_2(0) = \frac{1}{\pi}\int_0^\infty \left(\frac{\tanh\theta}{\theta}\right)^2 d\theta = \int_0^\infty \frac{y^2 dy}{2\sinh \pi y/2} = \frac{14\zeta(3)}{\pi^3}. \tag{95}$$

The formulae for moments and cumulants are equivalent to classical series expansions of the hyperbolic functions [28, p. 35] involving the secant and tangent numbers A_m and Bernoulli numbers B_n. For instance, by differentiation of the expansion of $\coth\theta$ displayed in (83),

$$\left(\frac{\theta}{\sinh\theta}\right)^2 = \sum_{n=0}^{\infty}(2n-1)(-1)^{n+1}B_{2n}2^{2n}\frac{\theta^{2n}}{(2n)!} \qquad (0<|\theta|<\pi) \tag{96}$$

from which we read the moments of $\hat{S}_2$. Table 6 reveals some remarkable similarities between moments and cumulants, especially for $\hat{C}_2$ and $\hat{S}_2$. These observations lead to the following

Theorem 8 *(i) Let X be a random variable with all moments finite and all odd moments equal to* 0*. Then*

$$X \stackrel{d}{=} \hat{C}_2 \iff \kappa_{2n+2}(X) = 2E(X^{2n}) \qquad (n = 0, 1, 2, \ldots) \tag{97}$$

while

$$X \stackrel{d}{=} \hat{S}_2 \iff E(X^2) = \frac{2}{3} \text{ and } \kappa_{2n}(X) = \frac{E(X^{2n})}{n(2n-1)} \qquad (n = 1, 2, \ldots). \tag{98}$$

(ii) Let X be a random variable with all moments finite. Then

$$X \stackrel{d}{=} C_2 \iff \kappa_{n+1}(X) = \frac{E(X^n)}{n + \frac{1}{2}} \qquad (n = 0, 1, 2, \ldots) \tag{99}$$

while

$$X \stackrel{d}{=} S_2 \iff E(X) = \frac{2}{3} \text{ and } \kappa_n(X) = \frac{E(X^n)}{n(2n-1)} \qquad (n = 1, 2, \ldots). \tag{100}$$

Remarks. Similar but less pleasing characterizations could be formulated for other variables featured in Table 6. For instance, the results for $\hat{C}_1$ and C_1 would involve the ratio A_{2n}/A_{2n-1} for which there is no simple expression. In Section 7 we interpret the identities (99) and (100) in terms of Brownian motion. Later in this section we give several variations of these identities.
Proof. Each of the four implications $\Longrightarrow$ is found by inspection of Table 6. These properties determine the moments of these four distributions uniquely because for any random variable X_1 with all moments finite, the moments $E(X_1^n), n = 1, 2, \ldots$ and cumulants $\kappa_n := \kappa_n(X_1), n = 1, 2, \ldots$ determine each other via the recursion (18) with $t = 1$. Since each of the four distributions involved has a convergent moment generating function, each of these distributions is uniquely determined by its moments. $\square$

In the previous theorem the four distributions involved were characterized without assuming infinite divisibility, but assuming all moments finite. The following corollary presents corresponding results assuming infinite divisibility, but with only a second moment assumption for most parts.

Corollary 9 *Let $(X_t, t \geq 0)$ be the Lévy process associated with a finite Kolmogorov measure K_X via the Kolmogorov representation (12), and let U be a random variable with uniform distribution on $[0,1]$, with U independent of X_2. Then*
(i) For each fixed $t > 0$, assuming that the distribution of X_t is symmetric,

$$X_t \stackrel{d}{=} \hat{C}_t \quad \Longleftrightarrow \quad K_X(dx) = P(X_2 \in dx) \tag{101}$$

while

$$X_t \stackrel{d}{=} \hat{S}_t \quad \Longleftrightarrow \quad \frac{d^2}{dx^2}\left(\frac{K_X(dx)}{dx}\right) = \frac{P(X_2 \in dx)}{dx} \tag{102}$$

where for the implication $\Longleftarrow$ in (102) it is assumed that K_X has a density $k(x) := K_X(dx)/dx$ with two continuous derivatives, and that both $k(x)$ and $k'(x)$ tend to 0 as $|x| \to \infty$.

(ii) For each fixed $t > 0$, without the symmetry assumption,

$$X_t \stackrel{d}{=} C_t \quad \Longleftrightarrow \quad K_X(dx) = xP(U^2 X_2 \in dx) \tag{103}$$

while

$$X_t \stackrel{d}{=} S_t \quad \Longleftrightarrow \quad E(X_2) = \frac{2}{3} \text{ and } K_X(dx) = xE[(1-U)X_2 1(U^2 X_2 \in dx)]1(x > 0) \tag{104}$$

Proof. Each of the implications $\Longrightarrow$ follows easily from corresponding results in the previous theorem, using (15). So do the converse implications, provided it is assumed that X_2 has all moments finite. That a second moment assumption is adequate for the converse parts of (101), (103) and (104) is a consequence of results proved later in the paper. We refer to Theorem 10 for (101), to Proposition 11 for 103, and to Proposition 12 for (104). □

6.1 Self-generating Lévy processes

Morris [44] pointed out the implication $\Longrightarrow$ in (97), and coined the term *self-generating* for a Lévy process (X_t) whose Kolmogorov measure K_X is a scalar multiple of the distribution of X_u for some $u \geq 0$:

$$\frac{K_X(dx)}{K_X(\mathbb{R})} = P(X_u \in dx). \tag{105}$$

To indicate the value of u and to abbreviate, say X is $SG(u)$. In particular, X is $SG(0)$ iff $\Psi(\theta) = i\theta c + \sigma^2\theta^2/2$, which is to say X is a Brownian motion with drift and variance parameters c and σ^2. We see from the Kolmogorov representation (12) that (X_t) is $SG(u)$ iff

$$\frac{\Psi''(\theta)}{\Psi''(0)} = \exp(u\Psi(\theta)). \tag{106}$$

Written in terms of $g(\theta) = \exp(\Psi(\theta))$ for $u = 2$, this is just the first differential equation for g in (69). To restate either Corollary 6 or (101), *the process* $X = \hat{C}$ *is the unique symmetric* $SG(2)$ *Lévy process with* $E(X_1^2) = 1$.

It is easily seen that for $u > 0, a > 0, b \neq 0$

$$(X_t, t \geq 0) \text{ is } SG(u) \text{ iff } (aX_{bt}, t \geq 0) \text{ is } SG(u/b). \tag{107}$$

Also, if X is $SG(u)$ and the moment generating function $M(\xi) := E[\exp(\xi X_1)] = g(-i\xi)$ is finite for some $\xi \in \mathbb{R}$, then the exponentially tilted process $(X_t^{(\xi)}, t \geq 0)$ with

$$P(X_t^{(\xi)} \in dx) = M^{-t}(\xi) e^{\xi x} P(X_t \in dx)$$

is easily seen to be $SG(\xi)$. The self-generating Lévy processes obtained from $\hat{C}$ by these operations of scaling and exponential tilting have been called *generalized exponential hyperbolic secant processes* [32, 44, 42]. But we prefer the briefer term *Meixner processes* proposed in [58], which indicates the relationship between these processes and the Meixner-Pollaczek polynomials, analogous to the well known relationship between Brownian motion and the Hermite polynomials [58, 57]. See also Grigelionis [29]. Another self-generating family, which is a weak limit of the Meixner family [44], is the family of gamma processes $(b\Gamma_{at}, t \geq 0)$ for $0 \neq b \in \mathbb{R}$ and $a > 0$. Then, the orthogonal polynomials involved are the Laguerre polynomials [57]. Morris [44, p. 74, Theorem 1] states that the Poisson and negative binomial processes are self-generating, but this is clearly not the case. Rather, the collection of examples mentioned above is exhaustive:

Theorem 10 *The only Lévy processes* (X_t) *with the self-generating property* (105) *for some* $u \geq 0$ *are Brownian motions (with* $u = 0$*), and Meixner and Gamma processes (with* $u > 0$*).*

Proof. The characterization for $u = 0$ is elementary, so consider X which is $SG(u)$ for some $u > 0$. Observe first that X cannot have a Gaussian component, or equivalently that K_X has no mass at 0. For a Gaussian component would make X_u have a density, implying $P(X_u = 0) = 0$ in contradiction to (105). Similarly, X cannot have a finite Lévy measure (in particular X cannot have a lattice distribution) because then $P(X_u = 0) > 0$ which would force K_X to have an atom at 0. By use of the scaling transformation (107), the problem of characterizing all Lévy processes X that are $SG(u)$ for arbitrary $u > 0$ is reduced to the problem of characterizing all Lévy processes X that are $SG(u)$ for some particular u, and the choice $u = 2$ is most convenient. Also, by suitable choice of a in (107) we reduce to (106) with $\Psi''(0) = -1$. So it is enough to find all characteristic exponents $\Psi(\theta)$ such that

$$-\Psi''(\theta) = \exp(2\Psi(\theta)) \text{ with } \Psi(0) = 0. \tag{108}$$

Set

$$D(\theta) := 1/E[\exp(i\theta X_1)] = \exp[-\Psi(\theta)] \tag{109}$$

so (108) is equivalent to

$$DD'' - (D')^2 = 1 \text{ with } D(0) = 1. \tag{110}$$

According to Kamke [36, p.571, 6.111] the general solution of (110) is

$$D_b(\theta) = \frac{\cosh(\theta \cosh b + b)}{\cosh b}$$

for some $b \in \mathbb{C}$, including the limit case when $\cosh b = 0$. In particular, for $b = ia$ with $a \in (-\pi/2, \pi/2)$ we find

$$D_{ia}(\theta) = \frac{\cosh(\theta \cos a + ia)}{\cos a} \tag{111}$$

corresponding to a Meixner process, and the limit case $a = \pm\pi/2$ corresponds to $\pm\Gamma$ for Γ the standard gamma process. Other choices of $a \in \mathbb{R}$ yield the same examples, by symmetries of cosh and cos. To complete the argument, it suffices to show that $1/D_{ia}(\theta)$ is not an infinitely divisible characteristic function if $a \notin \mathbb{R}$. For D derived by (109) from a Lévy process X we have

$$D'(0) = -i\mu \text{ where } \mu = E(X_1) \in \mathbb{R}$$

whereas

$$D'_{ia}(0) = \sinh(ia) = i \sin a.$$

This eliminates the case when $\sin a \notin \mathbb{R}$, and it remains to deal with the case $\sin a \in \mathbb{R}\backslash[-1, 1]$. In that case $\cos^2 a = 1 - \sin^2 a < 0$ implying that $\cos a = iv$ for some real $v \neq 0$. But then, since cosh is periodic with period $2\pi i$, the function $D_{ia}(\theta)$ in (111) is periodic with period $2\pi/v$, hence so is $1/D_{ia}(\theta)$. If $1/D_{ia}(\theta)$ were the characteristic function of X_1, the distribution of X_1 would be concentrated on a lattice, hence the Lévy measure of X would be finite. But then X could not be self-generating, as remarked at the beginning of the proof. $\square$

6.2 The law of C_2

We start by observing from (3), (1) and (12) that

$$E[e^{-\lambda T_1}] = \frac{\tanh\sqrt{2\lambda}}{\sqrt{2\lambda}} = \frac{d}{d\lambda}\left(\log(\cosh\sqrt{2\lambda})\right) = \int_0^\infty e^{-\lambda x} x^{-1} K_C(dx) \tag{112}$$

where K_C is the Kolmogorov measure of (C_t). Thus we read from Table 4 that

$$\frac{P(T_1 \in dx)}{dx} = \frac{K_C(dx)}{x\,dx} = \sum_{n=1}^{\infty} e^{-\pi^2(n-\frac{1}{2})^2 x/2}. \tag{113}$$

According to (113) and the property of C displayed in (103), if U has uniform distribution on $[0,1]$ and U and C_2 are independent, then

$$T_1 \stackrel{d}{=} U^2\, C_2. \tag{114}$$

This also has a Brownian interpretation, indicated in Section 7.2. But in this section we maintain a more analytic perspective, and use these identities in distribution to provide some further characterizations of the law of C_2. See Section 6.4 for more about the distribution of T_1. For a non-negative random variable X with $E(X) < \infty$ let X^* denote a random variable with the *size-biased* or *length-biased* distribution of X, that is

$$P(X^* \in dx) = xP(X \in dx)/E(X).$$

As discussed in [62, 4], the distribution of X^* arises naturally both in renewal theory, and in the theory of infinitely divisible laws. For $\lambda \geq 0$ let $\varphi_X(\lambda) := E[e^{-\lambda X}]$. Then the Laplace transform of X^* is $E[e^{-\lambda X^*}] = -\varphi_X'(\lambda)/E(X)$ where φ_X' is the derivative of φ_X. According to the Lévy-Khintchine representation, the distribution of X is infinitely divisible iff

$$-\frac{\varphi_X'(\lambda)}{\varphi_X(\lambda)} = c + \int_0^\infty x e^{-\lambda x}\Lambda(dx)$$

for some $c \geq 0$ and some Lévy measure Λ, that is iff

$$E[e^{-\lambda X^*}] = \varphi_X(\lambda)\varphi_Y(\lambda)$$

where Y is a random variable with

$$P(Y \in dy) = (c\delta_0(dy) + \Lambda(dy))/E(X). \tag{115}$$

Thus, as remarked by van Harn and Steutel [62], for a non-negative random variable X with $E(X) < \infty$, the equation

$$X^* \stackrel{d}{=} X + Y$$

is satisfied for some Y independent of X if and only if the law of X is infinitely divisible, in which case the distribution of Y is given by (115) for Λ the Lévy measure of X. See also [4] for further discussion. In this vein, we find the following characterizations of the law of C_2:

Proposition 11 *For a non-negative random variable X with Laplace transform $\varphi(\lambda) := E[\exp(-\lambda X)]$, the following conditions are equivalent:*
(i) $X \stackrel{d}{=} C_2$, *meaning that* $\varphi_X(\lambda) = \left(1/\cosh\sqrt{2\lambda}\right)^2$;
(ii) $\varphi = \varphi_X$ *solves the differential equation*

$$-\frac{\varphi'(\lambda)}{\varphi(\lambda)} = 2\int_0^1 \varphi(\lambda u^2)du; \tag{116}$$

(iii) $E(X) = 2$ *and*

$$X^* \stackrel{d}{=} X + U^2\tilde{X} \tag{117}$$

where $X, \tilde{X}$ and U are independent random variables with $\tilde{X} \stackrel{d}{=} X$ and U uniform on $[0,1]$;
(iv) $E(X) = 2$ *and the function* $\psi(\theta) := 1/\sqrt{\varphi_X(\frac{1}{2}\theta^2)}$ *satisfies the differential equation*

$$\psi''\psi - (\psi')^2 = 1 \quad \textit{on } (0,\infty). \tag{118}$$

Proof. This is quite straightforward, so we leave the details to the reader. For orientation relative to previous results, we note from (114) that $\varphi(\lambda) := \varphi_{C_2}(\lambda) = (1/\cosh\sqrt{2\lambda})^2$ satisfies the differential equation (116), and the equation (118) was already encountered in (110).

6.3 The law of S_2

The following proposition is a companion of Proposition 11 for S_2 instead of C_2.

Proposition 12 *For a non-negative random variable X with Laplace transform $\varphi_X(\lambda) := E[\exp(-\lambda X)]$ and $E(X) = 2/3$, the following conditions are equivalent:*
(i) $X \stackrel{d}{=} S_2$, *that is* $\varphi_X(\lambda) = (\sqrt{2\lambda}/\sinh\sqrt{2\lambda})^2$.

(ii) *if* $X_1, X_2, \ldots$ *are independent random variables with the same distribution as* X*, and* $M_1 > M_2 > \ldots$ *are the points of a Poisson point process on* $(0,1)$ *with intensity measure* $2m^{-2}(1-m)dm$*, then*

$$\sum_i M_i^2 X_i \stackrel{d}{=} X; \tag{119}$$

(iii) *the function* $\varphi = \varphi_X$ *solves the following differential equation for* $\lambda > 0$*:*

$$\frac{\varphi'(\lambda)}{\varphi(\lambda)} = 2\int_0^1 (1-m)\varphi'(\lambda m^2)dm; \tag{120}$$

(iv) *if* X, X^* *and* H *are independent,* $P(X^* \in dx) = xP(X \in dx)/E(X)$ *and*

$$P(H \in dh) = (h^{-1/2} - 1)dh \qquad\qquad (0 < h < 1) \tag{121}$$

then

$$X^* \stackrel{d}{=} X + HX^*; \tag{122}$$

(v) *the function* $\psi(\theta) := \theta / \sqrt{\varphi_X(\frac{1}{2}\theta^2)}$ *satisfies the differential equation*

$$\psi''\psi - (\psi')^2 = -1 \quad \textit{on } (0,\infty). \tag{123}$$

Proof.
(i) $\Longrightarrow$ (ii). By a simple computation with Lévy measures, the nth cumulant of $\sum_i M_i^2 X_i$ is $E(X^n)/(n(2n-1))$. Compare with (98), to see that if $X \stackrel{d}{=} S_2$ then $\sum_i M_i^2 X_i$ and X have the same cumulants, hence also the same distribution.
(ii) $\Longrightarrow$ (i). By consideration of Lévy measures as in the previous argument, this is the same implication as $\Longleftarrow$ in (104) whose proof we have deferred until now. This is easy if all moments of X are assumed finite, by consideration of cumulants. But without moment assumptions we can only complete the argument by passing via conditions (iii) to (v), which we now proceed to do.
(ii) $\Longrightarrow$ (iii). The identity (119) implies that the law of X is infinitely divisible, with probability density f and Lévy density ρ which are related as follows. From (119), we deduce that the Laplace transform

$$\varphi(\lambda) := E[\exp(-\lambda X)] = \exp\left(-\int_0^\infty (1-e^{-\lambda x})\rho(x)dx\right)$$

satisfies

$$\varphi(\lambda) = \exp\left(-\int_0^\infty dy f(y) \int_0^1 (1-e^{-\lambda y m^2})\frac{2}{m^2}(1-m)dm\right)$$

Using Fubini's theorem, and making the change of variables $y = x/m^2$, this yields

$$\rho(x) = 2\int_0^1 \frac{dm}{m^4}(1-m)f(x/m^2), \tag{124}$$

and (120) follows using

$$-\frac{\varphi'(\lambda)}{\varphi(\lambda)} = \int_0^\infty xe^{-\lambda x}\rho(x)dx.$$

(iii) $\Longleftrightarrow$ (iv). Rewrite (iii) as

$$\frac{3}{2}\varphi'(\lambda) = \varphi(\lambda)\int_0^1 dh(h^{-1/2}-1)\frac{3}{2}\varphi'(\lambda h) \tag{125}$$

But since

$$-\frac{3}{2}\varphi'(\lambda) = \frac{3}{2}E(Xe^{-\lambda X}) = E(e^{-\lambda X^*})$$

formula (125) is the Laplace transform equivalent of (iv).
(iii) $\Longleftrightarrow$ (v). A simple argument using integration by parts shows that (iii) holds iff $\varphi = \varphi_X$ solves the following differential equation for $\lambda > 0$:

$$-\sqrt{\lambda}\frac{\varphi'(\lambda)}{\varphi(\lambda)} = \frac{1}{2}\int_0^\lambda \frac{dx}{x^{3/2}}(1-\varphi(x)). \tag{126}$$

Straightforward but tedious computations show that φ with $\varphi(0) = 1, \varphi'(0) = -2/3$ solves (126) if and only if ψ satisfies (123) with $\psi(0) = 0, \psi'(0) = 1$.
(v) $\Longleftrightarrow$ (i). According to Kamke [36, 6.110], the solutions ψ of (123) are determined by

$$C_1\psi(x) = \sinh(C_1 x + C_2)$$

for two complex constants C_i. Since $\psi > 0$ with $\psi(0+) = 0, \psi'(0) = 1$, it follows that $\psi(x) = \sinh x$. □

Remarks. The fact that $\varphi(\lambda) := 2\lambda/\sinh^2\sqrt{2\lambda}$ is a solution of the differential equation (126) appears in Yor [70, §11.7,Cor. 11.4.1], with a different probabilistic interpretation. Comparison of formula (83) with formula (11.38) of [70] shows that if W is a random variable with $P(W \in dt) = 3t\Lambda_S(dt)$, where $\Lambda_S(dt)$ is the Lévy measure of the process S with $S_1 \stackrel{d}{=} T_1(R_3)$, then W has the same distribution as the total time spent below level 1 by a 5-dimensional Bessel process started at 1. This is reminiscent of the Ciesielski-Taylor identities relating the distribution of functionals of Bessel processes of different dimensions [7, 17].

As another remark, note that by iteration of (122) followed by an easy passage to the limit,

$$X^* \stackrel{d}{=} X + H_1 X_1 + (H_1 H_2) X_2 + \cdots + (H_1 H_2 \cdots H_n) X_n + \cdots \tag{127}$$

where $X, X_1, X_2, \dots$ and $H_1, H_2, \dots$ are independent, with the X_i distributed like X and the H_i distributed like H. This is reminiscent of constructions considered in [63] and [51]. Regarding the stochastic equation $X^* \stackrel{d}{=} X + HX^*$ considered here, given a distribution of H one may ask whether there exists such a distribution of X. It is easily shown that if H is uniform on $[0,1]$, then X must have an exponential distribution. See Pakes [45] for a closely related result.

6.4 The laws of T_1 and T_2

A formula for the density of T_1 was given already in (113). Either from (113) and the partial fraction expansion of $\theta^{-1}\tanh\theta$, or from (114), we find that the Mellin transform of T_1 involves the zeta function:

$$E[T_1^s] = \int_0^\infty x^{s+1}\Lambda_C(dx) = (4^{s+1}-1)\frac{2^{s+1}}{\pi^{2s+2}}\Gamma(s+1)\zeta(2s+2). \qquad (\Re s > -\tfrac{1}{2}). \tag{128}$$

We deduce from (3), (1) and $\tanh^2\theta = 1 - 1/\cosh^2\theta$ that the distributions of T_2 and C_2 are related by the formula

$$P(T_2 \in dx)/dx = \tfrac{1}{2}P(C_2 > x) \qquad\qquad (x > 0). \tag{129}$$

In terms of renewal theory [24, p. 370], if interarrival times are distributed like C_2, the limit distribution of the residual waiting time is the law of T_2. Formula (129) allows the Mellin transform of T_2 to be derived from the Mellin transform of C_2. The result appears in Table 1. By inspection of the Mellin transforms of T_1 and T_2, we see that if T_1^* has the size-biased distribution of T_1, and $U_{1/3}$ is a random variable independent of T_1^* with $0 \le U_{1/3} \le 1$ and $P(U_{1/3} \le u) = u/3$ for $0 \le u < 1$, then

$$U_{1/3}\, T_1^* \stackrel{d}{=} T_2. \tag{130}$$

By consideration of a corresponding differential equation for the Laplace transform, as in Propositions 11 and 12, we see that the distribution of T_1 on $(0,\infty)$ is uniquely characterized by this property (130) (with T_2 the sum of two independent random variables with the same distribution as T_1) and $E(T_1) = 2/3$.

6.5 The laws of $\hat{S}_1$ and $\hat{S}_2$

We see from the density of $\hat{S}_1$ displayed in Table 6 that $\pi\hat{S}_1$ has the *logistic distribution*

$$P(\pi\hat{S}_1 > x) = (1 + e^x)^{-1} \qquad (x \in \mathbb{R}),$$

which has found numerous applications [39]. Aldous [2] relates this both distribution and that of $\pi\hat{S}_2$ to asymptotic distributions in the random assignment problem. According to [2, Theorem 2 and Lemma 6] the function

$$h(x) := P(\pi\hat{S}_2 > x) = \frac{e^{-x}(e^{-x} - 1 + x)}{(1 - e^{-x})^2} \qquad (x > 0)$$

is the probability density on $(0, \infty)$ of the limit distribution as $n \to \infty$ of $n\varepsilon_{1,\pi_n(1)}$, where $\varepsilon_{i,j}, i, j = 1, 2, \ldots$ is an array of independent random variables with the standard exponential distribution $P(\varepsilon_{i,j} > x) = e^{-x}$ for $x \geq 0$, and π_n is the permutation which minimizes $\sum_{i=1}^n \varepsilon_{i,\pi(i)}$ over all permutations π of $\{1, \ldots, n\}$.

For $p > 0$ we can compute

$$2\int_0^\infty px^{p-1}h(x)dx = E\left(|\pi\hat{S}_2|^p\right)$$

using (27), (21), and the Mellin transform of $\hat{S}_2$ given in Table 1. We deduce that the density $h(x)$ is characterized by the remarkably simple Mellin transform

$$\int_0^\infty x^{p-1}h(x)dx = (p-1)\Gamma(p)\zeta(p) \qquad (\Re p > 0) \tag{131}$$

where the right side is interpreted by continuity at $p = 1$. Aldous [2, Lemma 6] gave the special cases of this formula for $p = 1$ and $p = 2$.

7 Brownian Interpretations

For a stochastic process $X = (X_t, t \geq 0)$ let

$$H_a(X) := \inf\{t : X_t = a\}$$

denote the hitting time of a by X. We now abbreviate $H_1 := H_1(|\beta|)$ where β is a standard one-dimensional Brownian motion with $\beta_0 = 0$, and set

$$G_1 := \sup\{t < H_1 : \beta_t = 0\}, \tag{132}$$

so G_1 is the time of the last zero of β before time H_1. It is well known [17] that

$$H_1 \stackrel{d}{=} C_1 \tag{133}$$

and

$$H_1 - G_1 \stackrel{d}{=} S_1$$

where G_1 is independent of $H_1 - G_1$ by a last exit decomposition. Hence in view of (86)

$$G_1 \stackrel{d}{=} T_1. \tag{134}$$

This Brownian interpretation of the decomposition $C_1 \stackrel{d}{=} T_1 + S_1$, where T_1 is independent of S_1, was discovered by Knight [38].

7.1 Brownian local times and squares of Bessel processes

Let $L_t^x, t \geq 0, x \in \mathbb{R}$ be the process of Brownian local times defined by the occupation density formula

$$\int_0^t f(\beta_s) ds = \int_{-\infty}^{\infty} f(x) L_t^x dx \tag{135}$$

for all non-negative Borel functions f, and almost sure joint continuity in t and x. See [53, Ch. VI] for background, and proof of the existence of Brownian local times. According to results of Ray [52], Knight [37] and Williams [66, 67, 68], there are the identities in distribution

$$(L_{H_1}^a, 0 \leq a \leq 1) \stackrel{d}{=} (R_{2,1-a}^2, 0 \leq a \leq 1)$$

and

$$(L_{H_1}^a - L_{G_1}^a, 0 \leq a \leq 1) \stackrel{d}{=} (r_{2,a}^2, 0 \leq a \leq 1)$$

where for $\delta = 1, 2, \ldots$ the process $R_\delta^2 := (R_{\delta,t}^2, t \geq 0)$ is the square of a δ-*dimensional Bessel process* $BES(\delta)$,

$$R_{\delta,t} := \left(\sum_{i=1}^{\delta} \beta_{i,t}^2 \right)^{1/2}$$

where $(\beta_{i,t}, t \geq 0)$ for $i = 1, 2, \ldots$ is a sequence of independent one-dimensional Brownian motions, and $r_\delta := (r_{\delta,u}, 0 \leq u \leq 1)$ is the δ-*dimensional Bessel bridge* defined by conditioning $R_{\delta,u}, 0 \leq u \leq 1$ on $R_{\delta,1} = 0$. Put another way, r_δ^2 is the sum of squares of δ independent copies of the standard Brownian bridge. As observed in [59] and [47], the definition of the processes R_δ^2 and r_δ^2 can be extended to all positive real δ in such a way that $R^2_{\gamma+\delta}$ is distributed as the sum of R^2_γ and an independent copy of R^2_δ. It then follows from results of Cameron-Martin [13] and Montroll [43, (3.41)] for $\delta = 1$ that there are the identities in law

$$(\int_0^1 R^2_{\delta,u}du, \delta \geq 0) \stackrel{d}{=} (C_{\delta/2}, \delta \geq 0) \tag{136}$$

and

$$(\int_0^1 r^2_{\delta,u}du, \delta \geq 0) \stackrel{d}{=} (S_{\delta/2}, \delta \geq 0). \tag{137}$$

In particular, for $\delta = 2$, we see from these identities and the Ray-Knight theorems that

$$\int_0^1 R^2_{2,u}du \stackrel{d}{=} H_1 \stackrel{d}{=} C_1 \text{ while } \int_0^1 r^2_{2,u}du \stackrel{d}{=} H_1 - G_1 \stackrel{d}{=} S_1.$$

More generally, for an arbitrary positive measure μ on $[0,1]$, the Laplace transforms of $\int_0^1 R^2_{\delta,u}\mu(du)$ and $\int_0^1 r^2_{\delta,u}\mu(du)$ can be characterized in terms of the solutions of an appropriate Sturm-Liouville equation. See [13, 43, 48] or [53, Ch. XI] for details and further developments.

7.2 Brownian excursions

Consider now the excursions away from 0 of the reflecting Brownian motion $|\beta| := (|\beta_t|, t \geq 0)$. For $t > 0$ let $G(t) := \sup\{s \leq t : \beta_s = 0\}$ and $D(t) := \inf\{s > t : \beta_s = 0\}$. So $G(t)$ is the time of the last zero of β before time t, and $D(t)$ is the time of the first zero of β after time t. The path fragment

$$(\beta_{G(t)+v}, 0 \leq v \leq D(t) - G(t))$$

is then called the *Brownian excursion straddling time t.* The works of Lévy[41] and Williams [66] show that the distribution of the Brownian excursion straddling time t is determined by the identity

$$\left(\frac{|\beta_{G(t)+u(D(t)-G(t))}|}{(D(t)-G(t))^{1/2}}, 0 \leq u \leq 1 \right) \stackrel{d}{=} (r_{3,u}, 0 \leq u \leq 1) \tag{138}$$

and the normalized excursion on the left side of (138) is independent of $D(t) - G(t)$. See also [15, 53] for further treatment of Brownian excursions, and [26] for a recent study of the moments of $\int_0^1 r_{3,s}ds$.

Recall from around (86) that H_1 is the first hitting time of 1 by the reflecting Brownian motion $|\beta|$, that $G_1 := G(H_1)$, and that $H_1 \stackrel{d}{=} C_1$ and $H_1 - G_1 \stackrel{d}{=} S_1$. According to Williams [66, 67], the random variable

$$U := \max_{0 \le t \le G_1} |\beta_t|$$

has uniform distribution on $[0,1]$, and if T_* denotes the almost surely unique time at which $|\beta|$ attains this maximum value over $[0, G_1]$, then the processes

$$(U^{-1}|\beta(tU^2)|, 0 \le t \le T_*/U^2) \text{ and } (U^{-1}|\beta(G(H_1) - tU^2)|, 0 \le t \le (G_1 - T_*)/U^2)$$

are two independent copies of $(|\beta_t|, 0 \le t \le H_1)$, independent of U. Thus

$$G_1 = U^2(G_1/U^2)$$

where U is independent of G_1/U^2 and G/U^2 is explicitly represented as the sum of two independent copies of $H_1 \stackrel{d}{=} C_1$, so

$$\frac{G_1}{U^2} = \frac{T_*}{U^2} + \frac{G_1 - T_*}{U^2} \stackrel{d}{=} C_2. \tag{139}$$

This provides a Brownian derivation of the identity (114). Let $H_x(R_3)$ be the first hitting time of x by R_3. It is known [17, 38] that

$$H_1 - G_1 \stackrel{d}{=} H_1(R_3) \stackrel{d}{=} S_1. \tag{140}$$

The first identity of laws in (140) is the identity in distribution of lifetimes implied by the following identity in distribution of processes, due to Williams [66, 67]:

$$(|\beta_{G_1+t}|, 0 \le t \le H_1 - G_1) \stackrel{d}{=} (R_{3,t}, 0 \le t \le H_1(R_3)). \tag{141}$$

Let $D_1 := D(H_1)$ be the time of the first return of β to 0 after the time H_1 when $|\beta|$ first reaches 1. According to the strong Markov property of Brownian motion, the process $(|\beta_{H_1+u}|, 0 \le u \le D_1 - H_1)$ is just a Brownian motion started at 1 and stopped when it first reaches 0, and this process is independent of the process in (141). Thus the excursion of $|\beta|$ straddling time H_1 is decomposed into two independent fragments, a copy of R_3 run until it first reaches 1, followed by an unconditioned Brownian motion for the return trip back to 0. Let

$$M := \max_{H_1 \le t \le D_1} |\beta_t|.$$

As shown by Williams [68] (see also [54]), the identity in distribution (141) together with a time reversal argument shows that conditionally given $M = x$, the excursion of $|\beta|$ straddling H_1 decomposes at its maximum into two independent copies of $(R_{3,t}, 0 \leq t \leq H_x(R_3))$ put back to back. Together with Brownian scaling and (140), this implies the identity

$$(M, D_1 - G_1) \stackrel{d}{=} \left(V^{-1}, V^{-2} S_2\right) \tag{142}$$

where V is independent of S_2 with uniform distribution on $[0,1]$. In particular,

$$\frac{D_1 - G_1}{M^2} \stackrel{d}{=} S_2 \tag{143}$$

which should be compared with (139).

7.3 Poisson interpretations

By exploiting the strong Markov property of Brownian motion at times when it returns to its starting point, Lévy[41], Itô[34] and Williams [68] have shown how to decompose the Brownian path into a Poisson point process of excursions, and to reconstruct the original path from these excursions. See Rogers-Williams [55], Revuz-Yor [53], Blumenthal [11], Watanabe [64], Ikeda-Watanabe [33] for various accounts of this theory and its generalizations to other Markov processes. Here we give some applications of the Poisson processes derived from the heights and lengths of Brownian excursions.

For β a standard one-dimensional Brownian motion, with past maximum process $\overline{\beta}_t := \max_{0 \leq s \leq t} \beta_s$ let

$$R_t := \overline{\beta}_t - \beta_t \qquad (t \geq 0).$$

According to a fundamental result of Lévy[41], [53, Ch VI, Theorem 2.3], there is the identity in law

$$(R_t, t \geq 0) \stackrel{d}{=} (|\beta_t|, t \geq 0)$$

so the structure of excursions of R and $|\beta|$ away from zero is identical. As explained in Williams [68], Lévy's device of considering R instead of $|\beta|$ simplifies the construction of various Poisson point processes because the process $(\overline{\beta}_t, t \geq 0)$ serves as a *local time process at 0* for $(R_t, t \geq 0)$. Consider now the excursions of R away from 0, corresponding to excursions of β below its continuously increasing past maximum process $\overline{\beta}$. Each excursion of R away from 0 is associated with some unique level ℓ, the constant value of $\overline{\beta}$ for the duration of the excursion, which equals $H_{\ell+} - H_\ell$ where we now abbreviate H_ℓ for $H_\ell(\beta)$ rather than $H_\ell(|\beta|)$.

Proposition 13 (Biane-Yor[9]) *The random counting measure N on $(0,\infty)^3$ defined by*

$$N(\cdot) := \sum_{\ell} 1((\ell, H_{\ell+} - H_\ell, \ell - \min_{H_\ell \le u \le H_{\ell+}} \beta_u) \in \cdot), \tag{144}$$

where the sum is over the random countable set of ℓ with $H_{\ell+} - H_\ell > 0$, is Poisson with intensity measure $d\ell\mu(dv \times dm)$ where (v, m) denotes a generic value of

$$(H_{\ell+} - H_\ell, \ell - \min_{H_\ell \le u \le H_{\ell+}} \beta_u)$$

and

$$\mu(dv \times dm) = \frac{dv}{\sqrt{2\pi} v^{3/2}} P(\sqrt{v}\,\overline{r}_{3,1} \in dm) = \frac{dm}{m^2} P(m^2 S_2 \in dv) \tag{145}$$

where $\overline{r}_{3,1} := \max_{0 \le u \le 1} r_{3,u}$ is the maximum of a 3-dimensional Bessel bridge (or standard Brownian excursion) and S_2 is the sum of two independent random variables with the same distribution as S_1 and $T_1(R_3)$.

See also [49] for further discussion of the *agreement formula* (145) and some generalizations.

Consider now the counting measure $N_3(\cdot)$ on $(0,\infty)^3$ defined exactly like $N(\cdot)$ in (144), but with the underling Brownian motion β replaced by a BES(3) process R_3. That is

$$N_3(\cdot) := \sum_{\ell} 1((\ell, H_{3,\ell+} - H_{3,\ell}, \ell - \min_{H_{3,\ell} \le u \le T_{3,\ell+}} R_{3,u}) \in \cdot\ ),$$

where $H_\ell := T_\ell(R_3)$.

From Proposition 13 and the McKean-Williams description of BES(3) as a conditioned Brownian motion, we deduce the following:

Corollary 14 *The point process N_3 on $(0,\infty)^3$ is Poisson with intensity measure*

$$dh 1_{(m<h)} \mu(dv \times dm)$$

where μ is the intensity measure on $(0,\infty)^2$ defined in (145).

Proof. The Poisson character of the point process follows easily from the strong Markov property of R_3. To identify the intensity measure, it is enough to consider its restriction to $(a,b) \times (0,\infty)^2$ for arbitrary $0 < a < b < \infty$. The restriction of N_3 to $(a,b) \times (0,\infty)^2$ is generated by R_3 after time $H_{3,a}$ and before time $H_{3,b}$, during which random interval R_3 evolves like β conditioned to hit b before 0. The Poisson measure N derived from

β in Proposition 13 is then conditioned to have no points (h, v, m) such that $h \in (a, b)$ and $m \geq h$. The conditioned Poisson measure is just Poisson with a restricted intensity measure, and the conclusion follows. $\square$

Since

$$H_{3,1} = \sum_{0<\ell<1} (H_{3,\ell+} - H_{3,\ell})$$

where the sum is over the countable random set of ℓ with $0 < \ell < 1$ and $H_{3,\ell+} - H_{3,\ell} > 0$, and these $H_{3,\ell+} - H_{3,\ell} > 0$ are the points of a Poisson point process, whose intensity is the Lévy measure $\Lambda_S(dt)$ of the common distribution of $H_{3,1}$ and S_1. Hence for each non-negative Borel function g

$$E\left[\sum_{0<\ell<1} g(H_{3,\ell+} - H_{3,\ell})\right] = \int_0^\infty g(t)\Lambda_S(dt) \tag{146}$$

In particular, for $g(x) = x^{s/2}$ we deduce from Table 4 the following probabilistic interpretation of Riemann's ξ function:

$$E\left[\sum_{0<\ell<1} (H_{3,\ell+} - H_{3,\ell})^{s/2}\right] = \left(\frac{2}{\pi}\right)^{s/2} \frac{2\xi(s)}{s(s-1)} \qquad (\Re(s) > 1). \tag{147}$$

See also [35, §4.10] and [12] for more general discussions of the Poisson character of the jumps of the first passage process of a one-dimensional diffusion.

If $\tilde{M}_1 > \tilde{M}_2 > \ldots$ are the ranked values of the maxima of the excursions of $\overline{R}_3 - R_3$ away from 0 up to time $H_1(R_3)$, and $\tilde{V}_i$ is the length of the excursion whose maximum is $\tilde{M}_i$, then we have the representation

$$H_1(R_3) = \sum_{i=1}^\infty \tilde{V}_i = \sum_{i=1}^\infty \tilde{M}_i^2 S_{2,i} \tag{148}$$

where the $S_{2,i}$ are independent copies of S_2, with the $S_{2,i}$ independent also of the $\tilde{M}_i$. Now the $\tilde{M}_i$ are the ranked points of a Poisson process on $(0,1)$ with intensity measure $m^{-2}(1-m)dm$. If the last sum in (148) is considered for $\tilde{M}_i$ that are instead the ranked points of a Poisson process on $(0,1)$ with intensity measure $2m^{-2}(1-m)dm$, the result is distributed as the sum of two independent copies of $H_1(R_3)$, that is like S_2. Together with the fact that $E(S_2) = 2E[H_1(R_3)] = 2/3$, the above argument shows that $X = S_2$ satisfies condition (ii) of Proposition 12. Indeed, it was by this argument that we first discovered this property of the law of S_2.

7.4 A path transformation

There are several known constructions of the path of a $\mathrm{BES}_0(3)$ process, or segments thereof, from the path of a one-dimensional Brownian motion β. It will be clear to readers familiar with Itô's excursion theory that the previous discussion can be lifted from the description of the point processes of heights and lengths of excursions of R_3 below its past maximum process to a similar description of a corresponding point process of excursions defined as a random counting measure on $(0,1) \times \Omega$ where Ω is a suitable path space. Essentially, the conclusion is that the point process of excursions of R_3 below its past maximum process is identical in law to the point process obtained from the excursions of β below its past maximum process by deletion of every excursion whose height exceeds its starting level, meaning that the path of β hits zero during that excursion interval. Since the path of R_3 can be reconstructed from its process of excursions below its past maximum process, we obtain the following result, found independently by Jon Warren (unpublished).

Theorem 15 *For a standard one-dimensional Brownian motion β, let $\overline{\beta}_t := \max_{0 \le u \le t} \beta_u$ and let $R_t := \overline{\beta}_t - \beta_t$. Let (G_s, D_s) be the excursion interval of R away from 0 that straddles time s. Let $M_s := \max_{G_s \le u \le D_s} R_u$ be the maximum of R over this excursion interval, and*

$$U_t := \int_0^t 1(M_s \le \overline{\beta}_s)\, ds \text{ and } \alpha_u := \inf\{t : U_t > u\}$$

Then the process $(\beta_{\alpha_u}, u \ge 0)$ is a $BES_0(3)$.

Note that the continuous increasing process U in this result is anticipating. See also Section 4 of Pitman [46] for closely related results.

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