STATIONARY MARKOV PROCESSES RELATED TO STABLE ORNSTEIN-UHLENBECK PROCESSES AND THE ADDITIVE COALESCENT

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Abstract

We consider some classes of stationary, counting–measure–valued Markov processes and their companions under time–reversal. Examples arise in the Lévy–Itô decomposition of stable Ornstein–Uhlenbeck processes, the large–time asymptotics of the standard additive coalescent, and extreme value theory. These processes share the common feature that points in the support of the evolving counting–measure are born or die randomly, but each point follows a deterministic flow during its life–time.

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1 Introduction

For $0 < \alpha \leq 2$, let $(S_u)_{u \geq 0}$ be a real-valued α-stable Lévy process with $S_0 = 0$. As shown by Breiman (1968), the process $(X_t)_{t \in \mathbb{R}}$ defined by

$$X_t := e^{-t/\alpha} S_{e^t}$$

is a two-sided stationary Markov process, called an *α-stable Ornstein-Uhlenbeck (OU) process*. For $\alpha = 2$ and S a standard Brownian motion, X is the usual Gaussian OU process. The OU process derived from a symmetric α-stable Lévy process for $0 < \alpha < 2$ was studied and characterised by Adler et al. (1990). Recently, in connection with the asymptotic distribution of the maximum of normalised sums of i.i.d. random variables in the domain of attraction of a stable law, Bertoin (1997) studied features of the OU process derived from an α-stable subordinator for $0 < \alpha < 1$. In this case S is a positive increasing process, while X is a strictly positive process with paths of locally bounded variation and only positive jumps. For the subordinator S there is the well known Lévy-Itô representation

$$S_u = \sum_{0 < v \leq u} \Delta S_v \tag{1}$$

where $\Delta S_v = S_v - S_{v-}$ and $\{(v, \Delta S_v) : v > 0,\, \Delta S_v > 0\}$ is the random set of points of a Poisson point process (PPP) on $]0, \infty[\times]0, \infty[$ with an intensity measure $c\, dv\, s^{-(\alpha+1)} ds$ for some constant $c > 0$. Since $\Delta X_w = e^{-w/\alpha} \Delta S_{e^w}$ the change of variables

$$(w, x) = (\log v, v^{-1/\alpha} s); \quad (v, s) = (e^w, e^{w/\alpha} x)$$

transforms (1) into

$$X_t = \sum_{w \leq t} e^{-(t-w)/\alpha} \Delta X_w \tag{2}$$

where $\{(w, \Delta X_w) : w \in \mathbb{R},\, \Delta X_w > 0\}$ is the set of points of a PPP on $\mathbb{R}\times]0, \infty[$ with an intensity measure $c\, dw\, x^{-(\alpha+1)} dx$ that is the push–forward of $c\, dv\, s^{-(\alpha+1)} ds$ by the change of variables. The representation (2) of the positive α-stable OU process X admits several possible interpretations. For instance, X might be regarded as a shot-noise process, or as a storage process Brockwell et al. (1982). Here we interpet X_t as the total mass at time t of an infinite system of masses in a stochastic equilibrium. Each point $(w, \Delta X_w)$ represents a mass of magnitude ΔX_w entering the system at time w; once it has entered, each mass is subject to deterministic exponential decay at rate $1/\alpha$ per unit time. Let $\mathbf{X}_t$ be the simple point process on $]0, \infty[$ which describes the distribution of masses present at time t. So $\mathbf{X}_t$ has a point $e^{-(t-w)/\alpha}\Delta X_w$ for each $w \leq t$ with $\Delta X_w > 0$. Then $(\mathbf{X}_t)_{t\in\mathbb{R}}$ is a stationary, time–homogeneous, Markov process. For each $t \in \mathbb{R}$ the point process $\mathbf{X}_t$ is a PPP whose intensity measure is the α-stable Lévy measure $cx^{-(\alpha+1)}dx$, and $X_t = \int_0^\infty x\mathbf{X}_t(dx)$ is the sum of all masses present at time t.

Let $\hat{\mathbf{X}}_t := \mathbf{X}_{(-t)-}$, $t \in \mathbb{R}$, be the time–reversal of $(\mathbf{X}_t)_{t\in\mathbb{R}}$. Then $(\hat{\mathbf{X}}_t)_{t\in\mathbb{R}}$ is also a stationary, time–homogeneous, Markov process. For each $t \in \mathbb{R}$ the point process $\hat{\mathbf{X}}_t$ is a PPP with intensity measure $cx^{-(\alpha+1)}dx$. Points in the evolving support die at rate 1, up to its death time each point undergoes exponential increase at rate $1/\alpha$, and points behave independently.

The counting–measure–valued processes $\mathbf{X} := (\mathbf{X}_t)_{t\in\mathbb{R}}$ and $\hat{\mathbf{X}} := (\hat{\mathbf{X}}_t)_{t\in\mathbb{R}}$ are a typical example of the general class of time–reverse pairs of stationary measure–valued Markov processes which is the subject of this paper.

Our interest in such processes arose from study of another sort of measure–valued process describing the evolution of a system of masses subject to coalescent collisions. Let $\mathcal{S}^{\downarrow}$ denote the ranked infinite simplex, that is, the set of all probability measures $(x_1, x_2, \dots)$ on the positive integers such that $x_1 \geq x_2 \geq \dots$. Regard an element of $\mathcal{S}^{\downarrow}$ as a fragmentation of a unit mass into clusters of masses x_i. The *ranked additive coalescent* $(X^{\downarrow}(t))_{t\geq 0}$ is the $\mathcal{S}^{\downarrow}$-valued Markov process in which each pair of mass clusters $\{x_i, x_j\}$ merges to form of cluster of mass $x_i + x_j$ at rate $x_i + x_j$, and after such a merger the masses are relabelled so that they are again in ranked order. See Evans and Pitman (1997) for details of the construction of such Markov processes, and references to their application to physical and chemical processes of coagulation, condensation and polymerisation. As shown in Pitman (1996) and Aldous and Pitman (1997) the additive coalescent arises from the evolution of tree components in a random graph process, and has asymptotic properties related to the $\frac{1}{2}-$stable subordinator and to Aldous's continuum random tree.

Let H be a PPP on $\mathbb{R}_+$ with intensity $(2\pi)^{-1/2}v^{-3/2}\,dv$ and write $\Sigma = \int v\,H(dv)$. We can think of Σ as the value at time 1 of a $\frac{1}{2}-$stable subordinator and H as the Poisson process of sizes of jumps made by the subordinator in the time interval $[0,1]$. Let $V = (V_1, V_2, \dots)$ be the $\mathcal{S}^{\downarrow}$-valued random variable such that $\Sigma V_1 > \Sigma V_2 > \dots$ are the locations of points of H, and write Q_s, $s \in \mathbb{R}$, for the conditional distribution of V given $\Sigma = e^{2s}$. It was shown in Evans and Pitman (1997) that the weak limit as $n \to \infty$ of the ranked additive coalescent started at time $-\frac{1}{2}\log n$ with initial state the uniform distribution on $\{1, \dots, n\}$ is a ranked additive coalescent $(X^{\downarrow}(s))_{s\in\mathbb{R}}$ such that the distribution of $X^{\downarrow}(s)$ is Q_s for every $s \in \mathbb{R}$. This limiting process is called the *standard additive coalescent* in Aldous and Pitman (1997) . There the following result is given regarding the asymptotic distribution of $X^{\downarrow}(s)$ as $s \to \infty$: the distribution of

$$e^{2s}(1 - X_1^{\downarrow}(s), X_2^{\downarrow}(s), X_3^{\downarrow}(s), \dots)$$

converges as $s \to \infty$ to that of $(\Sigma, \Sigma V_1, \Sigma V_2, \dots)$. The limiting behaviour as $s \to \infty$ of the process

$$(e^{2(s+t)}(X_2^{\downarrow}(s+t), X_3^{\downarrow}(s+t), \dots))_{t\geq 0}$$

is also easy to describe. When s is large, $X_1^{\downarrow}(s+t) \approx 1$ and $X_k^{\downarrow}(s+t) \approx 0$, $k \geq 2$, so that the masses $X_2^{\downarrow}(s+t), X_3^{\downarrow}(s+t), \dots$ are coalescing with each other at a negligible rate whilst each one is being removed by a coalescence with $X_1^{\downarrow}(s+t)$ at approximate rate 1. Therefore, if we build a random measure $\mathbf{Y}_t^{(s)}$ on $]0,\infty[$ by placing unit mass at each point $e^{2(s+t)}X_2^{\downarrow}(s+t), e^{2(s+t)}X_3^{\downarrow}(s+t), \dots$, it follows that $(\mathbf{Y}_t^{(s)})_{t\geq 0}$ converges in distribution as $s \to \infty$ to a stationary, measure–valued, time–homogeneous, Markov process $\mathbf{Y}^{\infty} := (\mathbf{Y}_t^{\infty})_{t\geq 0}$ with the same description as $(\hat{\mathbf{X}}_t)_{t\geq 0}$, where $\hat{\mathbf{X}}$ is as above with $\alpha = \frac{1}{2}$.

Further examples of the class of processes considered in this paper can be derived from functional forms of classical limit theorems in extreme value theory, as presented in Resnick (1987). We thank David Aldous for pointing out this connection to us. To illustrate, let $(\xi_k)_{k=1}^{\infty}$ be i.i.d. exponentially distributed random variables with common mean 1. Define a counting–measure $\mathbf{V}_t$, $t \geq 0$, by placing a unit mass at each of the points $\xi_{\lfloor e^t \rfloor} - t$ for $1 \le k \le \lfloor e^t \rfloor$. As $s \to \infty$ the process $(\mathbf{V}_{s+t})_{t \ge 0}$ converges in distribution to a stationary, measure–valued, time–homogeneous Markov process $(\mathbf{W}_t)_{t \ge 0}$ with the following description. The marginal distribution of $\mathbf{W}_t$ is that of a PPP on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$. As time evolves, points appear in space and time according to a PPP with intensity $e^{-x}\,dx\,dt$, $x \in \mathbb{R}$, $t > 0$, and after they are born points move with constant velocity -1. Similar results hold for sequences of i.i.d. random variables in the domain of attraction of the other classical extreme value limit distributions. In particular, the process $\mathbf{X}$ derived earlier from the positive α-stable OU process could be obtained this way.

Define a flow $(\varphi_t)_{t \in \mathbb{R}}$ on $]0, \infty[$ by $\varphi_t(v) := v e^{t/\alpha}$. It is apparent that the reason the processes $\mathbf{X}$ and $\hat{\mathbf{X}}$ (and $\mathbf{Y}^\infty$) are stationary is that for each $t \in \mathbb{R}$ the push–forward of the measure $cv^{-(\alpha+1)}\,dv$ by the map φ_t is the measure $e^t cv^{-(\alpha+1)}\,dv$. Similarly, if we define a flow $(\psi_t)_{t \in \mathbb{R}}$ on $\mathbb{R}$ by $\psi_t(x) := x + t$, then the push–forward of the measure $e^{-x}\,dx$ by the map ψ_t is the measure $e^t e^{-x}\,dx$, and this is what leads to the stationarity of $\mathbf{W}$. We observe in Section 3 that any similarly related measure and flow on an arbitrary measurable space give rise to a stationary, measure–valued, time–homogeneous, Markov processes with structure similar to that of $\mathbf{X}$, $\hat{\mathbf{X}}$, $\mathbf{Y}^\infty$ and $\mathbf{W}$.

What is not so obvious is that if we fix an integer n and let $\mathbf{Y}_t^n$ be the measure that assigns unit mass to each of the n largest points of $\mathbf{Y}_t^\infty$, then $\mathbf{Y}^n := (\mathbf{Y}_t^n)_{t \ge 0}$ is itself a stationary, measure–valued, time–homogeneous, Markov process with a simple, explicit description in the framework of jumping Markov processes introduced by Jacod and Skorokhod (1996), following the study of piecewise deterministic Markov processes by Davis (1984). We prove this fact in Section 6 and generalise it to similarly constructed processes for other suitably related pairs of measures and flows on arbitrary measurable spaces. In Section 7 we show that the time–reversal $\hat{\mathbf{Y}}^n := (\hat{\mathbf{Y}}_t^n)_{t \ge 0}$ of $\mathbf{Y}^n$ and its generalisations are also jumping Markov processes with simple, explicit descriptions.

Finally, we establish in Section 8 the generalisation of the result that for each n the distribution of $\mathbf{Y}_0^n$ is the unique stationary distribution for the semigroups of both $\mathbf{Y}^n$ and $\hat{\mathbf{Y}}^n$. Moreover, we show that if we start a Markov process with the same semigroup as either $\mathbf{Y}^n$ or $\hat{\mathbf{Y}}^n$ in any initial state, then the distribution of the process at time t

converges in total variation as $t \to \infty$ to the distribution of $\mathbf{Y}_0^n$.

2 Measures and flows

Hypothesis 1 Consider a measurable space $(E, \mathcal{E})$ equipped with a flow $\varphi : E \times \mathbb{R} \to E$ of measurable bijections of E into itself. That is, φ is $(\mathcal{E} \times \mathcal{B}(\mathbb{R}))\backslash\mathcal{E}$ - measurable, and, if we put $\varphi_t(v) = \varphi(v, t)$, then φ_0 is the identity map and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for $s, t \in \mathbb{R}$. Suppose further that μ is a non-trivial, σ-finite measure on E with the property that $\varphi_t(\mu)$, the push–forward of μ by φ_t, coincides with the measure $e^t\mu$ for all $t \in \mathbb{R}$. From now on we will suppose that we are always dealing with a flow φ and a measure μ that are related in this way.

Example 2 Take $E = \mathbb{R}^d$, μ to be Lebesgue measure, and $\varphi(v, t) = e^{-t/d}v$.

Example 3 Fix $-\infty \leq a < b \leq \infty$ and put $E =]a, b[$. Suppose that μ is a Borel measure on E with the following properties:

(i) μ is diffuse,

(ii) $\mu(E) = \infty$,

(iii) $\mu(]a', b[) < \infty$, $\forall a' > a$,

(iv) $\mu(]a', b'[) > 0$, $\forall a \leq a' < b' \leq b$.

Take φ to be the unique function with the property

$$\mu(]\varphi(v, t), b[) = e^{-t}\mu(]v, b[).$$

For example, if $E =]0, \infty[$ and $\mu(dv) = cv^{-(\alpha+1)}dx$ for some $c, \alpha > 0$, then $\varphi(v, t) = ve^{t/\alpha}$. The case $c = (2\pi)^{-1/2}$ and $\alpha = 1/2$ was the one encountered in Section 1 in connection with asymptotics of the standard additive coalescent.

3 Construction of the infinite particle system

Recall our standing Hypothesis 1. We want to generalise the definition of the process $\mathbf{Y}^\infty$ of Section 1 by building a stationary, measure–valued process with the following properties:

(i) the marginal distribution of the process at each time is Poisson with intensity μ,

(ii) each atom lives for an exponentially distributed amount of time with mean 1,

(iii) up to its death time, each atom follows the deterministic flow φ,

(iv) atoms behave independently.

Consider the following construction. Call an integer-valued, σ-finite measure with atoms of mass 1 a *simple point measure (SPM)*. Given a SPM $\mathbf{z}$ on $E \times \mathbb{R}_+$ with the property that $A \mapsto \mathbf{z}(A \times [0, s])$ is a SPM on E for each $s \in \mathbb{R}_+$, define for each $t \in \mathbb{R}$ another SPM $\pi_t(\mathbf{z})$ on E by

$$\pi_t(\mathbf{z})(A) = \mathbf{z}(\varphi_{-t}(A) \times [0, e^{-t}[). \tag{3}$$

Now let λ denote Lebesgue measure on $\mathbb{R}_+$. Suppose that $\mathbf{Z}$ is a PPP on $E \times \mathbb{R}_+$ with intensity $\mu \otimes \lambda$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $\pi_t(\mathbf{Z})$, $t \geq 0$, has the properties listed above. Because of its status as generalisation of the process in Section 1, we will denote $\pi_t(\mathbf{Z})$ as $\mathbf{Y}_t^\infty$, $t \geq 0$.

4 Construction of the finite particle system

Now that we have generalised the process $\mathbf{Y}^\infty$, we will generalise the construction of the process $\mathbf{Y}^n$ from Section 1.

Hypothesis 4 From now on, suppose that L is a measurable subset of E with the properties:

(i) $0 < \mu(L) < \infty$,

(ii) $\varphi_s(L) \supseteq \varphi_t(L)$ for all $s < t$,

(iii) $\varphi_t(L) = \bigcap_{s<t} \varphi_s(L)$ for all $t \in \mathbb{R}$.

Set $K(s) = \varphi_{-s}(L)$ for $s \in \mathbb{R}$, so that $K(s) \subseteq K(t)$ for all $s < t$ and $K(t) = \bigcap_{u>t} K(u)$ for all $t \in \mathbb{R}$. Put

$$K(-\infty) = \bigcap_{s\in\mathbb{R}} K(s),$$
$$K(\infty) = \bigcup_{s\in\mathbb{R}} K(s),$$
$$\bar{K}(t) = K(\infty)\backslash K(t),\ t \in [-\infty, \infty].$$

For a positive integer n and a SPM $\mathbf{x}$ on E, put

$$\sigma_n(\mathbf{x}) := \inf\{s \in \mathbb{R} : \mathbf{x}(K_s) \geq n\}. \tag{4}$$

Set

$$\kappa_n(\mathbf{x}) := K(\sigma_n(\mathbf{x}))$$
$$\bar{\kappa}_n(\mathbf{x}) := \bar{K}(\sigma_n(\mathbf{x})).$$

For each n define a SPM $\gamma_n(\mathbf{x})$ on E by

$$\gamma_n(\mathbf{x}) := \mathbf{x}(\cdot \cap \kappa_n(\mathbf{x})), \tag{5}$$

where we adopt the conventions $\inf \mathbb{R} = -\infty$ and $\inf \emptyset = \infty$. Note that $\varphi_t(\gamma_n(\mathbf{x})) = \gamma_n(\varphi_t(\mathbf{x}))$ for all $t \in \mathbb{R}$ and all positive integers n.

For $(\mathbf{Y}_t^\infty)_{t\geq 0}$ constructed in Section 3, put

$$\mathbf{Y}_t^n = \gamma_n(\mathbf{Y}_t^\infty) = (\gamma_n \circ \pi_t)(\mathbf{Z}).$$

It follows from the stationarity of $(\mathbf{Y}_t^\infty)_{t\geq 0}$ that $(\mathbf{Y}_t^n)_{t\geq 0}$ is also a stationary process for each positive integer n.

Example 5 Return to Example 2 above. One can take $L \subset \mathbb{R}^d$ to be any compact set that has 0 in its interior and is star-shaped with respect 0. In particular, if $L = \{v \in \mathbb{R}^d : |v| \leq 1\}$, then $\mathbf{Y}_t^n$ is the simple point process whose points are the n points of $\mathbf{Y}_t^\infty$ closest to 0.

Example 6 Return to Example 3 above. Set $L = [a', b[$ for some fixed $a' \in]a, b[$. Then $\mathbf{Y}_t^n$ is the simple point process whose points are the n largest points of $\mathbf{Y}_t^\infty$. When $E =]0, \infty[$ and $\mu(dv) = (2\pi)^{-1/2} v^{-3/2}\, dv$, we recover the process the $\mathbf{Y}^n$ considered in Section 1.

5 The canonical case

It is clear that $\mathbf{Y}^n$ is unchanged if we replace $\mathbf{Y}_t^\infty$ by $(\mathbf{Y}_t^\infty \cap K(\infty)) \backslash K(-\infty)$. We will therefore suppose without loss of generality from now on that $K(\infty) = E$ and $K(-\infty) = \emptyset$. In this case we can define a measurable injection $\psi : E \to \mathbb{R} \times L$ by $\psi(v) = (\tau(v), \varphi_{\tau(v)}(v))$ where $\tau(v) = \inf\{t \in \mathbb{R} : \varphi_t(v) \in L\} = \inf\{t \in \mathbb{R} : v \in K(t)\}$. The pushed–forward process $\tilde{\mathbf{Y}}_t^n := \psi(\mathbf{Y}_t^n)$, $t \geq 0$, is defined in the same manner as $\mathbf{Y}^n$ but with the defining ingredients $(E, (\varphi_t)_{t \in \mathbb{R}}, L, \mu)$ replaced by $(\tilde{E}, (\tilde{\varphi}_t)_{t \in \mathbb{R}}, \tilde{L}, \tilde{\mu})$, where $\tilde{E} = \mathbb{R} \times L$, $\tilde{\varphi}_t((s, v)) = (s - t, v)$, $\tilde{L} =]-\infty, 0] \times L$, and $\tilde{\mu}(ds, dv) = \mu(L)\, e^s ds \otimes \tilde{\nu}(dv)$ for a certain probability measure $\tilde{\nu}$ on L that is concentrated on $K(0) \backslash \bigcup_{t<0} K(t)$. We will call such a special case of the general construction a *canonical* case. Any instance of the general construction is just an instance of a canonical case in disguise.

6 Markov property of the finite particle system

We want to show that $(\mathbf{Y}_t^n)_{t \geq 0}$ (and later its time-reversal) is a time–homogeneous, strong Markov process. This will be a consequence of corresponding properties of the PPP $\mathbf{Z}$. To state these properties, let $\mathcal{F}_t^\circ$, $t \in \mathbb{R}$, be the sub-σ-field of $\mathcal{F}$ generated by the random variables $\mathbf{Z}(A \cap (K(t) \times \mathbb{R}_+))$, $A \in \mathcal{E} \times \mathcal{B}(\mathbb{R}_+)$, and let $\mathcal{F}_t := \mathcal{F}_{t+}^\circ \vee \mathcal{N}$, where $\mathcal{N}$ is the sub-σ-field of $\mathcal{F}$ generated by the $\mathbb{P}$-null sets. Similarly, let $\mathcal{G}_s^\circ$, $s \in \mathbb{R}_+$ be the sub-σ-field of $\mathcal{F}$ generated by the random variables $\mathbf{Z}(A \cap (E \times [0, s]))$, $A \in \mathcal{E} \times \mathcal{B}(\mathbb{R}_+)$, and let $\mathcal{G}_s := \mathcal{G}_{s+}^\circ \vee \mathcal{N}$.

Lemma 7 *(i) Suppose that T is a $\mathbb{R}$-valued stopping time for the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$. Then $\mathbf{Z}(\cdot \cap (\bar{K}(T) \times \mathbb{R}_+))$ is conditionally independent of $\mathcal{F}_T$ given T, and the* *conditional distribution of* $\mathbf{Z}(\cdot \cap (\bar{K}(T) \times \mathbb{R}_+))$ *given* $T = t$ *is the distribution of* $\mathbf{Z}(\cdot \cap (\bar{K}(t) \times \mathbb{R}_+))$.

(ii) Suppose that S *is a* $\mathbb{R}_+$*-valued stopping time for the filtration* $(\mathcal{G}_s)_{s\in\mathbb{R}_+}$. *Then* $\mathbf{Z}(\cdot \cap (E\times]S,\infty[))$ *is conditionally independent of* $\mathcal{G}_S$ *given* S, *and the conditional distribution of* $\mathbf{Z}(\cdot\cap(E\times]S,\infty[))$ *given* $S = s$ *is the distribution of* $\mathbf{Z}(\cdot\cap(E\times]s,\infty[))$.

Proof. (i) The result is clear for constant T, and hence for T that take on finitely many values. By our standing Hypotheses 1 and 4, $\mu(\bigcup_{s<t} K(s)) = \lim_{s\uparrow t} \mu(K(s)) = \lim_{s\uparrow t} e^s \mu(L) = e^t \mu(L) = \mu(K(t))$ and $K(t) = \bigcap_{s>t} K(s)$. Hence,

$$\mu(\bigcap_{s<t} \bar{K}(s)\backslash \bar{K}(t)) = 0 = \mu(\bar{K}(t)\backslash \bigcup_{s>t} \bar{K}(s)).$$

Thus,

$$\lim_{s\to t} \mathbb{P}\{\mathbf{Z}(\cdot \cap (\bar{K}(s) \times \mathbb{R}_+)) \neq \mathbf{Z}(\cdot \cap (\bar{K}(t) \times \mathbb{R}_+))\} = 0,$$

so that the distribution of $\mathbf{Z}(\cdot \cap (\bar{K}(t) \times \mathbb{R}_+))$ is continuous in total variation in t. A standard approximation argument similar to the one used to prove the strong Markov property for Feller processes (cf. Sections III.8,9 of Rogers and Williams (1994)), enables one use this continuity to pass to general T.
(ii) The proof is similar.

□

We also need the following result, which is elementary and well known.

Lemma 8 *Suppose that* $R_1, \ldots, R_n$ *are i.i.d. uniform random variables on* $[0, a[$ *for some* a. *Then the random variable* $-(\log(\bigvee_{i=1}^n R_i) - \log a)$ *is exponentially distributed with mean* $1/n$, *and the conditional distribution of the random measure* $\sum_{i=1}^n \delta_{R_i}(\cdot \cap [0, \bigvee_{i=1}^n R_i[)$ *given* $\bigvee_{i=1}^n R_i = r$ *is that of* $\sum_{i=1}^{n-1} \delta_{\tilde{R}_i}$, *where* $\tilde{R}_1, \ldots, \tilde{R}_{n-1}$ *are i.i.d. uniform random variables on* $[0, r[$.

Write M_n for the set of finite, simple point measures $\mathbf{x}$ on E such that

$$\mathbf{x}(K(t)\backslash \bigcup_{s<t} K(s)) \in \{0, 1\} \text{ for all } t \in \mathbb{R}$$

and

$$\mathbf{x}(E) = n.$$

Equip M_n with the σ-field $\mathcal{M}_n$ generated by the maps $\mathbf{x} \mapsto \mathbf{x}(B)$, $B \in \mathcal{E}$. It is clear from considering the canonical case that by modifying $\mathbf{Z}$ on a $\mathbb{P}$-null set we can ensure that $\mathbf{Y}_t^n \in M_n$ for all $t \geq 0$. Then the map $(t,\omega) \mapsto \mathbf{Y}_t^n(\omega)$ from $\mathbb{R}_+ \times \Omega$ into M_n is $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F})\backslash\mathcal{M}_n$ - measurable.

The structure of $\mathbf{Y}^n$ is quite simple. Fix n and put

$$\begin{aligned}
S_0 &:= 0, \\
T_0 &:= \inf\{t \in \mathbb{R} : \mathbf{Z}(K(t) \times [0,1[) = n\}, \\
S_{k+1} &:= \inf\{s > S_k : \mathbf{Z}(K(T_k) \times [0,e^{-s}[) = n-1\},\ k \geq 0, \\
T_{k+1} &:= \inf\{t > T_k : \mathbf{Z}(K(t) \times [0,e^{-S_{k+1}}[) = n\} \\
&= \inf\{t > T_k : \mathbf{Z}((K(t)\backslash K(T_k)) \times [0,e^{-S_{k+1}}[) = 1\}, k \geq 0.
\end{aligned}$$

While the definition of both S_k and T_k depends on n, to simplify displays this dependence is not carried in the notation. By construction,

$$\mathbf{Y}_s^n = \varphi_s(\mathbf{Z}((\cdot \cap K(T_k)) \times [0,e^{-S_k}[)),\ S_k \leq s < S_{k+1}.$$

Consequently,

$$\mathbf{Y}_s^n = \varphi_{s-r}(\mathbf{Y}_r^n),\ S_k \leq r \leq s < S_{k+1}.$$

From Lemma 7 and Lemma 8,

$$\mathbb{P}\{S_{k+1} - S_k > s \mid S_0, \ldots, S_k, \mathbf{Y}_{S_0}^n, \ldots, \mathbf{Y}_{S_k}^n\} = e^{-ns}.$$

Moreover,

$$\mathbb{P}\{\mathbf{Y}_{S_{k+1}}^n \in B \mid S_0, \ldots, S_{k+1}, \mathbf{Y}_{S_0}^n, \ldots, \mathbf{Y}_{S_k}^n\} = G_n(\varphi_{S_{k+1}-S_k}(\mathbf{Y}_{S_k}^n), B),$$

where the kernel G_n is defined as follows. Given $\mathbf{x} \in M_n$, let $\bar{\mathbf{Y}}^{n,\mathbf{x}}$ be a PPP on E with intensity $\mu(\cdot \cap \bar{\kappa}_n(\mathbf{x}))$. Denote the points of $\mathbf{x}$ by $\{v_1, \ldots, v_n\}$. Write $\bar{\mathbf{x}}_i$ for the SPM whose points are $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$. Let I_n be uniformly distributed on $\{1, \ldots, n\}$ and independent of $\bar{\mathbf{Y}}^{n\mathbf{x}}$. Then $G_n(\mathbf{x}, \cdot)$ is the distribution of $\gamma_n(\bar{\mathbf{x}}_{I_n} + \bar{\mathbf{Y}}^{n,\mathbf{x}})$.

Example 9 In the setting of Example 3, label the points of $\mathbf{x}$ as $v_1 > \cdots > v_n$. Then $G_n(\mathbf{x}, \cdot)$ is the distribution of $\bar{\mathbf{x}}_{I_n} + \delta_{W_n}$, where W_n is independent of I_n with distribution $\mathbb{P}\{W_n \leq w\} = \exp(-\mu(]w, v_n[))$, $w < v_n$.

It follows from these observations that $\mathbf{Y}^n$ is for each n a time–homogeneous, strong Markov process with a corresponding collection of laws $(P^{n,\mathbf{x}})_{\mathbf{x} \in M_n}$ and transition semigroup $(P_t^n)_{t \geq 0}$ defined as follows. On the same probability space that $\mathbf{Z}$ is defined, define an independent collection $R_1, \ldots, R_n$ of i.i.d. random variables uniformly distributed on $[0, 1[$. Consider $\mathbf{x} \in M_n$ with points $\{v_1, \ldots, v_n\}$. Put $\bar{\mathbf{Z}}^{n,\mathbf{x}} := \mathbf{Z}(\cdot \cap (\bar{\kappa}_n(\mathbf{x}) \times [0, 1[))$. Set $\mathbf{Z}^{n,\mathbf{x}} := \sum_{i=1}^n \delta_{(v_i, R_i)} + \bar{\mathbf{Z}}^{n,\mathbf{x}}$ and $\mathbf{X}_t^{n,\mathbf{x}} = (\gamma_n \circ \pi_t)(\mathbf{Z}^{n,\mathbf{x}})$, $t \geq 0$. Then $P^{n,\mathbf{x}}$ is the distribution of the M_n-valued process $\mathbf{X}^{n,\mathbf{x}}$ and $P_t^n(\mathbf{x}, \cdot)$ is the distribution of the M_n-valued r.v. $\mathbf{X}_t^{n,\mathbf{x}}$.

In fact, $(P^{n,\mathbf{x}})_{\mathbf{x} \in M_n}$ is for each $n = 1, 2, \ldots$ the collection of laws of a quasi–Hunt jumping Markov process in the sense of Jacod and Skorokhod (1996). The corresponding *local characteristics* are as follows. The *deterministic evolution* f is the flow on M_n obtained by pushing–forward using the flow φ on E. The cumulative *jump rates* $(\ell_{\mathbf{x}})_{\mathbf{x} \in M_n}$ are given by $\ell_{\mathbf{x}}(t) = nt$, and the *jump kernel* Γ is just G_n.

7 Time reversal

Recall that $\mathbf{Y}_t^n = (\gamma_n \circ \pi_t)(\mathbf{Z})$, $t \geq 0$, where π_t and γ_n are defined in (3) and (5), respectively. The time-reversal of $\mathbf{Y}^n$ is therefore the process $(\hat{\mathbf{Y}}_t^n)_{t \geq 0} = ((\gamma_n \circ \hat{\pi}_t)(\mathbf{Z}))_{t \geq 0}$, where we set

$$\hat{\pi}_t(\mathbf{z})(A) = \mathbf{z}(\varphi_t(A) \times [0, e^t])$$

for a SPM $\mathbf{z}$ on $E \times \mathbb{R}_+$ with the property that $A \mapsto \mathbf{z}(A \times [0, s])$ is a SPM on E for each $s \in \mathbb{R}_+$.

The structure of $\hat{\mathbf{Y}}^n$ is also relatively simple. Fix n and put

$$\begin{aligned}
\hat{S}_0 &= 0, \\
\hat{T}_0 &= \inf\{t \in \mathbb{R} : \mathbf{Z}(K(t) \times [0,1]) = n\}, \\
\hat{S}_{k+1} &= \inf\{s > \hat{S}_k : \mathbf{Z}(K(\hat{T}_k) \times [0, e^s]) = n+1\} \\
&= \inf\{s > \hat{S}_k : \mathbf{Z}(K(\hat{T}_k) \times]e^{\hat{S}_k}, e^s]) = 1\},\ k \geq 0, \\
\hat{T}_{k+1} &= \inf\{t < \hat{T}_k : \mathbf{Z}(K(t) \times [0, e^{\hat{S}_{k+1}}]) = n\},\ k \geq 0.
\end{aligned}$$

By construction,

$$\hat{\mathbf{Y}}^n_s = \hat{\varphi}_s(\mathbf{Z}((\cdot \cap K(\hat{T}_k)) \times [0, e^{\hat{S}_k}])),\ \hat{S}_k \leq s < \hat{S}_{k+1},$$

where we put $\hat{\varphi}_s = \varphi_{-s}$. Consequently,

$$\hat{\mathbf{Y}}^n_s = \hat{\varphi}_{s-r}(\hat{\mathbf{Y}}^n_r),\ \hat{S}_k \leq r \leq s < \hat{S}_{k+1}.$$

From Lemma 7,

$$\mathbb{P}\{\hat{S}_{k+1} - \hat{S}_k > s \mid \hat{S}_0, \ldots, \hat{S}_k, \hat{\mathbf{Y}}^n_{\hat{S}_0}, \ldots, \hat{\mathbf{Y}}^n_{\hat{S}_k}\} = \exp(-\mu(\kappa_n(\hat{\mathbf{Y}}^n_{\hat{S}_k})) \int_0^s e^r\, dr)$$

and

$$\mathbb{P}\{\hat{\mathbf{Y}}^n_{\hat{S}_{k+1}} \in B \mid \hat{S}_0, \ldots, \hat{S}_{k+1}, \hat{\mathbf{Y}}^n_{\hat{S}_0}, \ldots, \hat{\mathbf{Y}}^n_{\hat{S}_k}\} = \hat{G}_n(\hat{\varphi}_{\hat{S}_{k+1} - \hat{S}_k}(\hat{\mathbf{Y}}^n_{\hat{S}_k}), B),$$

where the kernel $\hat{G}_n$ is defined as follows. Given $\mathbf{x} \in M_n$, let v be the unique atom of $\mathbf{x}$ such that $v \notin K(u)$ for any $u < \sigma_n(\mathbf{x})$. Let V_n be an E-valued random variable with distribution $\mu(\cdot \cap \kappa_n(\mathbf{x}))/\mu(\kappa_n(\mathbf{x}))$. Then $\hat{G}_n(\mathbf{x}, \cdot)$ is the distribution of $\mathbf{x} - \delta_v + \delta_{V_n}$.

It is clear that $\hat{\mathbf{Y}}^n$ is for each n a time–homogeneous, strong Markov process with a corresponding collection of laws $(\hat{P}^{n,\mathbf{x}})_{\mathbf{x} \in M_n}$ and transition semigroup $(\hat{P}^n_t)_{t \geq 0}$ defined as follows. On the same probability space that $\mathbf{Z}$ is defined, define an independent collection $R_1, \ldots, R_n$ of i.i.d. random variables uniformly distributed on $[0,1[$. Consider $\mathbf{x} \in M_n$ with points $\{v_1, \ldots, v_n\}$. Put $\tilde{\mathbf{Z}}^{n,\mathbf{x}} := \mathbf{Z}(\cdot \cap (\kappa_n(\mathbf{x}) \times]1, \infty[))$. Set $\hat{\mathbf{Z}}^{n,\mathbf{x}} = \sum_{i=1}^n \delta_{(v_i, R_i)} + \tilde{\mathbf{Z}}^{n,\mathbf{x}}$. and $\hat{\mathbf{X}}^{n,\mathbf{x}}_t := (\gamma_n \circ \hat{\pi}_t)(\hat{\mathbf{Z}}^{n,\mathbf{x}})$. Then $\hat{P}^{n,\mathbf{x}}$ is the distribution of the M_n-valued process $\hat{\mathbf{X}}^{n,\mathbf{x}}$ and $\hat{P}^n_t(\mathbf{x}, \cdot)$ is the distribution of the M_n-valued r.v. $\hat{\mathbf{X}}^{n,\mathbf{x}}_t$.

For each n the collection $(\hat{P}^{n,\mathbf{x}})_{\mathbf{x}\in M_n}$ is the collection of laws of a quasi–Hunt jumping Markov process with local characteristics $(\hat{f}_n, (\hat{\ell}_{n,\mathbf{x}})_{\mathbf{x}\in M_n}, \hat{\Gamma}_n)$ defined as follows. The deterministic evolution $\hat{f}_n$ is the flow on M_n obtained by pushing–forward using the flow $\hat{\varphi}$ on E. The cumulative jump rate $\hat{\ell}_{n,\mathbf{x}}(t)$ is $\mu(\kappa_n(\mathbf{x}))\int_0^t e^s\,ds$, and the jump kernel $\hat{\Gamma}_n$ is $\hat{G}_n$.

8 Ergodic behaviour

Theorem 10 *For each $n = 1, 2, \ldots$ the common distribution of $\mathbf{Y}_0^n$ and $\hat{\mathbf{Y}}_0^n$ is the unique stationary distribution for each of the semigroups $(P_t^n)_{t\geq 0}$ and $(\hat{P}_t^n)_{t\geq 0}$. Both $P_t^n(\mathbf{x},\cdot)$ and $\hat{P}_t^n(\mathbf{x},\cdot)$ converge in total variation to this stationary distribution as $t\to\infty$ for each $\mathbf{x}\in M_n$.*

Proof. To prove both assertions for $(P_t^n)_{t\geq 0}$, it suffices to show for each pair $\mathbf{x},\mathbf{y}\in M_n$ that the total variation distance between $P_t^n(\mathbf{x},\cdot)$ and $P_t^n(\mathbf{y},\cdot)$ converges to 0 as $t\to\infty$. By the coupling inequality (cf. Section V.54 Rogers and Williams (1987)), this in turn will follow if we can show that there is a $\mathbb{P}$-a.s. finite random time S such that $\mathbf{X}_t^{n,\mathbf{x}} = \mathbf{X}_t^{n,\mathbf{y}}$ for all $t\geq S$. From the construction of $\mathbf{X}^{n,\mathbf{x}}$ and $\mathbf{X}^{n,\mathbf{y}}$ we see that it suffices to take

$$\begin{aligned} S &= \inf\{t\geq 0 : (\mathbf{Z}^{n,\mathbf{x}} + \mathbf{Z}^{n,\mathbf{y}})(K(\sigma_n(\mathbf{x})\vee\sigma_n(\mathbf{y}))\times[0,e^{-t}[) = 0\} \\ &= \bigvee_{i=1}^n(-\log R_i)\vee\inf\{t\geq 0 : \mathbf{Z}(K((\sigma_n(\mathbf{x})\vee\sigma_n(\mathbf{y}))\backslash K(\sigma_n(\mathbf{x})\wedge\sigma_n(\mathbf{y})))\times[0,e^{-t}[) = 0\}. \end{aligned}$$

A similar coupling argument works for $(\hat{P}_t^n)_{t\geq 0}$. Choose u such that $\mathbf{x}(K(u)) = 0$ and $\mathbf{y}(K(u)) = 0$. From the construction of $\hat{\mathbf{X}}^{n,\mathbf{x}}$ and $\hat{\mathbf{X}}^{n,\mathbf{y}}$, we see that it suffices to take as the coupling time

$$S = \inf\{t\geq 0 : \mathbf{Z}(K(u)\times]1,e^t]) = n\}.$$

□

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