The SDE solved by local times of a Brownian excursion or bridge derived from the height profile of a random tree or forest *

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Abstract

Let B be a standard one-dimensional Brownian motion started at 0. Let $L_{t,v}(|B|)$ be the occupation density of $|B|$ at level v up to time t. The distribution of the process of local times $(L_{t,v}(|B|), v \geq 0)$ conditionally given $B_t = 0$ and $L_{t,0}(|B|) = \ell$ is shown to be that of the unique strong solution X of the Itô SDE

$$dX_v = \left\{ 4 - X_v^2 \left(t - \int_0^v X_u du \right)^{-1} \right\} dv + 2\sqrt{X_v} dB_v$$

on the interval $[0, V_t(X))$, where $V_t(X) := \inf\{v : \int_0^v X_u du = t\}$, and $X_v = 0$ for all $v \geq V_t(X)$. This conditioned form of the Ray-Knight description of Brownian local times arises from study of the asymptotic distribution as $n \to \infty$ and $2k/\sqrt{n} \to \ell$ of the height profile of a uniform rooted random forest of k trees labeled by a set of n elements, as obtained by conditioning a uniform random mapping of the set to itself to have k cyclic points. The SDE is the continuous analog of a simple description of a Galton-Watson branching process conditioned on its total progeny. A result is obtained regarding the weak convergence of normalizations of such conditioned Galton-Watson processes and height profiles of random forests to a solution of the SDE. For $\ell = 0$, corresponding to asymptotics of a uniform random tree, the SDE gives a new description of the process of local times of a Brownian excursion, implying Jeulin's description of these local times as a time change of twice a Brownian excursion. Another corollary is the Biane-Yor

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description of the local times of a reflecting Brownian bridge as a time changed reversal of twice a Brownian meander of the same length.

1 Introduction

This paper describes the local time process of a Brownian excursion or reflecting Brownian bridge as the solution of a stochastic differential equation (SDE). The equation is the continuous analog of a corresponding description of the height profile of a random tree or forest obtained by conditioning a discrete time Galton-Watson process on its total progeny. This result is a development of the deep connections between Brownian excursions and branching processes established over the past 45 years and summarized briefly in the next paragraph.

Harris [27] observed that excursions of the simple random walk embedded in a Brownian path could be recoded to form the random tree associated with a discrete time Galton-Watson branching process with geometric(1/2) offspring distribution. As pointed out by Kawazu-Watanabe [35], this branching structure of random walk excursions is implicit in Knight's approach by random walk approximation to the Ray-Knight description of Brownian local time processes [64, 39, 65]. The Ray-Knight theorems are thus linked to Feller's [20] diffusion approximation for a critical branching process. Le Gall [24, 25] connected these ideas to Williams' path decompositions of Brownian motion [75]. Neveu-Pitman [49, 48] showed how the family tree of a continuous time Yule process is embedded in a path governed by Itô's law of Brownian excursions conditioned to exceed a given height. Aldous [2, 3, 4] developed analogous results encoding a Brownian excursion of given length as a continuum random tree, in the context of a more general theory of continuum random trees as weak limits as $n \to \infty$ of combinatorially defined trees with n vertices. See also Le Gall [26]. There is much current interest in the use of Brownian and other local time processes as models of continuous state branching processes, and the applications of such processes to Markovian superprocesses. See for instance [21, 22] and papers cited there.

Let B denote a standard Brownian motion started at 0. Let

$$B^{\mathrm{br},t} := (B^{\mathrm{br},t}_s, 0 \le s \le t) \stackrel{d}{=} (B_s, 0 \le s \le t \,|\, B_t = 0)$$

denote a Brownian bridge of length t. Here and throughout the paper, ":=" means "equal by definition" and " $\stackrel{d}{=}$ " denotes equality in distribution of random varibles or processes. For a fixed time $T > 0$ let $G_T := \sup\{s : s \le T, B_s = 0\}$ be the last zero of B before time T and $D_T := \inf\{s : s \ge T, B_s = 0\}$ be first zero of B after time T. It is known that for each $0 < t \le T$ the law of $(B_s, 0 \le s \le G_T)$ given $G_T = t$ does not depend on T, and is that of $B^{\mathrm{br},t}$, the Brownian bridge of length t. A process with the law of $(|B|_{G_T+s}, 0 \le s \le D_T - G_T)$ given $D_T - G_T = t$, which also does not depend on T, is called a *Brownian excursion of length* t, denoted here by $B^{\mathrm{ex},t}$. A process with the law of $(|B|_{G_T+s}, 0 \le s \le T - G_T)$ given $T - G_T = t$, which again does not depend on T, is called a *Brownian meander of length* t. It is well known that these Brownian bridges, excursions and meanders of length t can be constructed by Brownian scaling from the corresponding *standard* processes of length 1. If the length of one of these processes is not mentioned, it is assumed to be 1. See [65, 5] for background and further references to these processes.

For a suitable continuous function f with domain containing $[0,t]$, let $L_{t,v}(f)$ denote the *local time of f up to time t at level v* as defined by the *occupation density formula*

$$\int_0^t g(f_u)du = \int_{-\infty}^{\infty} g(v)L_{t,v}(f)dv \tag{1}$$

for every non-negative Borel measurable g, and continuity in v. Let $B^{|\mathrm{br}|,t}$ be a *reflecting Brownian bridge (RBB)* of length t

$$B^{|\mathrm{br}|,t} := (B_s^{|\mathrm{br}|,t}, 0 \le s \le t) := (|B_s^{\mathrm{br},t}|, 0 \le s \le t)$$

The abbreviation

$$L_{t,v}^{|\mathrm{br}|} := L_{t,v}(B^{|\mathrm{br}|,t})$$

will be used throughout the paper for the local time up to time t at level v of a RBB of length t. Using the Ray-Knight description of Brownian local times, Williams' [75] time reversal theorem, and an identity of σ-finite measures related to Brownian excursions, Biane-Yor [7] showed that the process $(L_{1,v}^{|\mathrm{br}|}, v \ge 0)$ of local times of RBB is a time change of the time reversal of twice a Brownian meander. Corollary 16 below recalls the precise statement of this result.

Section 3 reviews the appearance of the process $(L_{1,v}^{|\mathrm{br}|}, v \ge 0)$ as the asymptotic distribution for large n of the height profile of the random forest generated by a uniform random mapping of an n-element set to itself [1, 17]. The density of a limit law derived by Proskurin [63] from the height profile of this random forest was identified with the density of $L_{1,v}^{|\mathrm{br}|}$ by Aldous-Pitman [1]. See also Takács [70, 71] for a derivation of this density by a more straightforward random walk approximation. Recent work of Drmota-Gittenberger [17] develops the approach of [1] by use of a generating function analysis of the finite dimensional distributions of the height profile of a random mapping. After passage to the limit this yields a formula for the characteristic function of the finite dimensional distributions of $(L_{1,v}^{|\mathrm{br}|}, v \ge 0)$ in terms of a contour integral in the complex plane with a rather complicated integrand. The purpose of this paper is to record another description of the process $(L_{1,v}^{|\mathrm{br}|}, v \ge 0)$ in terms of the process X introduced in the following lemma, which is easily verified by techniques of stochastic calculus [65].

Lemma 1 *Let β be a Brownian motion. For each $\ell \ge 0$ and $t > 0$, there exists a unique strong solution X of the Itô SDE*

$$X_0 = \ell; \qquad dX_v = \delta_v(X)dv + 2\sqrt{X_v}d\beta_v \tag{2}$$

where

$$\delta_v(X) := 4 - X_v^2\left(t - \textstyle\int_0^v X_u du\right)^{-1} \tag{3}$$

with the convention that the equation for X is to be solved only on $[0, V_t(X))$ and that $X_v = 0$ for $v \ge V_t(X)$ where

$$V_t(X) := \inf\{v : \textstyle\int_0^v X_u du = t\}. \tag{4}$$

Definition 2 *For each $t > 0$ and $\ell \geq 0$ let $X := (X_{\ell,t,v}, v \geq 0)$ denote a process with same distribution as the solution X of the above SDE. Also, let $X_{0,0,v} := 0$ for all $v \geq 0$.*

The following proposition records some basic properties of this process X which follow easily from its definition by standard techniques of stochastic calculus.

Proposition 3 *The process $X := (X_{\ell,t,v}, v \geq 0)$ defined by Lemma 1 enjoys the following properties.*
(i) *For each $\ell \geq 0$ and $t > 0$ the random time $V_t(X)$ is strictly positive and finite a.s., and the left limit of X at time $V_t(X)$ exists and equals 0 a.s.. Consequently, the process X has continuous paths almost surely.*
(ii) *For each $\ell \geq 0$ and $t > 0$*

$$X_{\ell,t,0} = \ell \text{ and } \int_0^\infty X_{\ell,t,v} dv = t \text{ a.s.} \tag{5}$$

(iii) *For each $t > 0$ the collection of laws of $(X_{\ell,t,v}, v \geq 0)$ for $\ell \geq 0$ is determined by the collection of laws of $(X_{\ell,1,v}, v \geq 0)$ for $\ell \geq 0$ via the formula*

$$(X_{\ell,t,v}, v \geq 0) \stackrel{d}{=} (\sqrt{t} X_{\ell/\sqrt{t},1,v/\sqrt{t}}, v \geq 0) \tag{6}$$

(iv) *Let $E := [0,\infty) \times (0,\infty) \cup \{(0,0)\}$ and for $w := (\ell,t) \in E$ let Q^w denote the law of the process*

$$W := (W_{\ell,t,v}, v \geq 0) := ((X_{\ell,t,v}, t - \int_0^v X_{\ell,t,u} du), v \geq 0) \tag{7}$$

on the space of continuous E-valued paths. Then $(Q^w, w \in E)$ is the collection of laws of a strong Markov process W with state space E with $(0,0)$ as an absorbing state which is reached in finite time Q^w a.s. for all $w \in E$.

The *scaling property* (iii) implies that in formulating results about X there is no loss of generality in supposing that $t = 1$. However, this reduction tends to obscure basic properties of X such as the Markov property (iv) of W, where it is essential to involve both ℓ and t. Figure 1 displays approximations to trajectories of the process $(X_{\ell,t,v}, v \geq 0)$ for $\ell = 0, 1, 2, 3$ and $t = 1$, obtained by computer simulation. The main result of the paper is the following theorem, which together with Lévy's [45] well known formula

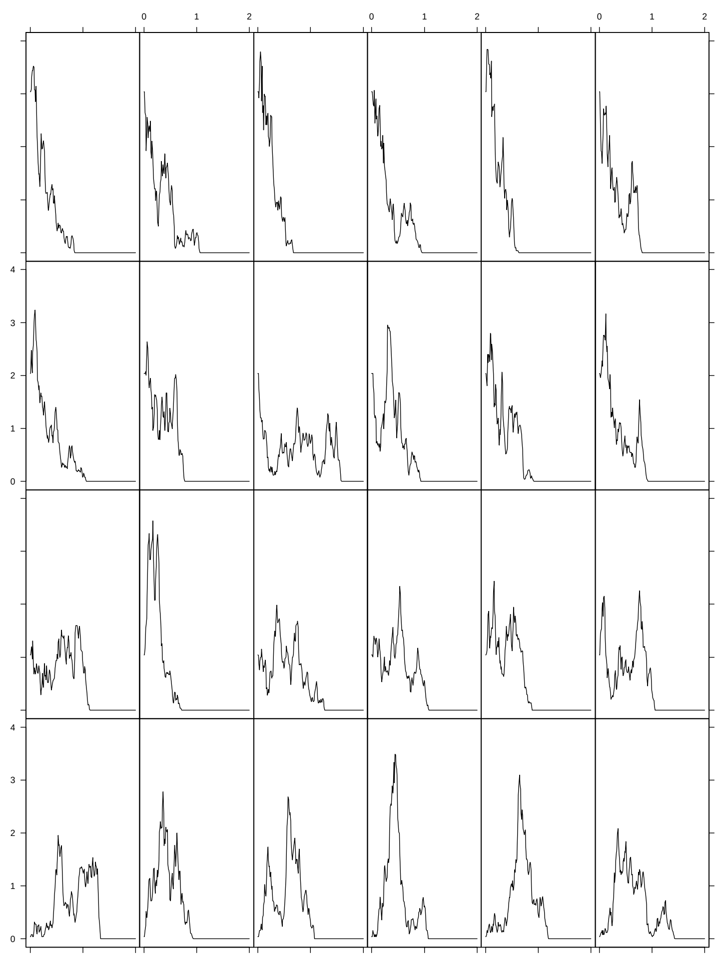
$$P(L_{t,0}^{|\mathrm{br}|}/2 > x) = e^{-x^2/2} \quad \text{for } x \geq 0 \tag{8}$$

determines the distribution of the process $(L_{t,v}^{|\mathrm{br}|}, v \geq 0)$.

Theorem 4 *For each $t > 0$ the collection of laws of $(X_{\ell,t,v}, v \geq 0)$ on the path space $C[0,\infty)$, parameterized by $\ell \geq 0$, is the unique weakly continuous version of the conditional law of $(L_{t,v}^{|\mathrm{br}|}, v \geq 0)$ given $(L_{t,0}^{|\mathrm{br}|} = \ell)$:*

$$(L_{t,v}^{|\mathrm{br}|}, v \geq 0 \,|\, L_{t,0}^{|\mathrm{br}|} = \ell) \stackrel{d}{=} (X_{\ell,t,v}, v \geq 0) \tag{9}$$

Figure 1 *Simulated trajectories of the local time process of a reflecting Brownian bridge of length 1.* Each panel shows an exact computer simulation of the height profile of a uniform random forest of k rooted trees with $n = 2500$ vertices, generated from binomial random variables via Lemma 9, and scaled as in (15) to approximate $(X_{\ell,1,v}, v \geq 0)$ governing the local times of a reflecting Brownian bridge of length 1 given local time ℓ at 0. In each panel, v ranges from 0 to 2 on a horizontal scale, and the vertical scale for local time ranges from 0 to 4. The area under each trajectory differs from negligibly from 1. Each row shows 6 repetitions for a given initial value $\ell = 2k/\sqrt{n} = k/25$, with $\ell = 1/25$ in the bottom row approximating the local time process of a Brownian excursion, and $\ell = 1, 2, 3$ in rows above.



It is intuitively clear, and made rigorous by Lemma 12, that conditioning a RBB of length t to have zero local time at 0 should produce a Brownian excursion of length t. Thus the particular case $\ell = 0$ of Theorem 4 yields a new characterization of the process of local times of a Brownian excursion:

Corollary 5 *The process of local times up to time t of a Brownian excursion of length t is identical in law to the process defined by the solution of the SDE* (2) *for $\ell = 0$:*

$$(L_{t,v}(B^{\mathrm{ex},t}), v \geq 0) \stackrel{d}{=} (X_{0,t,v}, v \geq 0) \tag{10}$$

If the path dependent drift $\delta_v(X)$ in the SDE (2) is replaced by a constant drift δ, then (2) becomes the SDE governing a $BESQ_\ell^{(\delta)}$ process $(X_{\ell,v}^{(\delta)}, v \geq 0)$, that is a *Bessel squared process of dimension* δ started at ℓ. See [68, 60, 65]. This process appears for $\delta = 0, 2$ and 4 in the Ray-Knight description of Brownian local time processes, also for other real δ, both fixed and path dependent, as the distribution of local times of appropriately perturbed Brownian motions. See Section 7 for references to such results. For $\tau_\ell := \inf\{t : L_{t,0}(|B|) = \ell\}$, the Ray-Knight theorem

$$(L_{\tau_\ell,v}(|B|), v \geq 0) \stackrel{d}{=} (X_{\ell,v}^{(0)}, v \geq 0) \tag{11}$$

combined with the well known and easily rigorized identity in law

$$(B_s^{|\mathrm{br}|,t}, 0 \leq s \leq t \,|\, L_{t,0}^{|\mathrm{br}|} = \ell) \stackrel{d}{=} (|B_s|, 0 \leq s \leq \tau_\ell \,|\, \tau_\ell = t)$$

for $\ell, t > 0$ shows that Theorem 4 implies the following corollary, and vice versa:

Corollary 6 *For each $\ell > 0$*

$$\left(X_{\ell,v}^{(0)}, v \geq 0 \,\middle|\, \int_0^\infty X_{\ell,v}^{(0)} dv = t \right) \stackrel{d}{=} (X_{\ell,t,v}, v \geq 0) \tag{12}$$

for all $t > 0$ where the distribution of the right side provides the unique determination of the conditional distribution of the left side that is weakly continuous in t.

It will be shown elsewhere that formula (12) can be proved without consideration of Brownian local times by using the theory of enlargement of filtrations [33, 34, 77] to deduce how the conditioning on $\int_0^\infty X_{\ell,v}^{(0)} dv = t$ introduces a drift term into the SDE solved by $(X_{\ell,v}^{(0)}, v \geq 0)$.

The rest of this paper is organized as follows. Section 2 shows how the process defined by the SDE (2) arises as the weak limit of a suitably normalized Galton-Watson branching process conditioned on its total progeny. This observation yields a new result regarding the weak convergence of conditioned Galton-Watson processes and height profiles of random forests to a limit process which is identified with the process appearing in Theorem 4. Section 3 explains the connection with random forests and random mappings. Section 4 lays out a series of corollaries which amplify the meaning of Proposition 1 and Theorem 4 in various ways. One of these corollaries is the Biane-Yor description of the process of local times of $B^{|\mathrm{br}|,1}$. Another is Jeulin's corresponding description of the process of local times of a Brownian excursion. It will be seen that this line of reasoning can also be reversed to recover Theorem 4 from either of these descriptions. Section 5 deduces from these results some explicit formulae regarding the distribution of local times of the reflecting bridge. Section 6 records a variation of Theorem 4 for an unreflected bridge. Finally, some concluding remarks and further references to related work are made in Section 7.

2 The Branching Process Approximation

According to a result in the theory of branching processes, first indicated by Feller [19] and further developed by Lamperti [44, 43] and Lindvall [46], if $Z_k(h)$ for $h = 0, 1, 2, \ldots$ denotes the number of individuals in generation h of a Galton-Watson process started with k individuals in which the offspring distribution has mean 1 and finite variance $\sigma^2 > 0$, and $Z_k(h)$ is defined for all $h \geq 0$ by linear interpolation between integers, then as $m \to \infty$ and k varies with m in such a way that $(2k)/(\sigma m) \to \ell$,

$$\left(\frac{2}{\sigma m} Z_k(2mv/\sigma), v \geq 0\right) \xrightarrow{d} (X^{(0)}_{\ell,v}, v \geq 0) \tag{13}$$

where $(X^{(0)}_{\ell,v}, v \geq 0)$ is the $BESQ^{(0)}_\ell$ process defined by the SDE $X_0 = \ell$, $dX_v = 2\sqrt{X_v}\,d\beta_v$, and $\xrightarrow{d}$ is the usual notion of convergence of distributions on $C[0,\infty)$. To check the non-standard normalizations in (13), observe that if the process on the left side has value x at v such that $2mv/\sigma$ equals an integer h, then $Z_k(h) = x\sigma m/2$. The number $Z_k(h+1)$ in the next generation of the branching process therefore has variance $(x\sigma m/2)\sigma^2$. The increment of the process on the left side over the next v-increment of $\sigma/(2m)$ has this variance multiplied by $(2/\sigma m)^2$. So along the grid of multiples of $\sigma/(2m)$, the variance of increments of the normalized process on the left side per unit v-increment, from one grid point to the next, given the normalized process has value x at the first grid point, is

$$\left(\frac{x\sigma m}{2}\right)\sigma^2\left(\frac{2}{\sigma m}\right)^2\left(\frac{\sigma}{2m}\right)^{-1} = 4x = (2\sqrt{x})^2$$

in accordance with the $BESQ^{(0)}$ SDE. Kawazu-Watanabe[35] showed that if the branching process is modified by allowing an immigration term, then the weak limit is a $BESQ^{(\delta)}_\ell$ process with δ representing an asymptotic rate of immigration per unit time. They showed also that this result for $\delta = 0$ and $\delta = 2$ yields the basic Ray-Knight theorems when applied to the branching processes with geometric offspring distribution derived from upcrossings of the Brownian path. See Le Gall [24] for another exposition of this idea, which is applied here to deduce Theorem 4 from a corresponding conditioned limit theorem for branching processes.

Consider now the distribution of the process $(Z_{k,n}(h), h \geq 0)$ defined by conditioning $(Z_k(h), h \geq 0)$ on the event that its *total progeny* $\sum_{h=0}^\infty Z_k(h)$ equals n, so

$$(Z_{k,n}(h), h \geq 0) \stackrel{d}{=} (Z_k(h), h \geq 0 \,|\, \textstyle\sum_{h=0}^\infty Z_k(h) = n) \tag{14}$$

where it assumed now that the offspring distribution is aperiodic, so the conditioning event has strictly positive probability for all sufficiently large n.

Theorem 7 *As $n \to \infty$ and $(2k)/(\sigma\sqrt{n}) \to \ell$ for some $\ell \geq 0$*

$$\left(\frac{2}{\sigma\sqrt{n}} Z_{k,n}(2\sqrt{n}v/\sigma), v \geq 0\right) \xrightarrow{d} (X_{\ell,1,v}, v \geq 0) \tag{15}$$

where $(X_{\ell,1,v}, v \geq 0)$ is the process constructed by Lemma 1 for $t = 1$.

Before discussing the proof of this result, the following remarks clarify its relation to the results in the introduction and to some earlier results in the literature. In view of (13) for $m = \sqrt{n}$, in the asymptotic regime with $(2k)/(\sigma\sqrt{n}) \to \ell$,

$$\frac{1}{n}\sum_{h=0}^{\infty} Z_k(h) \approx \int_0^{\infty} \frac{2}{\sigma\sqrt{n}} Z_k(2\sqrt{n}v/\sigma)dv \tag{16}$$

in the sense that the difference converges in probability to 0 as $n \to \infty$. So it is to be expected that if the limit process in (15) exists, then it must be the limit process in (13) conditioned to have integral equal to 1. That is precisely the claim of Corollary 6 for $t = 1$, provided $\ell > 0$. If $\ell = 0$ this line of reasoning fails because the limit process in (13) is identically equal to 0, but see Section 7 for further discussion. In view of the scaling properties of the processes involved, it is clear that Corollary 6 can be deduced from the combination of (13) and (15) for any particular choice of offspring distribution for the Galton-Watson process. Then Theorem 4 can be deduced from Corollary 6 via the Ray-Knight theorem (11), as indicated in the introduction.

In the particular case of Theorem 7 when $k = 1$ does not vary with n, so $\ell = 0$, convergence of one-dimensional distributions in (15), with a different characterization of the limit law, was obtained by Kennedy [36, Th 3]. See also Kolchin [40, Th. 2.5.6], who refers to Stepanov [69] for an earlier form of this convergence of one-dimensional distributions in a combinatorial setting which is equivalent to the conditioned branching process with a Poisson offspring distribution, as discussed in Section 3. The result of Theorem 7 for $k = 1$ and $\ell = 0$ was anticipated by Aldous [3, Conjecture 4], and proved by Drmota-Gittenberger [16], with the limiting process $(X_{0,1,v}, v \geq 0)$ defined by the process of local times of a Brownian excursion rather than by an SDE. Corollary 5 follows by comparison of this case of Theorem 7 with the result of Drmota-Gittenberger [16], or with the weaker integrated form of this result found by Aldous [3, Cor. 3], which is enough to characterize the limit process, assuming it exists, as the process of local times of a Brownian excursion. The sentence following (27) below gives an alternative proof of Corollary 5.

The key to the proof of Theorem 7 is provided by following lemma. Note that if $Z_{k,n}(h)$ is interpreted as the number of vertices at level h in a forest with n vertices defined by a collection of k family trees, one for each initial individual in the Galton-Watson process, then for $h = 0, 1, \ldots$ the random variable

$$A_{k,n}(h) := n - \sum_{i=0}^{h} Z_{k,n}(h) \tag{17}$$

represents the number of vertices in the forest strictly above level h.

Lemma 8 *Let $X_1, X_2, \ldots$ be a sequence of independent random variables with some distribution p on $\{0,1,2,\ldots\}$, and set $S_j = X_1 + \cdots + X_j$. Fix $1 \le k < n$ with $P(S_n = n-k) > 0$. Let $(Z_{k,n}(h), h = 0,1,2,\ldots)$ be a Galton-Watson branching process with offspring distribution p started with k individuals and conditioned to have total progeny n. Let $Z(h), h = 0,1,\ldots$ be a sequence of non-negative integer random variables, and set*

$$A(h) := n - \sum_{i=0}^{h} Z(h) \text{ and } W(h) := (Z(h), A(h)).$$

Then

$$(Z(h), h = 0,1,2,\ldots) \stackrel{d}{=} (Z_{k,n}(h), h = 0,1,2,\ldots) \tag{18}$$

if and only if the sequence $(W(h), h = 0,1,2,\ldots)$ is a Markov chain with state space

$$E' := (\{1,2,\ldots,\} \times \{0,1,2,\ldots,\}) \cup \{(0,0)\}$$

initial state $(k, n-k)$, and the following stationary transition probabilities: for all $h = 0,1,2,\ldots$ and all $(z,a) \in E'$ with $a > 0$ and $(z_1, a_1) \in E'$ with $z_1 + a_1 = a \ge 1$

$$P(W(h+1) = (z_1, a_1) \,|\, W(h) = (z,a)) = \frac{z_1(z+a)}{za} P(S_z = z_1 \,|\, S_{z+a} = a) \tag{19}$$

whereas $P(W(h+1) = (0,0) \,|\, W(h) = (z,0)) = 1$, and all other transition probabilities are zero.

Proof. This is easily verified by a computation using the Markov property of the branching process, Bayes rule and the well known formula of Dwass [18] for the distribution of the total progeny in the branching process starting with k individuals:

$$P\left(\sum_{h=0}^{\infty} Z_k(h) = n\right) = \frac{k}{n} P(S_n = n-k) \tag{20}$$

□

See [59] for a recent review of this fundamental formula (20) and its various probabilistic and combinatorial equivalents. As a check on formula (19), note that the sum of probabilities in (19) over all $0 \le z_1 \le a$ is 1 due to the well known formula

$$E(S_z \,|\, S_{z+a} = a) = \frac{za}{(z+a)}$$

which follows from exchangeability of the X_i. According to (19), given $W_{k,n}(h) = (z,a)$ the distribution of $Z_{k,n}(h+1)$ is obtained by size-biasing of the distribution of S_z given $S_{z+a} = a$, while $A_{k,n}(h+1) = a - Z_{k,n}(h+1)$. In particular, for a Poisson offspring distribution, the lemma yields:

Lemma 9 *Fix $1 \leq k < n$. A sequence $(Z(h), h = 0, 1 \ldots)$ has the same distribution as a Galton-Watson process with a Poisson offspring distribution started with k individuals and conditioned on total progeny equal to n, if and only if the sequence evolves by the following mechanism: $Z(0) = k$ and for each $h = 0, 1 \ldots$*

$$(Z(h+1) \,|\, Z(i), 0 \leq i < h, Z(h) = z, A(h) = a) \stackrel{d}{=} 1 + \mathrm{binomial}(a-1, z/(a+z)), \qquad (21)$$

where $A(h) := n - \sum_{i=0}^{h} Z(i)$ and $\mathrm{binomial}(m,p)$ *is a binomial(m,p) random variable, with the conventions* $\mathrm{binomial}(-1,p) = -1$ *and* $\mathrm{binomial}(0,p) = 0$.

Proof. For a Poisson offspring distribution the law of S_z given $S_{z+a} = a$ is binomial$(a, z/(z+a))$. It is elementary that a size-biased binomial(n,p) variable is 1 plus a binomial$(n-1,p)$ variable, and the conclusion follows by the previous lemma. □

Proof of Theorem 7 in the Poisson case. Consider the rescaled process on the left side of (15) in the Poisson case, so $\sigma = 1$, in an asymptotic regime with $n \to \infty$ and $2k/\sqrt{n} \to \ell$ for some $\ell \geq 0$. From (21), in the limit as n, z and a tend to ∞ with $2z/\sqrt{n} \to x$ and $a/n \to p$, for integer h the increment $\Delta_{k,n}(h) := Z_{k,n}(h+1) - Z_{k,n}(h)$ is such that the corresponding normalized increment $\Delta^*_{k,n}(h) := 2\Delta_{k,n}(h)/\sqrt{n}$ has the following conditional mean and variance given a history $(Z_{k,n}(i), 0 \leq i \leq h)$ with $W_{k,n}(h) = (z,a)$:

$$E(\Delta^*_{k,n}(h) \,|\, W_{k,n}(h) = (z,a)) = \frac{2}{\sqrt{n}} \left(1 + \frac{(a-1)z}{a+z} - z\right) \approx \left(4 - \frac{x^2}{p}\right) \frac{1}{2\sqrt{n}}$$

$$Var(\Delta^*_{k,n}(h) \,|\, W_{k,n}(h) = (z,a)) = \frac{4}{n} \frac{(a-1)za}{(a+z)^2} \approx 4x \frac{1}{2\sqrt{n}}$$

where the relative errors of approximation are negligible as $n \to \infty$, uniformly in h, provided $x < 1/\epsilon$ and $p > \epsilon$ which can be arranged by a localization argument, stopping the normalized process when either its value exceeds x or its integral exceeds $1 - p$. Since $\Delta^*_{k,n}(h)$ is the increment of the normalized process over a time interval of length $1/(2\sqrt{n})$, and the value of $p \approx A_{k,n}(h)/n$ can be recovered from the path of the normalized process with a negligible error via

$$p \approx \frac{A_{k,n}(h)}{n} = 1 - \frac{1}{n} \sum_{i=0}^{h} Z_{k,n}(h) \approx 1 - \int_0^{h/(2\sqrt{n})} \frac{2}{\sqrt{n}} Z_{k,n}(2\sqrt{n}v) \qquad (22)$$

these calculations show that the normalized process is governed asymptotically by the SDE (2)-(3). Since it is easily verified that the SDE has a unique strong solution, the conclusion follows by application of known results regarding the approximation of a Markov chains by the solution of an SDE [42, 41]. □

The above argument in the Poisson case is all that is needed for the proof of Theorem 4 indicated earlier. As that result is the main focus of this paper, details of the following argument are left to the reader. In the case $k = 1$ and $\ell = 0$ this argument simplifies the previous approach of Drmota-Gittenberger [16], because the work of proving both convergence of finite-dimensional distributions and tightness is all done in the general setting of approximating solutions to an SDE rather than in the specific setting of the conditioned branching process, where explicit computations involving the finite dimensional distributions are difficult.

Proof of Theorem 7 in the general case. For a general aperiodic offspring distribution with mean 1 and finite variance the same asymptotic conditional means and variances are obtained by a local normal approximation to the distribution of the random variables S_z and S_{z+a} appearing in formula (19). $\square$

3 Applications to Forests and Mappings.

It is known [40, 1] that each of the two processes $(Z_{k,n}(h), h \geq 0)$ defined in (i) and (ii) below has the distribution of a Galton-Watson branching process with Poisson offspring distribution started with k individuals and conditioned to have total progeny n. See [59] for a quick proof of this fact in case (i). This case can also be deduced by using classical enumerations of trees and forests reviewed in [58] to show the conditions of Lemma 9 are satisfied.

(i) In a random forest with uniform distribution on the set of all rooted forests of k trees labeled the set $[n] := \{1, \ldots, n\}$, let $Z_{k,n}(h)$ equal the number of vertices at height h above the roots. Call this process $(Z_{k,n}(h), h \geq 0)$ the *height profile of the forest.*

(ii) For M a mapping from $[n]$ to $[n]$, with iterates M^m for $m = 0, 1, 2, \ldots$, call $v \in [n]$ a *cyclic point* of M if $M^m(v) = v$ for some $m > 0$. Let cyclic(M) be the set of cylic points of M. For $v \in [n]$ let $h(v, M)$ be the least $m \geq 0$ such that $M^m(v) \in \text{cyclic}(M)$. So $h(v, M)$ is the height of v in the usual forest derived from M whose set of roots is cyclic(M). For $h = 0, 1, 2, \ldots$ let $Z_{*,n}(h)$ be the number of $v \in [n]$ such that $h(v, M_n) = h$, for M_n a random mapping from $[n]$ to $[n]$, with uniform distribution on the set of n^n such mappings, as studied in [40, 1]. Call this process $(Z_{*,n}(h), h \geq 0)$ the *height profile of the mapping forest.* Let $(Z_{k,n}(h), h \geq 0)$ be the height profile of the mapping forest conditioned on the event $(Z_{*,n}(0) = k)$ that M_n has exactly k cyclic points.

That $(Z_{k,n}(h), h \geq 0)$ has the same distribution in (ii) as in (i) is evident because given that M_n has k cyclic points, the forest generated by M_n is a uniform random forest of k rooted trees labeled by $[n]$, exactly as supposed in (i).

Corollary 10 *If $(Z_{k,n}(h), h \geq 0)$ is either*
(i) *the height profile of a uniform random forest of k rooted trees labeled by $[n]$, or*
(ii) *the height profile of the forest derived from a random mapping from $[n]$ to $[n]$ conditioned to have k cyclic points,*
then the distribution of the sequence $(Z_{k,n}(h), h \geq 0)$ is that described by Lemma 9, and in the limit regime as $n \to \infty$ and $2k/\sqrt{n} \to \ell \geq 0$

$$\left(\frac{2}{\sqrt{n}} Z_{k,n}(2\sqrt{n}v), v \geq 0\right) \xrightarrow{d} (X_{\ell,1,v}, v \geq 0) \tag{23}$$

Aldous-Pitman [1] showed how a uniform random mapping M_n can be recoded as a non-uniformly distributed random walk of $2n$ steps starting and ending at 0, with each tree component of the forest generated by M_n corresponding to an excursion of the walk away from 0, in such a way that as $n \to \infty$ the normalized walk converges in distribution to $B^{|\mathrm{br}|,1}$, a reflecting Brownian bridge of length 1. The following further corollary is now obtained by mixing the result of the previous corollary with respect to the distribution of the number $Z_{*,n}(0)$ of cyclic points of M_n. It is well known [1] that for all $x > 0$

$$\lim_{n \to \infty} P(Z_{*,n}/\sqrt{n} > x) = e^{-x^2/2} \tag{24}$$

So Theorem 4 and Corollary 10 combine with (24) and (8) to yield:

Corollary 11 Drmota-Gittenberger [17] *The normalized height profile of the forest derived from a uniform random mapping M_n converges weakly to the process of local times of a reflecting Brownian bridge of length 1:*

$$\left(\frac{2}{\sqrt{n}} Z_{*,n}(2\sqrt{n}v), v \geq 0 \right) \xrightarrow{d} (L_{1,v}^{|\mathrm{br}|}, v \geq 0) \tag{25}$$

A second proof of Theorem 4 can be given by comparison of Corollary 10 with the known result (25), or with the weaker integrated form of (25) given in [1]. A third proof of Theorem 4 could be given by first checking Theorem 7 for geometric instead of Poisson offspring distribution, then working with the branching process with geometric offspring distribution embedded in the excursions of a lattice walk, and appealing to the well known fact that a rescaled uniform reflecting lattice bridge of length $2n$ converges weakly to RBB when suitably scaled. See also Borodin [10, 11] for more about approximation of Brownian local times by random walks.

4 Related Results

This section presents a series of results related to Theorem 4. Those deduced from the theorem are labeled as corollaries, while those proved independently of the theorem are labeled lemmas. In particular, the Biane-Yor description of $(L_{t,v}^{|\mathrm{br}|}, v \geq 0)$ is obtained as a corollary. This description combined with the lemmas yields a fourth proof of Theorem 4.

Observe first that in terms of the local time representation of X provided by Theorem 4, the Markov property of the process W described in Proposition 3, which is the continuous analog of the Markov property of W in Lemma 8, amounts to the following equality of distributions on $C[0, \infty)$, where $\mathrm{dist}(X|Y)$ stands for the conditional distribution of X given Y: for all $t > r > 0, \ell \geq 0, v \geq 0$

$$\mathrm{dist}(L_{t,v+z}^{|\mathrm{br}|}, z \geq 0 \,|\, L_{t,u}^{|\mathrm{br}|}, 0 \leq u \leq v \text{ with } L_{t,v}^{|\mathrm{br}|} = \ell, t - \textstyle\int_0^v L_{t,u}^{|\mathrm{br}|} du = r)$$

$$= \mathrm{dist}(L_{r,z}^{|\mathrm{br}|}, z \geq 0 \,|\, L_{r,0}^{|\mathrm{br}|} = \ell) \tag{26}$$

Lemma 12 prepares for a refinement of this identity in law which is stated in Lemma 13.

Lemma 12 *For each $t > 0$ there exists on the path space $C[0,t]$ a unique family of conditional distributions for $(B_s^{|\mathrm{br}|,t}, 0 \le s \le t)$ given $L_{t,0}^{|\mathrm{br}|} = \ell$, say $(P^{\ell,t}, \ell \ge 0)$, that is weakly continuous in ℓ. In particular, the law $P^{0,t}$ is the law of a Brownian excursion of length t.*

Proof. The existence of such a continuous family $(P^{\ell,t}, \ell \ge 0)$ follows from the construction of the RBB by first constructing its zero set, then piecing together independent Brownian excursions over the maximal open intervals in the complement of the zero set. See [62] for an explicit description of the law of the ranked lengths of the complementary intervals given $L_{t,0}^{|\mathrm{br}|} = \ell$. Given the lengths. each interval is assigned an independent local time value with uniform distribution on $[0,\ell]$, and then the lengths are laid down in the order of the local time variables. It follows from this description that for each $\epsilon > 0$ there exists δ such that for $\ell < \delta$, with $P^{\ell,t}$ probability at least $1-\epsilon$, there is a complementary interval of length at least $t-\epsilon$, which implies easily that $P^{0,t}$ is the law of a Brownian excursion of length t. □

This construction of a Brownian excursion of length t by conditioning $B^{|\mathrm{br}|,t}$ on $L_{t,0}^{|\mathrm{br}|} = 0$ parallels similar constructions by conditioning a Brownian bridge $B^{\mathrm{br},t}$ of length t on $Z_t = 0$ for suitable Z_t, due to Blumenthal [9] for $Z_t := \inf_{0 \le s \le t} B_s^{\mathrm{br},t}$ and Chaumont [14] for $Z_t := \int_0^t 1(B_s^{\mathrm{br},t} \le 0)ds$.

If $B^{|\mathrm{br}|,\ell,t}$ denotes a process with law $P^{\ell,t}$, then Lemma 12 allows Theorem 4 to be recast as

$$(L_{t,v}(B^{|\mathrm{br}|,\ell,t}), v \ge 0) \stackrel{d}{=} (X_{\ell,t,v}, v \ge 0). \tag{27}$$

For $\ell = 0$, Corollary 5 is then recovered from Lemma 12. The law $P^{\ell,t}$ could also be constructed as in [1] as a weak limit from a uniform mapping M_n conditioned to have around $k = \ell\sqrt{n}/2$ cyclic points, or from a random rooted forest of k trees with n vertices, or by similar conditioning of a uniform lattice walk path of length $2n$ on its number of returns to 0.

Lemma 13 *Fix $t > 0$. For $v \ge 0$ let $Y^{v,-}$ denote the process with lifetime*

$$\zeta^{v,-} := \int_0^t 1(B_s^{|\mathrm{br}|,t} \le v)ds = \int_0^v L_{t,u}^{|\mathrm{br}|}du$$

defined by deleting the excursions of $B^{|\mathrm{br}|,t}$ above v and closing up the gaps, and let $Y^{v,+}$ denote the process with lifetime

$$\zeta^{v,+} := \int_0^t 1(B_s^{|\mathrm{br}|,t} > v)ds = t - \int_0^v L_{t,u}^{|\mathrm{br}|}du$$

defined by deleting all portions of the path of $B^{|\mathrm{br}|,t}$ below v, closing up the gaps, and finally subtracting v so the path starts and ends at 0. Then
(i) *the process $(L_{t,u}^{|\mathrm{br}|}, 0 \le u \le v)$ is the restriction to $[0,v]$ of the process of local times of $Y^{v,-}$ up to time $\zeta^{v,-}$.*
(ii) *the process $(L_{t,v+z}^{|\mathrm{br}|}, z \ge 0)$ is the process of local times at levels z of the process $Y^{v,+}$ up to time $\zeta^{v,+}$.*

(iii) *for $\ell' \geq 0, r > 0$*

$$\operatorname{dist}(Y^{v,+} \,|\, Y^{v,-} \text{ with } L_{t,v}^{|\mathrm{br}|} = \ell', \zeta^{v,+} = r) = P^{r,\ell'}.$$

(iv) *these results hold also for each $\ell \geq 0$ with the reflecting bridge $B^{|\mathrm{br}|,t}$ replaced by a bridge $B^{|\mathrm{br}|,\ell,t}$ with the law $P^{\ell,t}$ described in Lemma 12, and in particular for $\ell = 0$ with $B^{|\mathrm{br}|,t}$ replaced by $B^{\mathrm{ex},t}$, an excursion of length t.*

Proof. See Itô-McKean[31, §2.11] for details of the construction involved in the definition of the processes $Y^{v,-}$ and $Y^{v,+}$. Properties (i) and (ii) follow immediately from the construction. Property (iii) can be deduced from the structure of Brownian excursions and excursion filtrations exposed in [73, 30, 47, 66, 67]. Property (iv) then follows from (iii) and the definition of the conditioned bridge laws $P^{\ell,t}$. □

Consider now the family of laws $BES_x^{(3)}, x \geq 0$ of a *three-dimensional Bessel process* which may be constructed as the solution of the SDE

$$R_0 = x; \quad dR_t = R_t^{-1}\, dt \,+\, d\beta_t$$

for a Brownian motion β. For $t > 0$ let $R^{x,y,t}$ denote a *$BES^{(3)}$ bridge from x to y of length t*, that is a $BES_x^{(3)}$ process R conditioned on $R_t = y$, regarded as process parameterized by $[0,t]$. It is easily seen that such a process $R^{x,y,t}$ may be constructed for $0 \leq s \leq t$ by the formula

$$R_s^{x,y,t} := \sqrt{(x + (y-x)s/t + B_{1,s}^{\mathrm{br},t})^2 + (B_{2,s}^{\mathrm{br},t})^2 + (B_{3,s}^{\mathrm{br},t})^2}$$

where the $(B_{i,s}^{\mathrm{br},t}, 0 \leq s \leq t)$ for $i = 1,2,3$ are three independent copies of a one-dimensional Brownian bridge of length t. As a consequence of this description and Itô's formula, $R^{x,y,t}$ can also constructed as the solution over $[0,t]$ of the SDE

$$R_0 = x; \quad dR_s = \left(\frac{1}{R_s} + \frac{(y - R_s)}{(t-s)}\right) ds \,+\, d\gamma_s \tag{28}$$

for a Brownian motion γ. See also [75, 56, 28, 65] for background. The following lemma was suggested by the results of Jeulin [34] and Biane-Yor [7] presented in Corollary 16.

Lemma 14 *For $\ell \geq 0, t > 0$ let $R^{\ell,0,t}$ be the process derived from $(X_{\ell,t,v}, v \geq 0)$ via the formula*

$$2R_s^{\ell,0,t} := X_{\ell,t,v} \text{ for the least } v : \int_0^v X_{\ell,t,u} du = s, \text{ where } 0 \leq s \leq t \tag{29}$$

Then $R^{\ell,0,t}$ is a $BES^{(3)}$ bridge from ℓ to 0 of length t, and $(X_{\ell,t,v}, v \geq 0)$ can be recovered from $R^{\ell,0,t}$ via the formula

$$X_{\ell,t,v} = 2R_s^{\ell,0,t} \text{ for the least } s : \int_0^s \frac{dr}{2R_r^{\ell,0,t}} = v \tag{30}$$

Consequently, starting from any $BES^{(3)}$ bridge $R^{\ell,0,t}$ from ℓ to 0 of length t, the process X defined by (30) has the same distribution as X defined by the SDE (2).

Proof. The recipe (30) for inverting the time change (29) is easily checked, so it suffices to show that if $R := (R_s^{\ell,0,t}, 0 \leq s \leq t)$ solves the SDE (28), for $(x, y) = (\ell, 0)$, that is

$$dR_s = \left(\frac{1}{R_s} - \frac{R_s}{(t-s)} \right) ds + d\gamma_s$$

for some Brownian motion γ, then $X := (X_{\ell,t,v}, v \geq 0)$ defined by (30) solves the SDE (2) for some Brownian motion β. But from (29) and (30)

$$dX_v = 2\, dR_s \text{ where } s = \int_0^v X_u du$$

A level increment dv for X corresponds to a time increment $ds = X_v dv$ for R, and $R_s = X_v/2$, so

$$dX_v = 2 \left(\frac{1}{X_v/2} - \frac{X_v/2}{(t - \int_0^v X_u du)} \right) X_v dv + 2\sqrt{X_v}\, d\beta_v \tag{31}$$

for some other Brownian motion β, where the factor $\sqrt{X_v}$ appears in the diffusion term due to Brownian scaling, and the equation (31) simplifies to (2). As a technical point, the definition of β above the level $\int_0^t dr/2R_r^{\ell,0,t}$ when X hits 0 may require enlargement of the probability space. See [65, Ch. V] for a rigorous discussion of such issues. □

Lemma 15 *The laws of a Brownian excursion $B^{\mathrm{ex},t}$ and a Brownian meander $B^{\mathrm{me},t}$, each of length t, can be expressed as follows in terms of the laws of $BES^{(3)}$ bridges $R^{x,y,t}$:*
(i) D. Williams [74]

$$B^{\mathrm{ex},t} \stackrel{d}{=} R^{0,0,t} \tag{32}$$

(ii) Imhof [28] *The final value $B_t^{\mathrm{me},t}$ of the meander has the distribution of $\sqrt{t}R$ for R with the standard Rayleigh distribution $P(R > r) = \exp(-r^2/2), r > 0$, and*

$$(B^{\mathrm{me},t} \,|\, B_t^{\mathrm{me},t} = y) \stackrel{d}{=} R^{0,y,t} \tag{33}$$

The above results now combine easily to yield:

Corollary 16 *For a process $Y := (Y_s, 0 \leq s \leq t)$ admitting a local time process $(L_{t,v}(Y), v \geq 0)$, define a process $\hat{L}(Y) := \hat{L}_r(Y), 0 \leq r \leq t$ by $\hat{L}_r(Y) := L_{t,v(r)}(Y)$ where $v(r) := \sup\{y \geq 0 : \int_y^\infty L_{t,x} dx > r\}$. So $\hat{L}_r(Y)$ is the local time of Y at a level $v(r)$ above which Y spends time r.*
(i) Jeulin [34, p. 264] *If Y is a Brownian excursion of length t, then so is $\hat{L}(Y)/2$.*
(ii) Biane-Yor [7, Th. (5.3)] *If Y is a reflecting Brownian bridge of length t, then $\hat{L}(Y)/2$ is a Brownian meander of length t.*
(iii) *If Y has the law $P^{t,\ell}$ of $B^{|\mathrm{br}|,t}$ given $L_{t,0}^{|\mathrm{br}|} = \ell$, then $\hat{L}(Y)/2$ is a $BES^{(3)}$ bridge from 0 to ℓ of length t.*

Indeed, by combining Lemmas 14 and 15, for Y as in (iii) the process $\hat{L}(Y)/2$ is seen to be a time-reversed copy of $R^{\ell,0,t}$, that is a copy of $R^{0,\ell,t}$, by a well known property of one-dimensional diffusion bridges. Now (i) is seen to be the special case $\ell = 0$ of (iii) by Corollary 5, while (ii) is an integrated form of (iii) by (33) and the fact that $L_{t,0}^{|\mathrm{br}|}/2$ and $B_t^{\mathrm{me},t}$ have the same distribution. In view of Lemma 12, the Biane-Yor result (ii) can be disintegrated by conditioning on the local time of Y at 0 to recover (iii). Theorem 4 can then be deduced from (iii) by retracing the above argument via Lemmas 15 and 14. Theorem 4 can even be deduced from the special case (i) of (iii). For (i) implies the special case $(\ell = 0, t = 1)$ of Theorem 4 by the argument just indicated, hence the case $(\ell = 0, t > 0)$ by Brownian scaling. By application of Lemma 13 it is clear that the bivariate process W^* derived from the local time representation of a non-negative process Y of length t, say

$$W^* := ((L_{t,v}(Y), t - \textstyle\int_0^v L_{t,w}(Y)dw), v \geq 0)$$

is Markovian with the same transition probabilities whenever Y has the distribution $P^{\ell,t}$ for any $\ell \geq 0, t > 0$. Denote this process W^* by $(W^*_{\ell,t,v}, v \geq 0)$. For Y an excursion of length t, corresponding to $\ell = 0$, for each $v > 0$ the distribution of $W^*_{0,t,v}$ has a strictly positive density over $(0,\infty)\times(0,t)$ as well as an atom at the absorbing state $(0,0)$. Due to the Markov property of W^*, the transition mechanism of this process starting at any state (ℓ, s) with $\ell > 0$ and $s < t$ is therefore determined by the evolution of W^* starting in state $(0,t)$ corresponding to an excursion Y. But by inspection of the SDE in the excursion case, the same SDE must be solved starting in an arbitrary state (ℓ, s) with $\ell > 0$ and $s < t$. Since t was arbitrary, the conclusion of Theorem 4 follows.

5 Some explicit formulae

Previous results combined with existing results in the literature yield a number of explicit formulae regarding the distribution of the process of local times $(L_{1,v}^{|\mathrm{br}|}, v \geq 0)$ of a reflecting Brownian bridge of length 1. As a consequence of the Biane-Yor result of Corollary 16(ii),

$$\sup_{v\geq 0} L_{1,v}^{|\mathrm{br}|} \stackrel{d}{=} 2 \sup_{0\leq u\leq 1} B_u^{\mathrm{me}} \stackrel{d}{=} 4 \sup_{0\leq u\leq 1} B_u^{|\mathrm{br}|,1} \tag{34}$$

where B^{me} is a Brownian meander of length 1, the second equality is due to Kennedy [37], and the distribution of $\sup_{0\leq u\leq 1} B_u^{|\mathrm{br}|,1}$ is given by the well known Kolmogorov-Smirnov formula. Also by Corollary 16(ii),

$$(L_{1,0}^{|\mathrm{br}|}, \sup_{v\geq 0} L_{1,v}^{|\mathrm{br}|}) \stackrel{d}{=} 2(B_1^{\mathrm{me}}, \sup_{0\leq u\leq 1} B_u^{\mathrm{me}}) \tag{35}$$

The joint density of this distribution can be read from known results for the Brownian meander [29]. By conditioning (35) on $L_{1,0}^{|\mathrm{br}|}$, or by Lemma 14

$$(\sup_{v\geq 0} L_{1,v}^{|\mathrm{br}|} \,|\, L_{1,0}^{|\mathrm{br}|} = \ell) \stackrel{d}{=} \sup_{v\geq 0} X_{\ell,1,v} \stackrel{d}{=} 2 \sup_{0\leq u\leq 1} R_u^{0,\ell,1} \tag{36}$$

where $R^{0,\ell,1}$ is a three-dimensional Bessel bridge from 0 to ℓ of length 1. The density of this conditional distribution can be read either from the joint density in (35), or from the general formula of Kiefer [38] for the distribution of the maximum of a d-dimensional Bessel bridge.

For fixed v the distribution of

$$(L_{1,v}^{|\mathrm{br}|} \,|\, L_{1,0}^{|\mathrm{br}|} = \ell) \stackrel{d}{=} X_{\ell,1,v} \tag{37}$$

can be evaluated by Theorem 7 as the limit distribution of $2Z_{k,n}(2v\sqrt{n})/\sqrt{n}$ as $n \to \infty$ with $2k/\sqrt{n} \to \ell$ for $Z_{k,n}(h)$ as in Lemma 9 the number of vertices at level h in a uniform random forest of k rooted trees labeled by $[n]$. A formula for the density of this limit distribution was found by Pavlov [51, Th. 6], in terms of an integral with respect to a two-dimensional probability distribution with an explicit Fourier transform. For $\ell = 0$ more explicit formulae are known from the representation (10) of $X_{0,1,v}$ as the distribution of local time of an excursion at level v. See [16] for a review of various representations of this distribution, and transform expressions for the higher dimensional distributions. Drmota-Gittenberger [17] give similar transforms for the finite dimensional distributions of $(L_{1,v}^{|\mathrm{br}|}, v \geq 0)$. Presumably similar transforms can be given for the the finite-dimensional distributions of $(X_{\ell,1,v}, v \geq 0)$. Pavlov [52, Th.2] found a transform for the asymptotic distribution of the maximum height in a random forest of plane rooted trees, which with appropriate scaling can be interpreted via Theorem 7 for geometric offspring distribution as the distribution of

$$(\sup_{0 \leq u \leq 1} B_u^{|\mathrm{br}|,1} \,|\, L_{1,0}^{|\mathrm{br}|} = \ell) \stackrel{d}{=} \inf\{v > 0 : X_{\ell,1,v} = 0\} \stackrel{d}{=} \frac{1}{2}\int_0^1 \frac{du}{R_u^{0,\ell,1}} \tag{38}$$

This distribution does not seem to have been studied in the Brownian literature, except in the case $\ell = 0$ when it reduces to the distribution of the maximum of Brownian excursion [15, 37, 7, 3].

6 The local time process of an unreflected Brownian bridge.

Let

$$L_{t,v}^{\mathrm{br}} := L_{t,v}(B^{\mathrm{br},t})$$

denote the local time up to time t at level v of an unreflected Brownian bridge of length t. In principle, the law of the process $(L_{t,v}^{\mathrm{br}}, v \in \mathbb{R})$ is determined by Ray's [64] description for each $\theta > 0$ of the process of local times $(L_{T_\theta,v}(B), v \in \mathbb{R})$ for T_θ an exponential variable with rate $\theta^2/2$ independent of B. According to that description, which is reviewed from a modern perspective in Biane-Yor [8], conditionally given $B_{T_\theta} = 0$ and $L_{T_\theta,0}(B) = \ell$ the processes $(L_{T_\theta,v}(B), v \geq 0)$ and $(L_{T_\theta,-v}(B), v \geq 0)$ are independent copies of the time-homogeneous diffusion process $Y = (Y_{\ell,\theta,v}, v \geq 0)$ defined as the solution of

$$Y_0 = \ell, \quad dY_v = -2\theta Y_v\, dv + 2\sqrt{Y_v}\, d\beta_v$$

for a Brownian motion β. Effectively, this describes the distribution of local times of a Brownian bridge of random length with distribution that of T_θ given $B_{T_\theta} = 0$, which is easily seen to be the gamma$(1/2, \theta^2/2)$ distribution. By application of Brownian scaling, it is clear that the law of $(L^{\rm br}_{1,v}, v \in \mathbb{R})$, hence that of $(L^{\rm br}_{t,v}, v \in \mathbb{R})$ for each $t > 0$, is determined by this description of the law of $(L_{T_\theta,v}(B), v \in \mathbb{R} \,|\, B_{T_\theta} = 0)$ even for $\theta = 1$. However, it is not easy to use this description to deduce more explicit descriptions of the finite dimensional distributions of $(L^{\rm br}_{1,v}, v \in \mathbb{R})$. For instance, it would be painful to recover from Ray's result the formula

$$P(L^{\rm br}_{1,v} > x) = e^{-(2|v|+x)^2/2} \text{ for } x \geq 0, v \in \mathbb{R} \tag{39}$$

given by Lévy [45] for $v = 0$ and $x = 0$, and Borodin [12] for general x and v. That the higher-dimensional distributions of the process $(L^{\rm br}_{1,v}, v \in \mathbb{R})$ are not so simple is clear already from the complexity of Proskurin's formula [63, 1, 70, 71] for the density of $L^{|\rm br|}_{1,v} = L^{\rm br}_{1,v} + L^{\rm br}_{1,-v}$ for $v > 0$.

Let $-I$ denote the infemum of the standard bridge $B^{\rm br,1}$, so

$$-I := \inf_{0 \leq u \leq 1} B^{\rm br,1}_u = \inf\{r : L^{\rm br}_{1,r} > 0\} \text{ a.s.}$$

As observed by Biane [6],

$$(L^{\rm br}_{1,v-I}, v \geq 0) = (L_{1,v}(B^{\rm ex,1}), v \geq 0) \tag{40}$$

where $B^{\rm ex,1}$ is the standard Brownian excursion derived from $B^{\rm br,1}$ by Vervaat's [72] transformation. Combined with Corollary 5 this yields the following description of the process of bridge local times:

Corollary 17

$$(L^{\rm br}_{1,r}, r \in \mathbb{R}; -I) \stackrel{d}{=} (X_{0,1,(J+r)\vee 0}, r \in \mathbb{R}; J) \tag{41}$$

where on the right side $P(J \in du | X_{0,1,v}, v \geq 0) = X_{0,1,u} du, u > 0$.

Proof. In view of (40) it suffices to show that $P(-I \in du | B^{\rm ex,1}) = L_{1,u}(B^{\rm ex,1})(du)$, and this follows easily from Biane's observation that the time of the minimum of $B^{\rm br,1}$ is independent of $B^{\rm ex,1}$ with uniform distribution on $[0,1]$. □

See Chaumont [14] for related results. As shown by Lévy the random variable

$$A^{\rm br}_1 := \int_0^1 1(B^{\rm br,1}_u \geq 0) du = \int_0^\infty L^{\rm br}_{1,v} dv$$

has a uniform distribution on $[0,1]$. The joint distribution of $(L^{\rm br}_{1,0}, A^{\rm br}_1)$, while not as simple as its marginals, is in principle determined by transforms which can be read from Ray's description up to time T_θ, or from the Feynman-Kac formula [32]. By consideration of the kind of transformation between reflecting and unreflecting bridges described in Bertoin-Pitman [5, Lemma 5.2], Theorem 4 yields also the following corollary:

Corollary 18 *Conditionally given* $(L^{\rm br}_{1,0}, A^{\rm br}_1) = (\ell, a)$, *the processes* $(L^{\rm br}_{1,v}, v \geq 0)$ *and* $(L^{\rm br}_{1,-v}, v \geq 0)$ *are independent copies of* $(X_{\ell,a,v}, v \geq 0)$ *and* $(X_{\ell,1-a,v}, v \geq 0)$ *respectively.*

7 Concluding Remarks

Perkins [53] showed that for each fixed $t > 0$ the process of local times of B at levels v up to time t is a semi-martingale as v ranges over all real values, and he gave the semi-martingale decomposition of this process. Jeulin [34] gave a version of Perkins results that allows conditioning on B_t. Presumably, a similar description of the process $(L_{t,v}(|B|), v \geq 0)$ could be obtained, and then Theorem 4 should appear after conditioning on $B_t = 0$.

As remarked in the discussion below Theorem 15, the left side of formula (12) has no meaning for $\ell = 0$ and $t > 0$, even though the process $(X_{0,t,v}, v \geq 0)$ is a well defined process identical in law to the process of local times of $B^{\mathrm{ex},t}$, a Brownian excursion of length t. However, for $\ell > 0$ formula (12) amounts to the following identity of probability measures on $C[0,\infty)$:

$$Q_\ell^{(0)} = \int_0^\infty Q_{\ell,t}\, q_\ell^{(0)}(dt) \tag{42}$$

where $Q_\ell^{(0)}$ is the law of the $BESQ_\ell^{(0)}$ process starting at $\ell > 0$, where $Q_{\ell,t}$ is the law of $(X_{\ell,t,v}, v \geq 0)$ for $\ell \geq 0$ and $t > 0$, and $q_\ell^{(0)}$ denotes the distribution of $\int_0^\infty X_v\, dv$ for X with distribution $Q_\ell^{(0)}$, that is, for $t > 0$

$$q_\ell^{(0)}(dt) = P(\tau_\ell \in dt) = \frac{\ell}{\sqrt{2\pi}}\, t^{-3/2} e^{-\ell^2/(2t)} dt \tag{43}$$

where the first equality is read from the Ray-Knight theorem (11), and the second is Lévy's formula for the density of the stable(1/2) variable τ_ℓ. If this form (42) of formula (12) is divided by ℓ, and the limit taken as $\ell \downarrow 0$, the result is the following Corollary, where according to (5), the law $Q_{0,t}$ of $(X_{0,t,v}, v \geq 0)$ may be interpreted as the law of local times up to time t of a Brownian excursion of length t, as in [60, 61]:

Corollary 19 Pitman-Yor [60, 61] *The formula*

$$M := \int_0^\infty Q_{0,t}\, \frac{t^{-3/2} dt}{\sqrt{2\pi}} \tag{44}$$

defines a σ-finite measure on $C[0,\infty)$ under which the co-ordinate process is Markovian with the $BESQ^{(0)}$ semigroup, with almost every path starting at 0.

As shown in [60, (4.2)], this σ-finite law M is the distribution of the ultimate local time process $(L_{\infty,v}(\varepsilon), v \geq 0)$ for ε an element of $C[0,\infty)$ subject to Itô's σ-finite law of Brownian excursions. See [23, 57, 60, 61] for various developments and applications of this result to the Lévy-Itô representation of squared Bessel and related processes.

Le Gall-Yor [23] deduced from the Lévy-Itô representation of squared Bessel processes [60] that $BESQ_0^{(\delta)}$ for $\delta \geq 0$, can be constructed as the process of ultimate local times $(L_{\infty,v}(Y^{(\delta)}), v \geq 0)$ of $Y^{(\delta)}$ constructed from a reflecting Brownian motion $|B|$ as $Y_t^{(\delta)} = |B|_t + L_{t,0}(|B|)/\delta$ for $t \geq 0$. Carmona-Petit-Yor [13] found a similar construction of $BESQ_0^{(\delta)}$ for $\delta < 0$. See also [54, 55]. Norris-Rogers-Williams [50, Th. 2]showed that the distribution of a local time process derived from another kind of perturbed Brownian motion, with a drift depending on its local time process, can be characterized by a variation of the Bessel square SDE like (2), but with a different form of path dependent drift coefficient $\delta_v(X)$. See also [76] and papers cited there for various other Ray-Knight type descriptions of Brownian local time processes, and further references on this topic.

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