Random Brownian Scaling Identities and Splicing of Bessel Processes

by Jim Pitman and Marc Yor

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Department of Statistics
University of California
367 Evans Hall # 3860
Berkeley, CA 94720-3860

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Abstract

An identity in distribution due to F. Knight for Brownian motion is extended in two different ways: firstly by replacing the supremum of a reflecting Brownian motion by the range of an unreflected Brownian motion, and secondly by replacing the reflecting Brownian motion by a recurrent Bessel process. Both extensions are explained in terms of random Brownian scaling transformations and Brownian excursions. The first extension is related to two different constructions of Itô's law of Brownian excursions, due to D. Williams and J.-M. Bismut, each involving back-to-back splicing of fragments of two independent three-dimensional Bessel processes. Generalizations of both splicing constructions are described which involve Bessel processes and Bessel bridges of arbitrary positive real dimension.

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1 Introduction

Let $B := (B_t, t \geq 0) := (B(t), t \geq 0)$ be a standard one-dimensional Brownian motion started at 0, and let $M_t := \sup_{0 \leq s \leq t} |B_s|$. Let $(L_t, t \geq 0)$ be the usual local time process at 0 for B, and set $\tau = \inf\{t : L_t = 1\}$. For $\delta > 0$ let $R^{(\delta)}$ be a $\mathrm{BES}_0^{(\delta)}$ process, that is a Bessel process of dimension δ started at 0, which can be constructed for positive integer δ as the square root of the sum of squares of δ independent copies of B. For $x > 0$ let $T_x^{(\delta)}$ denote the hitting time of x by $R^{(\delta)}$. As observed by Biane [2], a result of Knight [14, Theorem 3] can be re-expressed as follows:

$$\frac{\tau}{M_\tau^2} \stackrel{d}{=} T_2^{(3)} \tag{1}$$

where $\stackrel{d}{=}$ denotes equality in distribution. Biane [2] and Vallois [29] explained Knight's identity (1) by decomposing the path of the randomly rescaled process

$$(B(uM_\tau^2)/M_\tau,\, 0 \leq u \leq \tau/M_\tau^2)$$

into various fragments, and rearranging these fragments to make a path with the same distribution as $(R_u^{(3)}, 0 \leq u \leq T_2^{(3)})$. Here we use similar techniques to obtain some extensions of Knight's identity which were announced without proof in [21]. We also relate these identities in distribution to splicing constructions involving Bessel processes and their bridges for arbitrary positive real dimension δ. See Section 2 for a brief review of the definition of these processes.

Let

$$S_t := \sup_{0 \leq s \leq t} B_s, \qquad I_t := -\inf_{0 \leq s \leq t} B_s \tag{2}$$

and let $A_t := S_t + I_t$, called the *amplitude* or *range* of the Brownian path B up to time t.

Theorem 1 *There is the identity in distribution*

$$\frac{\tau}{A_\tau^2} \stackrel{d}{=} T_1^{(3)} + \hat{T}_1^{(3)} \tag{3}$$

where $\hat{T}_1^{(3)}$ is an independent copy of $T_1^{(3)}$. Moreover, τ/A_τ^2 is independent of the random variable I_τ/A_τ, which has uniform distribution on $[0, 1]$.

Theorem 1 constitutes an extension of Knight's identity (1), because

$$\frac{\tau}{M_\tau^2} = \frac{\tau}{A_\tau^2}\left[\max\left(\frac{I_\tau}{A_\tau}, 1-\frac{I_\tau}{A_\tau}\right)\right]^{-2} \tag{4}$$

and (1) follows from Theorem 1 and (4) by a routine Laplace transform calculation. A pathwise explanation of Theorem 1 is provided by the following theorem, which is proved in Section 3.

For $t > 0$ and a continuous function f whose domain contains the interval $[0, t]$, let $T_{\inf}^{f,t}$ be the least s such that $f(s) = \inf_{0\le u\le t} f(u)$, and let $T_{\sup}^{f,t}$ the least s such that $f(s) = \sup_{0\le u\le t} f(u)$.

Theorem 2 *Let $\rho := T_{\inf}^{B,\tau}$, so $B_\rho = -I_\tau$. Define*

$$B^\#(t) := B_{\rho+t \pmod{\tau}} - B_\rho,\ \ 0 \le t \le \tau,$$

and let $\sigma := T_{\sup}^{B^\#,\tau}$, so $B^\#(\sigma) = A_\tau$. Then the two processes

$$R := (A_\tau^{-1} B^\#(uA_\tau^2), 0 \le u \le \sigma/A_\tau^2)$$

and

$$\widehat{R} := (A_\tau^{-1} B^\#((\tau-u)A_\tau^2), 0 \le u \le (\tau-\sigma)/A_\tau^2)$$

are independent copies of $(R_u^{(3)}, 0 \le u \le T_1^{(3)})$. Moreover, the pair of processes $(R, \hat{R})$ is independent of the random variable I_τ/A_τ whose distribution is uniform on $[0, 1]$.

See also [9, 10, 30] for other decompositions of the Brownian path involving the range process and $\mathrm{BES}^{(3)}$ pieces. These results are all closely related to Williams' [34] construction of Itô's law of Brownian excursions via back-to-back splicing of two independent $\mathrm{BES}^{(3)}$ fragments R and $\widehat{R}$ as in Theorem 2. To describe some more general splicing results, we consider the following construction:

Construction 3 Given two continuous path processes with random finite lifetimes and final value 1, say $R := (R(t), 0 \le t \le \eta)$ and $(\widehat{R} := (\widehat{R}(t), 0 \le t \le \widehat{\eta})$ with $R(\eta) = \widehat{R}(\widehat{\eta}) = 1$, construct a random element $r := (r(u), 0 \le u \le 1)$ of $C[0, 1]$ by first pasting R and $\widehat{R}$ back to back and then transforming the resulting path by Brownian scaling to have lifetime 1; that is

$$r(u) := \begin{cases} \zeta^{-1/2} R(u\zeta) & \text{if } 0 \le u \le V \\ \zeta^{-1/2}\widehat{R}((1-u)\zeta) & \text{if } V \le u \le 1 \end{cases} \tag{5}$$

where $\zeta := \eta + \widehat{\eta}$ and $V := \eta/\zeta$.

Observe that R and $\widehat{R}$ can be recovered from (r, V) via the formulae

$$\zeta = 1/r^2(V)$$

$$(R(t), 0 \leq t \leq \eta) = (r(t/\zeta)/r(V), 0 \leq t \leq V\zeta)$$

$$(\widehat{R}(t), 0 \leq t \leq \widehat{\eta}) = (r(1 - t/\zeta)/r(V), 0 \leq t \leq (1 - V)\zeta).$$

So any joint distribution of $(R, \widehat{R})$ determines a unique joint distribution of (r, V) with $r_V > 0$ a.s., and vice versa.

Our proof of Theorem 2 is based on case $\delta = 3$ of the following result of [21, 22]. Let $r^{(\delta)}$ be a standard $\mathrm{BES}^{(\delta)}$ bridge, starting at 0 at time 0 and ending at 0 at time 1.

Theorem 4 [21][22, Thm. 3.1] *For each real $\delta > 0$ the following conditions* (i) *and* (ii) *are equivalent:*

(i) *R and $\widehat{R}$ are two independent* $\mathrm{BES}_0^{(\delta)}$ *processes, each run until its first hit of 1;*

(ii) *The law of r is determined by the formula*

$$P(r \in dw) = 2^{1-\frac{\delta}{2}}\Gamma(\tfrac{\delta}{2})^{-1}(\sup_{0 \leq u \leq 1} w_u)^{\delta-2} P(r^{(\delta)} \in dw), \qquad w \in C[0,1] \tag{6}$$

and $V = T_{\text{sup}}^{r,1}$.

Formula (6) is meant to indicate the following absolute continuity relation between the laws of r and $r^{(\delta)}$ on $C[0,1]$: for every non-negative Borel measurable function F defined on $C[0,1]$,

$$P[F(r)] = P[D(r^{(\delta)})F(r^{(\delta)})]$$

where the density factor $D(w)$ at path w is $D(w) = 2^{1-\frac{\delta}{2}}\Gamma(\frac{\delta}{2})^{-1}(\sup_{0 \leq u \leq 1} w_u)^{\delta-2}$. Here P stands for the probability measure and expectation operator on some background probability space where processes under consideration are defined. Throughout the paper, similar notation will be used to describe absolute continuity relationships between the laws of various processes. See also [37] regarding other absolute continuity relationships related to random Brownian scaling operations.

For $\delta = 2$ the density factor in (6) reduces to 1, so condition (ii) of Theorem 4 reduces to

$$r \stackrel{d}{=} r^{(2)} \quad \text{and} \quad V = T^{r,1}_{\text{sup}}.$$

For $\delta = 3$, the standard $\text{BES}^{(3)}$ bridge $r^{(3)}$ has the same distribution as a *standard Brownian excursion* [32]. See [4, 35, 38] regarding the close connection between this case of Theorem 4 and the functional equation for Riemann's zeta function. In Section 4 we establish the following analog of Theorem 4 for splicing of two Bessel processes at their *last* hits of 1 instead of their *first* hits of 1.

Theorem 5 *For each $\delta > 2$ the following conditions* (i) *and* (ii) *are equivalent:*

(i) *R and $\widehat{R}$ are two independent $\text{BES}_0^{(\delta)}$ processes, each run until its last hit of 1.*

(ii) *the joint law of r and V is determined by the formula*

$$P(r \in dw, V \in dv) = c_\delta \, w_v^{\delta-4} P(r^{(\delta)} \in dw) \, dv \tag{7}$$

where $w \in C[0,1]$, $v \in (0,1)$, and

$$c_\delta = \frac{(\delta-2)^2}{\Gamma(\frac{\delta}{2})} 2^{-\frac{\delta}{2}} = \frac{\nu}{\Gamma(\nu)} 2^{1-\nu} \ \textit{with } \nu = (\delta-2)/2. \tag{8}$$

In this result the density factor in (7) reduces to 1 only if $\delta = 4$. Then condition (ii) simplifies to:

$$r \stackrel{d}{=} r^{(4)} \textit{ and } V \textit{ is independent of } r \textit{ with uniform distribution on } (0,1).$$

Theorems 4 and 5 are probabilistic equivalents of the following two theorems which express identities between σ-finite measures on appropriate spaces.

Theorem 6 [34, 19, 4, 22] *For each $\delta > 0$, on the space Ω_{ex} of continuous non-negative paths with a finite lifetime, starting and ending at 0, the same σ-finite measure Λ_δ is determined by either of the following two descriptions:*

Description I: Conditioning on the lifetime t: *First pick t according to the σ-finite density $2^{-\frac{\delta}{2}}\Gamma(\frac{\delta}{2})^{-1}t^{-\frac{\delta}{2}}dt$ on $(0,\infty)$; then given t, pick ω according to the distribution of a $\mathrm{BES}^{(\delta)}$ bridge from 0 to 0 over time t.*

Description II: Conditioning on the maximum m: *First pick m according to the σ-finite density $m^{1-\delta}dm$ on $(0,\infty)$; then given m, construct ω by joining back to back two independent $\mathrm{BES}_0^{(\delta)}$ processes, each run till it first hits m.*

For each $\epsilon \in (0,2)$ and $C > 0$, when the local time process of $\mathrm{BES}_0^{(\epsilon)}$ is normalized as occupation density relative to the speed measure $2Cx^{\epsilon-1}dx$ on $(0,\infty)$, Itô's law for excursions of $\mathrm{BES}_0^{(\epsilon)}$ away from 0 is $(2-\epsilon)^2 C\Lambda_{4-\epsilon}$.

In particular, for $\delta = 3$ the measure Λ_3 is Itô's law for excursions of $|B|$ away from zero for the local time process defined by occupation density of B at 0 relative to Lebesgue measure. Theorem 6 in this case was indicated by D. Williams [34, §II.67]. The extension to other dimensions δ was obtained in [19, 4, 22]. The last sentence of the theorem was indicated without attention to normalization constants in [19], and with an incorrect normalization constant (4 instead of 2) in [4, formula (3h)]. For $\delta \in (0,2] \cup [4,\infty)$ the measure Λ_δ is not an excursion law in the sense of Itô [11]. Nonetheless these measures have some interesting properties [19, 22]. Due to the Ray-Knight description of Brownian local times, the measure $4\Lambda_4$ is is the distribution of the square root of the total local time process of a path governed by the Brownian excursion law Λ_3. Consequently, Λ_4 appears also in the Lévy-Khintchine representation of the infinitely divisible family of squares of Bessel processes and Bessel bridges [19, 18]. As will be seen in Section 4, the simple form of Theorem 5 for $\delta = 4$ is also closely connected to the Ray-Knight description of Brownian local times.

The next theorem gives an alternative characterization of the measures Λ_δ for all $\delta > 2$ by generalizing a result of Bismut [5] for $\delta = 3$. The constant c_δ involved is the same as in (8).

Theorem 7 *For each $\delta > 2$ the same σ-finite measure M_δ on $\Omega_{\mathrm{ex}} \times (0,\infty)$ is determined by each of the following two ways of picking a point (ω, a) from $\Omega_{\mathrm{ex}} \times (0,\infty)$.*

Description I': Conditioning on the lifetime t of ω: *First pick t according to the σ-finite density $c_\delta t^{-\frac{\delta}{2}+1}dt$ on $(0,\infty)$; given t, pick ω from the distribution of a $\mathrm{BES}^{(\delta)}$ bridge from 0 to 0 over time t, pick u from the uniform probability distribution on $(0,t)$, independently of ω, and let $a = \omega(u)$.*

Description II': Conditioning on the level a: *First pick a according to the σ-finite density $2a^{3-\delta}da$ on $(0,\infty)$; then given a, construct ω by joining back to back two independent $\mathrm{BES}_0^{(\delta)}$ processes, each run till it last hits a.*

The marginal distribution of ω induced by M_δ has density $t(\omega)$ relative to $(\delta-2)^2\Lambda_\delta$, where $t(\omega)$ is the lifetime of the path ω.

For $\delta \in (2,4)$, Theorem 7 can be read from Theorem 6 by application to $\mathrm{BES}^{(4-\delta)}$ of a generalization of Bismut's result to an arbitrary recurrent strong Markov process, given in [17, §II]. In Section 4, Theorem 7 is deduced for all $\delta > 2$ by application of Theorem 5.

The rest of this paper is organized as follows. In Section 2 we briefly review the definition and basic properties of Bessel processes which underlie our study. Section 3 presents the proof of Theorem 2 followed by some variations of Knight's identity for one-dimensional Brownian motion. The splicing results of Theorems 5 and 7 are established in Section 4, followed in Section 5 by some corollaries for Bessel bridges. Section 6 presents another extension of Knight's identity, in which the reflecting Brownian motion $|B|$ is replaced by a recurrent $\mathrm{BES}^{(\delta)}$ process with dimension $\delta \in (0,2)$. For yet another extension of Knight's identity, involving the process $(|B_t| - \mu L_t, t \geq 0)$ for $\mu > 0$, see [36, Chapter 9] and [6].

2 Preliminaries on Bessel Processes.

The construction of $\mathrm{BES}_0^{(\delta)}$ as the radial part of a δ-dimensional Brownian motion for $\delta = 1, 2, 3\ldots$ makes evident the *Pythagorean property* of Bessel processes: for positive integers δ and ε, the sum of squares of independent $\mathrm{BES}^{(\delta)}$ and $\mathrm{BES}^{(\varepsilon)}$ processes is the square of a $\mathrm{BES}^{(\delta+\varepsilon)}$ process. As shown by Shiga-Watanabe [28], the family of $\mathrm{BES}^{(\delta)}$ processes for all real $\delta \geq 0$ can be constructed by extension of this Pythagorean property to all non-negative real δ and ε. See [19, 26, 22] for further background. Typical properties of Bessel processes are consequences of the Brownian representation for positive integer δ which have natural extensions to all $\delta > 0$. In particular, for each real $\delta > 0$ the $\mathrm{BES}_0^{(\delta)}$ process $R^{(\delta)}$ inherits the familiar *Brownian scaling property* from integer dimensions which underlies all the results of this paper: for every $c > 0$

$$(c^{-1/2}R_{ct}^{(\delta)}, t \geq 0) \stackrel{d}{=} (R_t^{(\delta)}, t \geq 0).$$

A *standard* $\mathrm{BES}^{(\delta)}$ *bridge*, denoted $r^{(\delta)}$, is a process

$$(r_u^{(\delta)}, 0 \leq u \leq 1) \stackrel{d}{=} (R_u^{(\delta)}, 0 \leq u \leq 1 | R_1^{(\delta)} = 0).$$

where $R^{(\delta)}$ is a $\mathrm{BES}_0^{(\delta)}$. Such a process is conveniently constructed as

$$r_u^{(\delta)} := (1-u)R_{u/(1-u)}^{(\delta)}, \qquad 0 \leq u < 1. \tag{9}$$

For an account of the basic properties of bridges derived from a nice Markov process such as $\mathrm{BES}^{(\delta)}$, see [7].

3 Results for one-dimensional Brownian motion

For a suitable real-valued path with either finite or infinite lifetime ζ, say $w = (w_t, 0 \leq t \leq \zeta)$, and a random time $T = T(w) \leq \zeta$, let $(L_T(w,x), x \in \mathbb{R})$ denote the process of local times of w at time T parameterized by the space variable x, as determined for all $x \in \mathbb{R}$ almost surely by the occupation density formula

$$\int_0^T f(w_s)ds = \int_{-\infty}^{\infty} f(x)L_T(w,x)dx$$

for all non-negative Borel functions f, and continuity in x. It is well known [26] that such a local time process exists for arbitrary $T(w) \leq \zeta$ and almost all w with respect to the laws of various processes under consideration here, such as fragments of Brownian motion, Brownian bridges, and Bessel processes. To illustrate the notation, for B a Brownian motion, the local time process of B at 0 is $(L_t, t \geq 0)$ defined by $L_t := L_t(B, 0)$.

3.1 Proof of Theorem 2.

Starting from the basic Brownian motion B and its inverse local time τ, it is easily checked that the pair of processes $(R, \hat{R})$ defined in Theorem 2 corresponds by Construction 3 to (r, V), where r is constructed from B via an intermediate process X, as in the statements of next two lemmas, and $V = T^{r,1}_{\sup}$. Also, $I_\tau / A_\tau = I_1(X)/A_1(X)$ as in Lemma 9. Theorem 2 then follows immediately by combination of the lemmas and the case $\delta = 3$ of Theorem 4.

Lemma 8 [3] *Let* $X(u) := B_{u\tau}/\sqrt{\tau}, 0 \le u \le 1$. *Then*

$$P(X \in d\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{L_1(\omega, 0)} P(B^{\mathrm{br}} \in d\omega), \qquad \omega \in C[0,1] \tag{10}$$

where B^{br} *is a standard Brownian bridge.*

For $\omega \in C[0,1]$ let

$$S_1(\omega) := \sup_{0 \le u \le 1} \omega_u; \quad I_1(\omega) = -\inf_{0 \le u \le 1} \omega_u; \quad A_1(\omega) = S_1(\omega) + I_1(\omega)$$

Lemma 9 *For a random element* X *of* $C[0,1]$ *let* $U := T^{X,1}_{\inf}$, *and define another random element* r *of* $C[0,1]$ *by*

$$r_t := X_{U+t(\mathrm{mod}\,1)} - X_U, \qquad 0 \le t \le 1.$$

If X *has the distribution* (10) *on* $C[0,1]$ *then* $I_1(X)/A_1(X)$ *and* r *are independent; the distribution of* $I_1(X)/A_1(X)$ *is uniform on* $[0,1]$, *while*

$$P(r \in d\omega) = \sqrt{2/\pi}\, S_1(\omega) P(r^{(3)} \in d\omega), \qquad \omega \in C[0,1]. \tag{11}$$

Proof. Note first from the construction of U and r that for any X with $X_0 = X_1 = 0$ there are the identities

$$I_1(X) = r_{1-U}; \quad A_1(X) = S_1(r); \quad L_1(X, 0) = L_1(r, r_{1-U}). \tag{12}$$

Let P govern X with distribution (10), and let P^{br} govern X as a standard Brownian bridge. As shown by Vervaat [31] and Biane [1], under P^{br} the random elements U and r are independent, with U uniform on $[0,1]$, and

r a standard BES$^{(3)}$ bridge. Let $c = \sqrt{2/\pi}$. Then for $w \in C[0,1]$ and $0 \le x \le S_1(w)$ we can use (12) to compute as follows:

$$\begin{aligned} P(I_1(X) \in dx, r \in dw) &= cP^{\mathrm{br}}(1/L_1(X,0); I_1(X) \in dx, r \in dw)) \\ &= cP^{\mathrm{br}}(1/L_1(w,x); r_{1-U} \in dx, r \in dw) \\ &= c(1/L_1(w,x))L_1(w,x)\, dx\, P^{\mathrm{br}}(r \in dw) \end{aligned}$$

where the last equality uses the fact that under P^{br} the variable $1-U$ has uniform distribution and is independent of r. It follows that for $x \ge 0$ and $w \in C[0,1]$

$$P(I_1(X) \in dx, r \in dw) = c1(0 < x < S_1(w))\, dx\, P^{\mathrm{br}}(r \in dw)$$

and the conclusions of the lemma are evident. $\square$

3.2 Some variations of Knight's identity.

Recall the well known formulae

$$P\exp\left(-\frac{\lambda^2}{2}T_x^{(1)}\right) = \frac{1}{\cosh(x\lambda)}; \quad P\exp\left(-\frac{\lambda^2}{2}T_x^{(3)}\right) = \frac{x\lambda}{\sinh(x\lambda)} \tag{13}$$

which are the particular cases $\delta = 1$ and $\delta = 3$ of the general formula for the Laplace transform of $T_x^{(\delta)}$ which appears later in equation (24). In view of the second formula in (13), Knight's identity (1) amounts to the formula

$$P\exp\left(-\frac{\lambda^2}{2}\frac{\tau}{M_\tau^2}\right) = \frac{2\lambda}{\sinh(2\lambda)} \qquad (\lambda > 0). \tag{14}$$

Use $\sinh(2\lambda) = 2(\cosh\lambda)(\sinh\lambda)$ and (13) to see that Knight's identity can be rewritten

$$\frac{\tau}{M_\tau^2} \overset{d}{=} T_1^{(1)} + T_1^{(3)} \tag{15}$$

where $T_1^{(1)}$ and $T_1^{(3)}$ are assumed independent. The fact that $T_2^{(3)} \overset{d}{=} T_1^{(1)} + T_1^{(3)}$ can be understood as follows:

$$T_2^{(3)} = (T_2^{(3)} - T_1^{(3)}) + T_1^{(3)}$$

where $T_2^{(3)} - T_1^{(3)}$ and $T_1^{(3)}$ are independent by the strong Markov property of $\mathrm{BES}_0^{(3)}$ at time $T_1^{(3)}$; that $T_2^{(3)} - T_1^{(3)} \stackrel{d}{=} T_1^{(1)}$ is implicit in the description of $\mathrm{BES}^{(3)}$ as a conditioned one-dimensional Brownian motion [15, 33].

Straightforward calculations based on Theorem 1 yield also the following variation of (15) and (3):

$$\frac{\tau}{S_\tau^2} = \frac{\tau}{A_\tau^2}\left(1 - \frac{I_\tau}{A_\tau}\right)^{-2} \stackrel{d}{=} T_1^{(3)} + \tau \tag{16}$$

where $T_1^{(3)}$ is assumed independent of τ, and τ has the same stable(1/2) distribution as both B_1^{-2} and the hitting time of 1 by B. Similarly, we find that

$$\frac{\tau}{(\min(S_\tau, I_\tau))^2} \stackrel{d}{=} 4(T_1^{(3)} + \tau). \tag{17}$$

The identities in law (1), (3), (16), and (17) exhibit some interesting features of the 4-dimensional random variable $(\tau^+, S_\tau, \tau^-, I_\tau)$, where $\tau^\pm = \int_0^\tau ds 1(B_s \in \mathbb{R}_\pm)$. We shall not attempt here the systematic description of this joint law, but see [26, p. 484, Ex. (4.24)] for some further results involving it. See also [22, 25, 23, 24] for the study of related laws of both heights and lengths of excursions of Brownian motion and Bessel processes.

4 Splicing of Bessel Bridges

The following lemma records a variation of the Ray-Knight description of the local time process of B in the space variable at the inverse local time τ:

Lemma 10 *The processes*

$$(L_\tau(B, S_\tau - v), 0 \le v \le S_\tau) \text{ and } (L_\tau(B, -I_\tau + v), 0 \le v \le I_\tau)$$

are independent copies of the square of $(R_v^{(4)}, 0 \le v \le \eta^{(4)})$ *where* $\eta^{(4)} := \sup\{t : R_t^{(4)} = 1\}$.

Proof. According to one of the Ray-Knight theorems, the processes

$$(L_\tau(B, x), x \ge 0) \text{ and } (L_\tau(B, -x), x \ge 0)$$

are two independent squares of Bessel processes of dimension 0, each started at 1, whose hitting times of 0 are S_τ and $-I_\tau$ respectively. The conclusion now follows by application of Williams' time-reversal theorem [33, 19]. $\Box$

On the other hand, we obtain the following corollary of Theorem 2. See also [19] for closely related appearances of the square of the standard $\mathrm{BES}^{(4)}$ bridge.

Corollary 11 *The process $(A_\tau^{-1} L_\tau(B, -I_\tau + uA_\tau), 0 \le u \le 1)$ is the square of a standard* $\mathrm{BES}^{(4)}$ *bridge; moreover this process is independent of the uniformly distributed random variable I_τ / A_τ.*

Proof. Let $B^{\#}$ be as constructed in Theorem 2, and let $B^*(v) := A_\tau^{-1} B^{\#}(vA_\tau^2)$ for $0 \le v \le \tau / A_\tau^2$. Then it is easily seen that for $0 \le u \le 1$

$$A_\tau^{-1} L_\tau(B, -I_\tau + uA_\tau) = A_\tau^{-1} L_\tau(B^{\#}, uA_\tau) = L(B^*, \tau/A_\tau^2, u) \qquad (18)$$

where on the right side, and in some following equations, we write $L(\omega, t, x)$ instead of $L_t(\omega, x)$. According to Theorem 2, the process B^* can be constructed by pasting back-to-back two independent copies of $(R_t^{(3)}, 0 \le t \le T_1^{(3)})$, and B^* is independent of I_τ / A_τ. It is implicit in Williams' path decompositions [32, 33] that the process $(L(R^{(3)}, T_1^{(3)}, u), 0 \le u \le 1)$ is the square of a standard $\mathrm{BES}^{(2)}$ bridge. Since the sum of squares of two independent standard $\mathrm{BES}^{(2)}$ bridges is the square of a standard $\mathrm{BES}^{(4)}$ bridge, the conclusion follows. $\Box$

The simplest case of Theorem 5, when $\delta = 4$, is now evident by comparison of the results of Lemma 10 and Corollary 11. The proof of Theorem 5 for general $\delta > 2$ is based on the known results stated in the following two lemmas.

For $\nu > 0$ let Γ_ν denote a random variable with the gamma(ν) density $\Gamma(\nu)^{-1} x^{\nu - 1} e^{-x}$ for $x > 0$.

Lemma 12 *For $\delta > 0$ let $\eta^{(\delta)} := \sup\{t : R_t^{(\delta)} = 1\}$.*

(i) [8] $\eta^{(\delta)} \stackrel{d}{=} 1/(2\Gamma_\nu)$ *where $\nu = (\delta - 2)/2$.*

(ii) [7] *Conditionally given $\eta^{(\delta)} = v$ the process $(R_t^{(\delta)}, 0 \le t \le v)$ is a* $\mathrm{BES}^{(\delta)}$ *bridge from $(0,0)$ to $(1, v)$.*

By the identity $r_v^{(\delta)} \stackrel{d}{=} \sqrt{2v(1-v)\Gamma_{\delta/2}}$ which follows from (9), for $z > 0$

$$P(r_v^{(\delta)} \in dz) = p_\delta(v,z)dz$$

where

$$p_\delta(v,z) := \frac{z^{2\nu+1}v^{-\nu-1}(1-v)^{-\nu-1}}{\Gamma(\nu+1)2^\nu} \exp\left(-\frac{z^2}{2v(1-v)}\right)$$

and $\nu = (\delta-2)/2$. The next lemma is an instance of Proposition 4 of [7]:

Lemma 13 [7] *Fix $\delta > 0$. For a process $r := (r_v, 0 \le v \le 1)$ with continuous paths and a random time V with values in $(0,1)$ the following conditions* (i) *and* (ii) *are equivalent:*
(i) *For $0 < v < 1$ and $z > 0$*

$$P(V \in dv, r_V \in dz) = \rho(v,z)\, dvdz$$

for some joint probability density function $\rho(v,z)$, and conditionally given $(V = v, r_V = z)$ the two processes $(r_u, 0 \le u \le v)$ and $(r_{1-u}, 0 \le u \le 1-v)$ are independent, with the first a $\mathrm{BES}^{(\delta)}$ bridge from $(0,0)$ to (v,z) and the second a $\mathrm{BES}^{(\delta)}$ bridge from $(0,0)$ to $(1-v,z)$.
(ii) *the joint law of r and V is given by the formula*

$$P(r \in dw, V \in dv) = f(v, w_v)P(r^{(\delta)} \in dw)\, dv \tag{19}$$

where $w \in C[0,1]$, $v \in (0,1)$, for some non-negative measurable function $f(v,z)$.

When these conditions hold, f and ρ are related by the formula

$$f(v,z) = \rho(v,z)p_\delta(v,z), \quad dv\, dz \text{ almost everywhere.}$$

Proof of Theorem 5. Because the transformation involved is a bijection, it suffices to show that (i) implies (ii). Suppose (i) holds. According to Lemma 12, conditionally on their lifetimes η and $\widehat{\eta}$ the processes R and $\widehat{R}$ are independent Bessel bridges from 0 to 1 with the given lengths. After the scaling operation to construct r the images of these processes are bridges of lengths η/ζ and $\widehat{\eta}/\zeta$ from 0 to $1/\sqrt{\zeta}$. Lemma 13 now yields the conclusion with $f_\delta(v,r_v)$ instead of $c_\delta r_v^{\delta-4}$ where $f_\delta(v,z)=\rho_\delta(v,z)/p_\delta(v,z)$ with $\rho_\delta(v,z)$ the joint density at (v,z) of

$$\left(\frac{\eta}{\eta+\widehat{\eta}},\frac{1}{\sqrt{\eta+\widehat{\eta}}}\right)=\left(\frac{\widehat{\Gamma}_\nu}{\Gamma_\nu+\widehat{\Gamma}_\nu},\left(\frac{1}{2\Gamma_\nu}+\frac{1}{2\widehat{\Gamma}_\nu}\right)^{-1/2}\right)$$

for Γ_ν and $\widehat{\Gamma}_\nu$ independent gamma(ν) variables with $\nu=(\delta-2)/2$. But elementary calculations show that $f_\delta(v,z)=c_\delta z^{\delta-4}$, and (ii) follows. □

Proof of Theorem 7. In Description II', given a denote by η_a and $\widehat{\eta}_a$ the last hitting times of a by the two independent $\mathrm{BES}_0^{(\delta)}$ processes. So by construction $a=\omega_v$ where $v:=\eta_a$, and the lifetime t of ω is $t=\eta_a+\widehat{\eta}_a$. Let $\overline{\omega}$ be the path ω standardized by Brownian scaling to have length 1. Note that $\overline{\omega}_v=a/\sqrt{t}$, and that (ω,a) is a measurable function of $(a,\overline{\omega},v)$. Description II' specifies the σ-finite marginal distribution $2a^{3-\delta}da$ for a, and given a a conditional probability distribution for $(\overline{\omega},v)$. By Brownian scaling and Theorem 5, this conditional probability distribution of $(\overline{\omega},v)$ given a does not depend on a, and is identical to the distribution of (r,V) described by formula (7). It follows from Fubini's theorem that for every non-negative measurable function $\Phi=\Phi(a,\overline{\omega},v)$, the integral of Φ with respect to the σ-finite joint distribution of $(a,\overline{\omega},v)$ determined by Description II' equals

$$P^{(\delta)}\int_0^\infty da\, 2a^{3-\delta}\, c_\delta r_U^{\delta-4}\Phi(a,r,U) \tag{20}$$

where $P^{(\delta)}$ denotes expectation with respect to a probability distribution which governs r as a standard $\mathrm{BES}^{(\delta)}$ bridge and U as an independent random variable with uniform distribution on $(0,1)$. The lifetime $t=t(\omega)$ is recovered from $(a,\overline{\omega},v)$ as $t=a^2/\overline{\omega}_v^2$. Apply (20) with $\Phi(a,\overline{\omega},v)=\Psi(a^2/\overline{\omega}_v^2,\overline{\omega},v)$, and make the change of variable $t=a^2/r_U^2$ in the integral, to deduce that for every non-negative measurable function $\Psi=\Psi(t,\overline{\omega},v)$, the integral of Ψ with respect to the distribution of $(t,\overline{\omega},v)$ induced by that of $(a,\overline{\omega},v)$ determined by Description II' equals

$$P^{(\delta)}\int_0^\infty dt\, c_\delta\, t^{1-\frac{\delta}{2}}\,\Psi(t,r,U). \tag{21}$$

But this is precisely the integral of $\Psi(t,\overline{\omega},v)$ with respect to the joint distribution of $(t,\overline{\omega},v)$ specified by Description I'. □

We now discuss further the correspondence between probability laws for $(R, \widehat{R})$ and for (r, V) induced by Construction 3. Instead of considering the distribution of (r, V) corresponding to R and $\widehat{R}$ which are independent copies of a $\mathrm{BES}_0^{(\delta)}$ run until its last hit of 1, we ask the following question: assuming that $\widehat{R}$ is an independent copy of R, how must R be distributed so that r is a standard $\mathrm{BES}^{(\delta)}$ bridge and V is independent of r? This question is answered by the following variation of Theorem 5, which coincides with that theorem for $\delta = 4$, but which is valid for all dimensions $\delta > 0$ rather than just $\delta > 2$.

Theorem 14 *For each $\delta > 0$ there is a unique distribution F_δ on $(0,1)$, and a unique distribution Q_δ for a process with finite lifetime, such that the following two conditions are equivalent:*

(i) *r is a standard $\mathrm{BES}^{(\delta)}$ bridge and V is independent of r with distribution F_δ;*

(ii) *R and $\widehat{R}$ are independent with common distribution Q_δ.*

The distribution F_δ is beta$(\delta/4, \delta/4)$; when R has distribution Q_δ the lifetime T of R is distributed like $(2\Gamma_{\delta/4})^{-1}$, and given T the process R is distributed like a $\mathrm{BES}^{(\delta)}$ bridge starting at $(0,0)$ and ending at $(T,1)$.

For the proof of this theorem, we introduce the following notation. For two random variables W and Y with $W > 0$ and $Y \geq 0$, call a process $(X_t, 0 \leq t \leq T)$ a $\mathrm{BES}_0^{(\delta)}(W, Y)$ bridge if $(T, X_T) \overset{d}{=} (W, Y)$ and given (T, X_T) the process X is distributed like a $\mathrm{BES}^{(\delta)}$ bridge starting at $(0,0)$ and ending at (T, X_T): that is, for all $v, y > 0$

$$(X_t, 0 \leq t \leq T \,|\, T = v, X_T = y) \overset{d}{=} (R_t^{(\delta)}, 0 \leq t \leq v \,|\, R_v^{(\delta)} = y)$$

For $\delta > 2$ let Q'_δ be the law of an unconditioned $\mathrm{BES}_0^{(\delta)}$ process R^δ stopped at its last hit of 1. According to Lemma 12,

$$Q'_\delta \text{ is the law of a } \mathrm{BES}_0^{(\delta)}((2\Gamma_{(\delta-2)/2})^{-1}, 1) \text{ bridge.}$$

The distribution Q_δ, defined in Theorem 14 for all $\delta > 0$ rather than just $\delta > 2$, is the law of a $\mathrm{BES}_0^{(\delta)}((2\Gamma_{\delta/4})^{-1}, 1)$ bridge. These distributions Q'_δ and Q_δ are mutually absolutely continuous for each $\delta > 2$. But they are identical only if $(\delta - 2)/2 = \delta/4$, that is $\delta = 4$. The following variation of Lemma 13 simplifies the proof of Theorem 14:

Lemma 15 *Fix $\delta > 0$. Suppose that r is a standard* $\mathrm{BES}^{(\delta)}$ *bridge and that $V \in (0,1)$ is independent of r. Then*

$$R \text{ is a } \mathrm{BES}_0^{(\delta)}(V/r_V^2, 1) \text{ bridge, } \widehat{R} \text{ is a } \mathrm{BES}_0^{(\delta)}((1-V)/r_V^2, 1) \text{ bridge,}$$

and these two processes are conditionally independent given (V, r_V).

Proof. This follows easily from the inhomogeneous Markov property of r, according to which for each fixed time $u \in (0,1)$, and $y \geq 0$, the two processes $(r_t, 0 \leq t \leq u)$ and $(r_{1-t}, 0 \leq t \leq 1-u)$ are conditionally independent given $r_u = y$, with the first process a $\mathrm{BES}_0^{(\delta)}(u, y)$ bridge and the second process a $\mathrm{BES}_0^{(\delta)}(1-u, y)$ bridge. See [7]. □

Proof of Theorem 14. For r and V as in Lemma 15, that lemma shows that R and $\widehat{R}$ are i.i.d.(independent and identically distributed), if and only if r_V^2/V and $r_V^2/(1-V)$ are i.i.d.. But from the representation (9),

$$(V, r_V^2) \stackrel{d}{=} (V, 2V(1-V)\Gamma_{\delta/2})$$

where $\Gamma_{\delta/2}$ is assumed independent of V. So r_V^2/V and $r_V^2/(1-V)$ are i.i.d. iff $(1-V)\Gamma_{\delta/2}$ and $V\Gamma_{\delta/2}$ are i.i.d.. It is well known and easily verified that this condition holds iff the distribution of V is beta$(\delta/4, \delta/4)$; then

$$r_V^2/V \stackrel{d}{=} r_V^2/(1-V) \stackrel{d}{=} 2V\Gamma_{\delta/2} \stackrel{d}{=} 2\Gamma_{\delta/4}$$

and the theorem follows. □

As a final variation on this theme, we record the following extension of Theorems 5 and 14.

Theorem 16 *Let $r^{(\delta)}$ be a standard* $\mathrm{BES}^{(\delta)}$ *bridge. For $\delta > 0$ and $a, b \in (0, \delta/2)$, the following conditions* (i) *and* (ii) *are equivalent:*

(i) *R and $\widehat{R}$ are independent, with R a* $\mathrm{BES}_0^{(\delta)}((2\Gamma_{\frac{\delta}{2}-a})^{-1}, 1)$ *bridge and $\widehat{R}$ a* $\mathrm{BES}_0^{(\delta)}((2\Gamma_{\frac{\delta}{2}-b})^{-1}, 1)$ *bridge;*

(ii) *the joint law of r and V is determined by the formula*

$$P(r \in dw, V \in dv) = c_{\delta,a,b}\, v^{a-1}(1-v)^{b-1} w_v^{\delta-2a-2b} P(r^{(\delta)} \in dw)\, dv \qquad (22)$$

where $w \in C[0,1]$, $v \in (0,1)$, and

$$c_{\delta,a,b} = 2^{a+b-\frac{\delta}{2}} \frac{\Gamma(a)\Gamma(b)\Gamma(\frac{\delta}{2})}{\Gamma(\frac{\delta}{2}-a)\Gamma(\frac{\delta}{2}-b)\Gamma(a+b)}.$$

Proof. This is obtained by the same method used to derive Theorem 5. Details are left to the reader. □

As a check on this theorem, for $\delta > 2$ and $a = b = 1$, in view of Lemma 12 we recover Theorem 5, while for $\delta > 0$ and $a = b = \delta/4$ we recover most of Theorem 14. See also [36, Section 3.7] for some related results.

5 Some identities in law for Bessel bridges

We indicate in this section some consequences for Bessel bridges of the splicing constructions considered in the previous section. Observe first, in the setting of Theorem 2, with notation from the proof of Corollary 11, that the lifetime τ/A_τ^2 of the process $(B^*(v), 0 \leq v \leq \tau/A_\tau^2)$ can be written as

$$\frac{\tau}{A_\tau^2} = \int_0^1 L(B^*, \tau/A_\tau^2, u)du.$$

But from (18)

$$L(B^*, \tau/A_\tau^2, I_\tau/A_\tau) = L(B, 0, \tau)/A_\tau = 1/A_\tau.$$

Thus τ can be recovered from the squared BES$^{(4)}$ bridge in (18) and the independent uniform random variable I_τ/A_τ as

$$\tau = \frac{1}{(L(B^*, \tau/A_\tau^2, I_\tau/A_\tau))^2} \int_0^1 L(B^*, \tau/A_\tau^2, u)du.$$

Similar considerations apply to the amounts of time τ_+ and τ_- that B spends in the intervals $(0, \infty)$ and $(-\infty, 0]$ respectively up to time τ. On the other hand, in the notation of Construction 3, we have

$$I_- := \frac{1}{r^4(V)} \int_0^V r^2(u)du = \int_0^\eta R^2(t)dt$$

$$I_+ := \frac{1}{r^4(V)} \int_V^1 r^2(u)du = \int_0^{\widehat{\eta}} \widehat{R}^2(t)dt$$

$$I := \frac{1}{r^4(V)} \int_0^1 r^2(u)du = I_- + I_+$$

Thus we deduce from Lemma 10 and Corollary 11 that for (r, V) and $(R, \widehat{R})$ as in Theorem 5 for $\delta = 4$, the random variables I_- and I_+ are independent with the same stable(1/2) distribution shared by τ_-, τ_+, $\tau/4$ and $1/(8\Gamma_{1/2})$, while I has the same distribution as τ and $1/(2\Gamma_{1/2})$. This is the case $p = 2$ of the following result:

Theorem 17 *Let r be a standard* $\mathrm{BES}^{(4)}$ *bridge, and V an independent random variable with uniform distribution on $(0,1)$. For $p > 0$ let*

$$J_-(p) := \int_0^V r_s^{2p-2} ds; \quad J_+(p) := \int_V^1 r_s^{2p-2} ds; \quad J(p) := \int_0^1 r_s^{2p-2} ds$$

Then

(i) *the random variables $p^2 J_-(p)/r_V^{2p}$ and $p^2 J_+(p)/r_V^{2p}$ are independent with the same distribution as $1/(2\Gamma_{1/p})$;*

(ii) *$J_-(p)/J(p)$ has a beta$(1/p, 1/p)$ distribution, and is independent of the random variable*

$$\frac{r_V^{2p} J(p)}{2p^2 J_-(p) J_+(p)}$$

which has a gamma$(2/p)$ distribution.

This result is obtained by combination of Theorem 5 with the following lemma:

Lemma 18 [26, p. 427, Prop. (1.11)] *Let $(R_\delta(t), t \geq 0)$ be a* $\mathrm{BES}^{(\delta)}$. *For all $\delta > 0$ and $p > 0$*

$$R_\delta^p(t) = \bar{R}_{\bar{\delta}}\left(p^2 \int_0^t ds R_\delta^{2p-2}(s)\right), \quad t \geq 0$$

where $\bar{\delta} = 2 + (\delta - 2)/p$ and $\bar{R}_{\bar{\delta}}$ is a $\mathrm{BES}^{(\bar{\delta})}$.

Proof of Theorem 17. In terms of R and $\widehat{R}$, two independent $\mathrm{BES}^{(\delta)}$ processes, with η and $\widehat{\eta}$ are their respective last hits of 1, from Theorem 5 for $\delta = 4$, Lemma 18 and Lemma 12, we find

$$\left(\frac{p^2 J_-(p)}{r_V^{2p}}, \frac{p^2 J_+(p)}{r_V^{2p}}\right) \stackrel{d}{=} \left(p^2 \int_0^\eta dt R_t^{2p-2}, p^2 \int_0^{\widehat{\eta}} dt \widehat{R}_t^{2p-2},\right) \stackrel{d}{=} \left(\frac{1}{2\Gamma_{1/p}}, \frac{1}{2\widehat{\Gamma}_{1/p}}\right)$$

where $\Gamma_{1/p}$ and $\widehat{\Gamma}_{1/p}$ are two independent gamma$(1/p)$ variables. This is (i), and (ii) follows by the elementary relations between beta and gamma variables. $\square$

The simplicity of this result should be compared with the complexity of the law of $J(p)$. See [12] and references therein for an approach to the law of $J(p)$ via Sturm-Liouville equations.

The following further corollary, whose proof is left to the reader, is a consequence of Theorem 5 and Lemma 18. In two particular cases, if $\delta = 4$ or $\varepsilon = 4$ we recover some instances of [26, p. 444, Theorem (3.5)].

Proposition 19 *Let $P^{(\delta)}$ be an expectation operator governing r as a standard* BES$^{(\delta)}$ *bridge, and let c_δ be as defined by* (8)*. Suppose $\delta > 2, p > 0$, and let $\varepsilon := 2 + (\delta - 2)/p$. Then for every non-negative measurable function f there is the equality*

$$c_\delta P^{(\delta)} \left[f \left(\int_0^1 du r_u^{2p-2} \right) \left(\int_0^1 du r_u^{\delta - 4} \right) \right]$$
$$= c_\varepsilon P^{(\varepsilon)} \left[f \left(\left(\int_0^1 du (p r_u)^{\frac{2}{p} - 2} \right)^{-p} \right) \left(\int_0^1 du r_u^{\varepsilon - 4} \right) \right].$$

6 An analog of Knight's identity for a recurrent Bessel process

Another generalization of Knight's identity is obtained by replacing $|B|$ by a BES$_0^{(\delta)}$ process $R := R^{(\delta)}$, for some $\delta \in (0, 2)$, when 0 is a recurrent point for R. Let $M_t := \sup_{0 \le s \le t} R_s$, let $(L_t, t \ge 0)$ be a local time process at 0 for R, and for $s \ge 0$ set $\tau_s = \inf\{t : L_t > s\}$. Note that while the definition of these processes (L_t) and (τ_s) in terms of R involves δ, this dependence is suppressed in the notation. It is known [16] that (τ_s) is a stable subordinator of index $\alpha := 1 - \frac{\delta}{2}$, that is

$$P^{(\delta)} \exp(-\lambda \tau_s) = \exp(-K s \lambda^\alpha) \qquad (\lambda > 0) \qquad (23)$$

where $P^{(\delta)}$ is an expectation operator governing R as a BES$_0^{(\delta)}$, and K is a constant depending on the normalization of the local time process. By scaling, the law of $\tau_s / M_{\tau_s}^2$ depends neither on s, nor on the choice of normalization of local time. So we now write simply τ instead of τ_s for some arbitrary fixed δ and s.

Theorem 20 *For $\delta \in (0,2)$ the distribution of τ/M_τ^2 for a $\mathrm{BES}_0^{(\delta)}$ process is identical to that of $T_1^{(\delta)} + T_1^{(4-\delta)}$ where $T_1^{(\delta)}$ and $T_1^{(4-\delta)}$ are independent.*

For all $\delta > 0$ there is the formula [13]

$$P^{(\delta)} \exp(-\tfrac{1}{2}\lambda^2 T_x^{(\delta)}) = \frac{(x\lambda)^\nu}{2^\nu \Gamma(\nu+1) I_\nu(x\lambda)} \quad \text{where } \nu := (\delta-2)/2 \tag{24}$$

and I_ν is the modified Bessel function of index ν. So Theorem 20 amounts to the formula

$$P^{(\delta)} \exp\left(-\frac{\lambda^2}{2}\frac{\tau}{M_\tau^2}\right) = \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{I_\alpha(\lambda)I_{-\alpha}(\lambda)} \quad \text{for } \alpha := (2-\delta)/2 \in (0,1). \tag{25}$$

We offer the following pathwise explanation of Theorem 20, in the same spirit as Theorem 2. See also [20, Sec. 6] for a similar construction. Let $\tilde{\tau} := \tau/M_\tau^2$ and

$$\tilde{R}_t := \frac{1}{M_\tau} R(tM_\tau^2), \;\; 0 \le t \le \tilde{\tau}, \tag{26}$$

so the process $(\tilde{R}_t, 0 \le t \le \tilde{\tau})$ begins and ends at 0, and has maximum value 1 at time $\tilde{\rho} := \rho/M_\tau^2$ where ρ is the a.s. unique time u in $(0,\tau)$ at which $R_u = M_\tau$. Now define a rearrangement $R^\#$ of the path of $\tilde{R}$, as follows: delete the excursion of $\tilde{R}$ straddling time $\tilde{\rho}$, close up the gap, and replace the deleted excursion at the end. Let

$$\tilde{D} := \text{ (first zero of } \tilde{R} \text{ after time } \tilde{\rho}) = D_\rho/M_\tau^2 \tag{27}$$

where D_ρ is the first zero of R after time ρ.

Theorem 21 *For R a $\mathrm{BES}_0^{(\delta)}$ with $\delta \in (0,2)$ the two processes*

$$(R^\#(t), 0 \le t \le \tilde{\tau} - (\tilde{D} - \tilde{\rho})) \text{ and } (R^\#(\tilde{\tau} - v), 0 \le t \le \tilde{D} - \tilde{\rho})$$

are independent; the first is distributed as a $\mathrm{BES}_0^{(\delta)}$ up to its hitting time of 1, and the second is distributed as a $\mathrm{BES}_0^{(4-\delta)}$ up to its hitting time of 1.

As a consequence of Theorem 21, for (τ, M_τ) derived from a $\mathrm{BES}_0^{(\delta)}$ process, the decomposition

$$\frac{\tau}{M_\tau^2} = \frac{\tau - (D_\rho - \rho)}{M_\tau^2} + \frac{(D_\rho - \rho)}{M_\tau^2}$$

expresses τ/M_τ^2 as the sum of two independent random variables, distributed like $T_1^{(\delta)}$ and $T_1^{(4-\delta)}$ respectively. Thus Theorem 20 is a consequence of Theorem 21. In the Brownian case ($\delta = 1$), this derivation of Knight's identity in the form (15) simplifies the closely related approaches of Vallois [29] and Biane[2].

Proof of Theorem 21. The following observations 1)-4) are consequences of Itô's excursion theory [11, 27] which are valid for any recurrent diffusion process R starting at 0 instead of a $\mathrm{BES}_0^{(\delta)}$ process R, with $\tau := \inf\{t : L_t > 1\}$ where L is a local time process of R at 0, with M the past maximum process of R, and $\rho \in (0, \tau)$ defined by $M_\tau = R_\rho$. The excursion interval of R containing ρ is denoted (G_ρ, D_ρ).

1) For each $x > 0$, conditionally given $M_\tau = x$, the excursion of R over (G_ρ, D_ρ) is distributed according to Itô's excursion law given an excursion of maximum height x, independently of the *residual process* with lifetime $\tau - (D_\rho - G_\rho)$ obtained from the process R on $[0, \tau]$ by excision of the excursion over (G_ρ, D_ρ).

2) Conditionally given $M_\tau = x$, the residual process is identical in law to

$$(R_t, 0 \leq t \leq G_{T_x} | L_{T_x} = 1),$$

where G_{T_x} is the time of the last zero of R before T_x, the first hitting time of x by R.

3) Conditionally given $M_\tau = x$, the excursion of height x over (G_ρ, D_ρ) may be decomposed at its maximum into two independent copies of $\hat{R}$ run till time $\hat{T}_x$ and joined back to back, where $\hat{T}_x$ is the hitting time of x by the diffusion $\hat{R}$ started at 0 obtained as "R conditioned to reach $+\infty$ before returning to 0" in the usual sense of h-processes. (Williams decomposition [33, 34]. For R a $\mathrm{BES}^{(\delta)}$, it is known [19] that $\hat{R}$ is a $\mathrm{BES}^{(4-\delta)}$.)

By combination of 1), 2) and 3) with the last exit decomposition of R at time G_{T_x}, which has a similar expression in terms of excursion theory, it is clear that

4) if the excursion attaining the maximum of R on $[0, \tau]$ is removed and tacked after the residual process, conditionally given $M_\tau = x$, this

rearranged process of lifetime τ decomposes at its maximum (at time $\tau - (D_\rho - \rho)$) into two independent processes, the first a copy of

$$(R_t, 0 \leq t \leq T_x | L_{T_x} = 1)$$

and the second a time-reversed copy of $(\hat{R}_t, 0 \leq t \leq \hat{T}_x)$.

Assuming now that R is a $\mathrm{BES}_0^{(\delta)}$ for some $\delta \in (0,2)$, the two processes considered in Theorem 21 are obtained from the two processes considered above by application of Brownian scaling to obtain processes with maximum value 1 instead of M_τ. Combined with the Brownian scaling property of R, the above argument shows

$$(R^{\#}(t), 0 \leq t \leq \tilde{\tau} - (\tilde{D} - \tilde{G}) | M_\tau = x) \stackrel{d}{=} (R_t; 0 \leq t \leq T_1 | L_{T_1} = x^{-2\alpha})$$

where L_{T_1} is the local time at 0 of R, a $\mathrm{BES}_0^{(\delta)}$, up to the time $T_1 = T_1^{(\delta)}$ that R first hits 1. Similarly

$$(R^{\#}(\tilde{\tau} - v), 0 \leq t \leq \tilde{D} - \tilde{\rho} | M_\tau = x) \stackrel{d}{=} (\hat{R}_t, 0 \leq t \leq \hat{T}_1).$$

where $\hat{T}_1 \stackrel{d}{=} T_1^{(4-\delta)}$ is the hitting time of 1 by the $\mathrm{BES}_0^{(4-\delta)}$ process $\hat{R}$. Since the distribution of the last path is independent of the value x of M_τ, the independence claimed in the theorem is immediate. To finish the argument, it only remains to check that the following relation holds for R a $\mathrm{BES}^{(\delta)}$:

$$M_\tau^{-2\alpha} \stackrel{d}{=} L_{T_1}, \text{ where } \alpha = (2-\delta)/2. \tag{28}$$

By Theorem 6, for $\mathrm{BES}^{(\delta)}$ the rate of excursions to hit x is $cx^{-2\alpha}$ for some c depending on the choice of normalization of local time, so L_{T_1} is exponential(c):

$$P(L_{T_1} > \ell) = e^{-c\ell}. \tag{29}$$

But also

$$P(M_\tau^{-2\alpha} > \ell) = P(M_\tau < \ell^{-1/2\alpha}) = e^{-c\ell} \tag{30}$$

where the second equality is a consequence of Itô's excursion theory: in a Poisson process with rate $c\ell^{-(1/2\alpha)(-2\alpha)} = c\ell$, the probability of no points in time 1 is $e^{-c\ell}$. □

Remark. Consider again for R a $\mathrm{BES}_0^{(\delta)}$ with $\delta \in (0,2)$ the process $\tilde{R}$ with lifetime $\tilde{\tau} = \tau/M_\tau^2$ derived by Brownian scaling R on $[0,\tau]$ to have maximum height 1, as in (26). It is natural to consider the decomposition of $\tilde{R}$ at its maximum time $\tilde{\rho} = \rho/M_\tau^2$. By symmetry under time reversal, $\tilde{R}$ decomposes at time $\tilde{\rho}$ into an exchangeable pair of processes, say Y with lifetime $\tilde{\rho}$ and $\hat{Y}$ with lifetime $\tilde{\tau} - \tilde{\rho}$, put back to back. A variation of the above argument identifies the common law of Y and $\hat{Y}$, and shows they are not independent. To see this, let $L := \tilde{L}_{\tilde{\rho}}$, the total local time at 0 of Y, $\hat{L} = \tilde{L}_{\tilde{\tau}} - L$, the total local time at 0 of $\hat{Y}$. A variation of the above argument shows that

$$\begin{aligned}(Y_t, 0 \le t \le \tilde{\rho} | L = \ell) &\stackrel{d}{=} (\hat{Y}_t, 0 \le t \le \tilde{\tau} - \tilde{\rho} | \hat{L} = \ell) \\ &\stackrel{d}{=} (R_t, 0 \le t \le T_1 | L_{T_1} = \ell).\end{aligned}$$

Moreover, Y and $\hat{Y}$ are conditionally independent given $(L, \hat{L})$. But it is easily seen that

$$(L, \hat{L}) \stackrel{d}{=} (UL_{T_1}, (1-U)L_{T_1}), \tag{31}$$

where U is uniform $[0,1]$ independent of L_{T_1}, which has exponential(c) distribution for some $c > 0$. Let $(R'_t, 0 \le t \le T'_1)$ be an independent copy of $(R_t, 0 \le t \le T_1)$. It follows easily that the law of $(Y, \hat{Y})$ is absolutely continuous with respect to that of $((R_t, 0 \le t \le T_1), (R'_t, 0 \le t \le T'_1))$, with density $c^{-1}(L_{T_1} + L'_{T'_1})^{-1}$ where $L'_{T'_1}$ is the local time of R' at 0 at time T'_1.

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Jim Pitman
Department of Statistics
University of California
367 Evans Hall # 3860
Berkeley, CA 94720-3860
U.S.A.

Marc Yor
Laboratoire de Probabilités
Université Pierre et Marie Curie
4, Place Jussieu - Tour 56
75252 Paris Cedex 05
France