Transforming spatial point processes into Poisson processes

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Abstract

In 1986, Merzbach and Nualart demonstrated a method of transforming a two-parameter point process into a planar Poisson process of unit rate, using random stopping sets. Merzbach and Nualart's theorem applies only to a special class of point processes, since it requires two restrictive conditions: the (F4) condition of conditional independence and the convexity of the 1-compensator. The (F4) condition was removed in 1990 by Nair, but the convexity condition remained. Here both the (F4) condition and the convexity condition are removed by making use of predictable sets rather than stopping sets. As with Nair's theorem, the result extends to point processes in higher dimensions.

1 Introduction.

Suppose N is a point process. Is it possible to rescale the domain in such a way that N is transformed into a Poisson process with rate 1?

When N is a simple point process on the line, the question is answered by Papangelou (1972) and by Brémaud (1972), using the characterization of the Poisson process of Watanabe (1964). Provided its compensator is continuous, any such N can be transformed into a unit rate Poisson process via random stopping

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times.

Now suppose that N is a multivariate point process, i.e. a countable sequence of point processes on the line. Meyer (1971) shows that provided an orthogonality condition is satisfied, N can be transformed into a sequence of independent unit rate Poisson processes on the line. This result is also proven by Aalen and Hoem (1978) for the self-exciting case, and an elegant proof by Brown and Nair (1988) generalizes Meyer's result to include a wide class of multivariate point processes.

When N is a point process on the plane, the situation is more complex. Merzbach and Nualart (1986) show that N can be transformed into a unit rate Poisson process using random stopping sets, provided several conditions are met. For instance, N must satisfy the conditional independence condition (F4) of Cairoli and Walsh (1975). The need for this condition is removed by Nair (1990). However, both Merzbach and Nualart (1986) and Nair (1990) assume the convexity of the 1-compensator. This convexity condition is rather stringent; for example self-exciting point processes generally do not satisfy this condition.

The current paper investigates transformations based on $\mathcal{F}^1$-predictable sets rather than stopping sets and eliminates the need for the conditions mentioned above. However, the existence of the $\mathcal{F}^1$- and $\mathcal{F}^2$-intensities is assumed, which is also required by Merzbach and Nualart (1986) but not by Nair (1990). As with the result of Nair (1990), the result extends to the case where N is a point process in $\mathbf{R}^k$, for $k > 2$.

The next section introduces planar point processes, predictable sets, stopping sets, and some related concepts. In section 3 the result on transforming planar point processes to Poisson processes is presented, and a brief comparison of the use of stopping sets and predictable sets is given. An example is provided in section 4. In section 5, the result is extended to point processes in higher dimensions.

2 Preliminaries

First, some notation. In what follows, z, z', and z'' represent elements of $\mathbf{R}^2_+$, the positive quadrant of the plane, while s, s', t, t', t'', x and y denote elements of $\mathbf{R}_+$.

Before defining planar point processes, some ordering of points in the plane is required. For $z = (s,t)$ and $z' = (s', t') \in \mathbf{R}^2_+$, say $z < z'$ if $s < s'$ and $t < t'$. Similarly, $z \leq z'$ if $s \leq s'$ and $t \leq t'$. Let $(z, z']$ denote the rectangular region in $\mathbf{R}^2_+$ consisting of all points greater than z and less than or equal to z'; i.e. $(z, z'] = \{z'' : z < z'' \leq z'\}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A *filtration* $\mathcal{F}(z)$ is a collection of sub-σ-fields of $\mathcal{F}$ which is increasing (i.e. $\mathcal{F}(z) \subseteq \mathcal{F}(z')$ for $z \leq z'$), right continuous (i.e. $\mathcal{F}(z) = \bigcap_{z'>z} \mathcal{F}(z')$) and complete (i.e. each $\mathcal{F}(z)$ contains the null sets of $\mathcal{F}$).

For $z = (s,t)$, define $\mathcal{F}^1(z)$ as $\bigcup_{t'\geq 0} \mathcal{F}(s,t')$. Similarly, $\mathcal{F}^2(z) = \bigcup_{s'\geq 0} \mathcal{F}(s',t)$. The focus of most of this paper is on properties related to the filtration $\mathcal{F}^1$; the definitions and results related to $\mathcal{F}^2$ are analogous.

The σ-field $\mathcal{P}^1$ generated by sets of the form $F \times (z,z']$, for $z \leq z'$ and $F \in \mathcal{F}^1(z)$, is called the *$\mathcal{F}^1$-predictable σ-field*, and an element of $\mathcal{P}^1$ is an *$\mathcal{F}^1$-predictable set*. A process X on $\mathbf{R}_+^2$ is called *$\mathcal{F}^1$-predictable* if it is $\mathcal{P}^1$-measurable; i.e. if $\{(\omega,z) : \omega \in \Omega, z \in \mathbf{R}_+^2, X(\omega,z) \in B\} \in \mathcal{P}^1$, for any Borel set $B \in \mathbf{R}_+^2$.

A mapping D from Ω to the closed subsets of $\mathbf{R}_+^2$ is called an *$\mathcal{F}^1$-stopping set* provided $z' \in D(\omega)$ implies $z \in D(\omega)$ for all $z < z'$, and $\{\omega : z \in D(\omega)\} \in \mathcal{F}^1(z)$ for all $z \in \mathbf{R}_+^2$.

A process $N(z)$ on $\mathbf{R}_+^2$ is *increasing* if $N(z,z'] \geq 0$ for every $z < z'$, and $N(s,0) = N(0,t) = 0$ for every s,$t \geq 0$. If an adapted process $N(z)$ on $\mathbf{R}_+^2$ taking values in $\mathbf{N} \cup \{\infty\}$ is right-continuous and increasing, then N is a *point process*.

For X a process on $\mathbf{R}_+^2$ and B a Borel set in $\mathbf{R}_+^2$, let $X(B)$ denote $\int 1_B(z)dX(z)$, provided the integral exists. In particular, if $z = (s,t) < (s',t') = z'$, then $X(z,z']$ can be written as

$$X(z') - X(s,t') - X(s',t) + X(z). \tag{2.1}$$

A point process is called *simple* if all its jumps are of size 1, i.e. if $\lim_{\Delta \downarrow 0} N((s-\Delta, t-\Delta), (s,t)] = 0$ or 1, for each s,$t \in \mathbf{R}_+$. A *Poisson process* on $\mathbf{R}_+^2$ is a simple point process N where for any disjoint Borel sets $B_1, ..., B_n$ in $\mathbf{R}_+^2$, $N(B_1)$, ..., $N(B_n)$ are independent Poisson random variables. If the mean of N satisfies $EN(B) = \mu(B)$ for any Borel set B, where μ denotes Lebesgue measure on $\mathbf{R}_+^2$, then N is said to have *unit rate*.

A *1-martingale* is an integrable $\mathcal{F}^1$-adapted process X where for each $z \leq z'$, $E[X(z,z']|\mathcal{F}^1(z)] = 0$. If N is an $\mathcal{F}^1$-adapted point process, then a *1-compensator* A of N is an increasing $\mathcal{F}^1$-predictable process so that $N - A$ is a 1-martingale. The existence and uniqueness of A for simple, integrable N are proven by Jacod (1975).

Suppose that A is the 1-compensator of N and that there exists an integrable, non-negative, real-valued, $\mathcal{F}^1$-predictable process λ such that with probability 1, for each $z \in \mathbf{R}_+^2$,

$$\int_{z'<z} \lambda(z')d\mu(z') = A(z), \tag{2.2}$$

where μ denotes Legesbue measure. Then λ is called an $\mathcal{F}^1$*-intensity* of N.

3 Transformation of Planar Processes

This section contains a result on transformations changing point processes on $\mathbf{R}_+^2$ into Poisson processes. First, recall the following lemma of Nair (1990), which generalizes a similar result in Brown, Ivanoff and Weber (1986) for 1-parameter point processes.

Lemma 3.1. *If N is an $\mathcal{F}^1$-adapted point process on $\mathbf{R}_+^2$ with deterministic, continuous 1-compensator A and at most one point on any vertical line, then N is a Poisson process whose mean corresponds to A.*

Theorem 3.2. *Suppose N is a simple $\mathcal{F}$-adapted point process on $\mathbf{R}_+^2$ with $\mathcal{F}^1$-intensity λ_1 and $\mathcal{F}^2$-intensity λ_2. If with probability one, $\int\limits_0^\infty \lambda_1(\omega, s, t)dt = \infty$ for all $s \in \mathbf{R}_+$, then there is a sequence of $\mathcal{F}^1$-predictable sets D_z such that $M(z) := \int \mathbf{1}_{D_z} dN$ is a Poisson process on $\mathbf{R}_+^2$ with unit rate.*

Proof. Define a process τ_z on $\mathbf{R}_+$ as follows. Fix $z = (s,t) \in \mathbf{R}_+^2$. For $s' \leq s$, let

$$\tau_z(\omega, s') = \inf \left\{ t' : \int\limits_0^{t'} \lambda_1(\omega, s', t'')dt'' > t \right\}, \tag{3.1}$$

with the convention that $inf\{\emptyset\} = \infty$. For $s' > s$, let $\tau_z(\omega, s') = 0$.

By assumption, for any $s' \in R_+$, $\int\limits_0^\infty \lambda_1(\omega, s', t'')dt'' = \infty$ a.s., so $\tau_z(\omega, s') < \infty$ a.s. In addition, observe that by definition of $\tau_z(s')$, for almost all $\omega \in \Omega$,

$$\int\limits_0^{\tau_z(\omega,s')} \lambda_1(\omega, s', t')dt' \geq t,$$

and since $\lambda_1 < \infty$, in fact equality holds, i.e.

$$\int\limits_0^{\tau_z(\omega,s')} \lambda_1(\omega, s', t')dt' = t. \tag{3.2}$$

Let $M(z) = \int \mathbf{1}_{D_z} dN$, where D_z is the random closed region bounded by τ_z and the axes. That is,

$$D_z = \left\{ (\omega, s', t') \in \Omega \times \mathbf{R}_+^2 : s' \leq s, \int\limits_0^{t'} \lambda_1(\omega, s', t'')dt'' \leq t \right\} \tag{3.3}$$

which may be abbreviated as

$$\{s' \leq s\} \cap \left\{ \int_0^{t'} \lambda_1(s', t'')dt'' \leq t \right\} \tag{3.4}$$

To see that D_z is $\mathcal{F}^1$-predictable, recall the elementary fact that if any process X_s is $\mathcal{G}$-measurable, for $0 \leq s \leq t$, then the integral $\int_0^t X_s ds$ is also $\mathcal{G}$-measurable (see e.g. Appendix A3.3 of Daley and Vere-Jones (1988)). From this and the fact that λ_1 is $\mathcal{P}^1$-measurable, it follows that $\left\{ \int_0^{t'} \lambda_1(s', t'')dt'' \leq t \right\} \in \mathcal{P}^1$. The set $\{s' \leq s\}$ is also trivially $\mathcal{F}^1$-predictable (i.e. $\mathcal{P}^1$-measurable). Therefore $D_z \in \mathcal{P}^1$.

In order to show that M is a Poisson process, it is first necessary to show that M is a well-defined point process. For each $\omega \in \Omega$, since $\mathbf{1}_{D_z}$ is nonnegative, the ordinary Lebesgue integral $\int \mathbf{1}_{D_z} dN = \int_s \int_t \mathbf{1}_{D_z}(s,t)dN(s,t)$ is clearly well-defined. Therefore the random measure M is well-defined.

Moreover, M is $\mathcal{F}^1$-adapted. Since D_z is an $\mathcal{F}^1$-predictable set, the indicator $\mathbf{1}_{D_z}$ is an $\mathcal{F}^1$-predictable (and hence $\mathcal{F}^1$-adapted) process. Since $\mathbf{1}_{D_z}$ and N are both $\mathcal{F}^1$-adapted, so is their product. Therefore the integral $\int_s \int_t \mathbf{1}_{D_z}(s,t)dN(s,t) \in \mathcal{F}^1$.

In order to be a point process, M must furthermore take values in $\mathbf{Z} \cup \{\infty\}$, be increasing, and be right-continuous. It is clear from the definition of M that M inherits from N the property of taking values in $Z^+ \cup \{\infty\}$.

To show that M is increasing, recall from equation (??) that for $(x, y) \leq (x', y') \in \mathbf{R}^2_+$,

$$\begin{aligned} M((x,y),(x',y')] &= M(x',y') - M(x,y') - M(x',y) + M(x,y) \\ &= \int \mathbf{1}_{D(x',y')} dN - \int \mathbf{1}_{D(x,y')} dN - \int \mathbf{1}_{D(x',y)} dN + \int \mathbf{1}_{D(x,y)} dN \\ &= \int \mathbf{1}_{D(x',y') \setminus D(x,y')} dN - \int \mathbf{1}_{D(x',y) \setminus D(x,y)} dN, \end{aligned} \tag{3.5}$$

the last equation following from the fact that $D_{(x,y')} \subseteq D_{(x',y')}$ and $D_{(x,y)} \subseteq D_{(x',y)}$, as is evident from the definition of D_z in (??) or (??). Further, since $y \leq y'$ and

$$D_{(x',y')} \setminus D_{(x,y')} = \{x < s' \leq x'\} \cap \left\{ \int_0^{t'} \lambda_1(s', t'')dt'' \leq y' \right\}$$

and

$$D_{(x',y)} \setminus D_{(x,y)} = \{x < s' \leq x'\} \cap \left\{ \int_0^{t'} \lambda_1(s', t'')dt'' \leq y \right\},$$

one sees that $D_{(x',y')} \setminus D_{(x,y')} \supseteq D_{(x',y)} \setminus D_{(x,y)}$. From the fact that the point process N is nonnegative, it follows that

$$\int \mathbf{1}_{D(x',y')\setminus D(x,y')} dN \quad \geq \quad \int \mathbf{1}_{D(x',y)\setminus D(x,y)} dN$$

which with (??) establishes that $M((x,y),(x',y')] \geq 0$. In order for M to be increasing, M must also satisfy:

$$M(s,0) = M(0,t) = 0 \tag{3.6}$$

for every $s,t \geq 0$. N is a point process, so (??) holds with M replaced by N. Since $M = \int 1_{D_z} dN$, (??) clearly holds for M as well.

In order to establish that M is a point process, only the right-continuity of M remains to be verified. Fix any $\omega \in \Omega$. From the definition of τ in (??) and the fact that $\lambda_1 < \infty$, it follows that for any $x \in \mathbf{R}_+$, $\tau_{z'}(\omega, x) \downarrow \tau_z(\omega, x)$ as $z' \downarrow z$. Consequently, for any $(x,y) \in \mathbf{R}_+^2$, $\mathbf{1}_{D_{z'}}(\omega, x, y) \downarrow \mathbf{1}_{D_z}(\omega, x, y)$ as $z' \downarrow z$, and the right-continuity of M follows by monotone convergence. Thus M is a well-defined, $\mathcal{F}^1$-adapted point process.

Using Lemma 2.1 of Nair (1990) and the assumption that the $\mathcal{F}^2$-intensity of N exists, N contains at most one point on any vertical line a.s. This is true also of M, since the definition of M implies that N has a point at (s,t) if and only if M has a point at $(s, \int_0^t \lambda_1(t')dt')$.

For $z = (s,t)$, Let $C(z) = \int 1_{D_z} dA$, where A is the 1-compensator of N, i.e. $A(z) = \int_0^s \int_0^t \lambda_1(s',t')dt'ds'$. In other words, $C(z) = \int_0^s \int_0^{\tau_z(s)} \lambda_1(s',t')dt'ds'$. Recall from (??) that for all $s' \leq s$, $\int_0^{\tau_z(s')} \lambda_1(s',t')dt' = t$, so $C(z) = \int_0^s t ds' = st$. Thus, C is the 1-compensator of the unit rate Poisson process.

For any $z = (s,t) \leq (s',t') = z'$, the random set $(D_z, D_z']$ defined as $D_{z'} \setminus \{D_{(s,t')} \cup D_{(s',t)}\}$ is $\mathcal{F}^1$-predictable, since it is the symmetric difference of $\mathcal{F}^1$-predictable sets. Further, for any $F \in \mathcal{F}^1(z)$, since $(F \times \mathbf{R}_+^2)$ is also $\mathcal{F}^1$-predictable, the set $(F \times \mathbf{R}_+^2) \cap (D_z, D_{z'}]$ is $\mathcal{F}^1$-predictable as well. Since the process $1_{(F\times\mathbf{R}_+^2)\cap(D_z,D_{z'}]}$ is thus $\mathcal{F}^1$-predictable and is clearly bounded and nonnegative, one may apply the martingale property (see e.g. equation (1) of Nair (1990)) to $N - A$, and obtain:

$$E[\int \mathbf{1}_{(F\times\mathbf{R}_+^2)\cap(D_z,D_{z'}]} dN] = E[\int \mathbf{1}_{(F\times\mathbf{R}_+^2)\cap(D_z,D_{z'}]} dA].$$

Since the above holds for all $F \in \mathcal{F}^1(z)$, it follows that

$$E[\int \mathbf{1}_{(D_z,D_{z'}]} dN | \mathcal{F}^1(z)] = E[\int \mathbf{1}_{(D_z,D_{z'}]} dA | \mathcal{F}^1(z)].$$

Therefore

$$E[M(z,z'] - C(z,z'] | \mathcal{F}^1(z)] = E[\int \mathbf{1}_{(D_z,D_{z'}]} dN - \int \mathbf{1}_{(D_z,D_{z'}]} dA | \mathcal{F}^1(z)] = 0.$$

Thus $M - C$ is a 1-martingale; i.e. C is the 1-compensator of M. From Lemma 3.1, the proof is complete. $\square$

Remark 3.3. The relation of Theorem 1 to the results of Nair (1990) and Merzbach and Nualart (1986) is of interest. The results of the previous authors involve transforming the point process N via a sequence of stopping sets. Notice that the sets D_z defined in Theorem 1 are generally not stopping sets, since they may fail to meet the requirement that if a stopping set contains a point z', then it must also contain all points less than z'.

Theorem 3.3 of Nair (1990) assumes that A is 1-convex, i.e. that $A(s+\Delta s, t) - A(s,t) \leq A(s+2\Delta s, t) - A(s+\Delta s, t)$, for all s, Δs, and $t \in R_+$. When λ_1 exists, the 1-convexity of A is equivalent to the assumption in Theorem 4 of Merzbach and Nualart (1986) that for any t, $\int_0^t \lambda_1(s,t')dt'$ is a nondecreasing function of s. This means that each function $\tau_z(s')$ is decreasing in s'. This condition ensures that D_z is an $\mathcal{F}^1$-stopping set.

Note that Theorem 5 of Merzbach and Nualart (1986) and Theorem 3.4 of Nair (1990) relax the convexity condition slightly. For example, in Theorem 5 of Merzbach and Nualart (1986), $\lambda_1(s,t)/\beta(s)$ is assumed nondecreasing in s, for all $t \geq 0$, where β is some positive decreasing function. Though a bit weaker than the convexity condition, this condition also holds only in rather special cases. The situation is the same in Nair (1990).

Remark 3.4. In applications, one often observes a point process N on a finite subregion $S \subset \mathbf{R}^2_+$, and typically the requirement in Theorem 3.2 that $\int_0^\infty \lambda_1(\omega,s,t)dt = \infty$ is not met. One can nevertheless apply Theorem 1 to the process $\bar{N} := N + N'$, where N' is an $\mathcal{F}$-adapted Poisson process independent of N with $\mathcal{F}^1$-intensity 1 on S^c and $\mathcal{F}^1$-intensity 0 on S. Let $\bar{\lambda}_1$ be the $\mathcal{F}^1$-intensity of the process $\bar{N}$. In transforming $\bar{N}$ as in Theorem 1, a point $z = (s,t)$ is moved to (s, $\int_0^t \bar{\lambda}_1(s,t')dt'$). Similarly, the region S corresponds in the transformed plane to a region $T := \{(s, \int_0^t \bar{\lambda}_1(s,t')dt') : (s,t) \in S\} \subset \mathbf{R}^2_+$. Note that the shape of T is random. If λ_1 is a left-continuous version of the $\mathcal{F}^1$-intensity of N, and S is the set of points under a left-continuous function $f(x)$, i.e. $S = \{(x,y) \in \mathbf{R}^2_+ : y \leq f(x)\}$, then from the definitions of T and λ_1 it follows that T is the set of points under some left-continuous function $g(x)$. Further properties of T may be a subject for future research.

Corollary 3.5. *Suppose that N satisfies the conditions of Theorem 3.2, and let λ_1^* be any integrable,* *nonnegative, $\mathcal{F}^1$-predictable process. Define D_z^* and M^* by*

$$\begin{aligned} D_z^* &:= \left\{ (\omega, s', t') \in \Omega \times \mathbf{R}_+^2 : s' \leq s, \int_0^{t'} \lambda_1^*(\omega, s', t'')dt'' \leq t \right\}, \\ M^*(z) &:= \int \mathbf{1}_{D_z^*} dN. \end{aligned}$$

Then M^* is a Poisson process on $\mathbf{R}_+^2$ with unit rate if and only if $\lambda_1^* = \lambda_1$, π-a.e., where π is the product measure $\mu \times P$, and μ denotes Lebesgue measure on $\mathbf{R}_+^2$.

Proof. In order for the statement to have meaning, it should first be noted that the set $\{\lambda_1^* = \lambda_1\}$ is always a π-measurable set. Since both λ_1^* and λ_1 are $\mathcal{F}^1$-predictable, so is $\lambda_1^* - \lambda_1$. Thus by the definition of predictability,

$$\{\lambda_1^* - \lambda_1 = 0\} \in \sigma(\{\mathcal{F}_z^1 \times (z, z']; z, z' \in \mathbf{R}_+^2\}) \subseteq \sigma(\mathcal{F} \times \mathcal{B}),$$

where $\mathcal{B}$ denotes the Borel subsets of $\mathbf{R}_+^2$. That is, $\{\lambda_1^* = \lambda_1\}$ is π-measurable.

If $\lambda_1^* = \lambda_1$ π-a.e., then λ_1^* is a version of the $\mathcal{F}^1$-intensity of N, so M^* is a unit-rate Poisson process by Theorem 3.2.

For the other direction, suppose that M^* is a unit-rate Poisson process, and that $\pi(\{\lambda_1^* \neq \lambda_1\}) > 0$. We must show that this leads to a contradiction.

Either $\pi(\{\lambda_1^* > \lambda_1\}) > 0$ or $\pi(\{\lambda_1^* < \lambda_1\}) > 0$; suppose that $\pi(\{\lambda_1^* > \lambda_1\}) > 0$. (The case where $\pi(\{\lambda_1^* < \lambda_1\}) > 0$ can be proven equivalently.)

Some definitions are required. For $z \in \mathbf{R}_+^2$, define sequences of sets I_z, J_z, J_z^*, and K_z as follows:

$$I_z := \{(\omega, z') : \lambda_1^*(\omega, z') > \lambda_1(\omega, z'), z' \leq z\}; \tag{3.7}$$

$$J_z := \left\{ (\omega, s', \int_0^{t'} \lambda_1(\omega, s', t'')dt'') : (\omega, s', t') \in I_z \right\}; \tag{3.8}$$

$$J_z^* := \left\{ (\omega, s', \int_0^{t'} \lambda_1^*(\omega, s', t'')dt'') : (\omega, s', t') \in I_z \right\}. \tag{3.9}$$

$$K_z := \{\omega \in \Omega : \mu(\{z' : \lambda_1^*(\omega, z') > \lambda_1(\omega, z'), z' \leq z\}) > 0\}; \tag{3.10}$$

Observe that I_z and $(K_z \times \mathbf{R}_+^2)$ are each $\mathcal{F}^1$-predictable sets, since λ_1^* and λ_1 are both $\mathcal{F}^1$-predictable processes. Similarly, since $\int_0^{t'} \lambda_1(s', t'')dt''$ and $\int_0^{t'} \lambda_1^*(s', t'')dt''$ are both $\mathcal{F}^1$-predictable, it follows that J_z and J_z^* are $\mathcal{F}^1$-predictable sets as well.

Let M be defined as in Theorem 3.2. Both M and M^* are well-defined point processes, as shown in the proof of Theorem 3.2.

Since M^* is by assumption a Poisson process with unit rate, and since the process $1_{J_z^* \cap (K \times \mathbf{R}_+^2)}$ is bounded, nonnegative and $\mathcal{F}^1$-predictable, applying the martingale property one obtains:

$$\begin{aligned} E \int 1_{J_z^* \cap (K_z \times \mathbf{R}_+^2)} dM^* &= \int 1_{J_z^* \cap (K_z \times \mathbf{R}_+^2)} d\mu(z) \\ &= E[\mu(J_z^*)|K_z], \end{aligned} \tag{3.11}$$

where the random variable $\mu(J_z^*)$ denotes the Lebesgue measure of the random subset $J_z^* \subset \mathbf{R}_+^2$.

Similarly, by Theorem 3.2 M is a Poisson process of unit rate, so

$$\begin{aligned} E \int 1_{J_z \cap (K_z \times \mathbf{R}_+^2)} dM &= \int 1_{J_z \cap (K_z \times \mathbf{R}_+^2)} d\mu(z) \\ &= E[\mu(J_z)|K_z]. \end{aligned} \tag{3.12}$$

Since $\pi(\{\lambda_1^* > \lambda_1\}) > 0$ by assumption, $P(\{\omega : \mu(\{\lambda_1^* > \lambda_1\})\}) > 0$, and since $K_{(t,t)} \uparrow \{\omega : \mu(\{\lambda_1^* > \lambda_1\})\}$ as $t \to \infty$, we must have $P(K_z) > 0$ for some z. Fix such a z, and choose any $\omega \in K_z$. For any $(\omega, s', t') \in I_z$, $\lambda_1^* > \lambda_1$, and therefore $\mu(J_z^*) > \mu(J_z)$. Thus,

$$E[\mu(J_z^*)|K_z] > E[\mu(J_z)|K_z]. \tag{3.13}$$

Combining (??), (??), and (??) yields:

$$E \int 1_{J_z^* \cap (K_z \times \mathbf{R}_+^2)} dM^* > E \int 1_{J_z \cap (K_z \times \mathbf{R}_+^2)} dM,$$

i.e.,

$$E[M^*(J_z^*)|K_z] > E[M(J_z)|K_z]. \tag{3.14}$$

For each $\omega \in \Omega$, $M^*(J_z^*) = N(I_z)$, by definition of M^* and J_z^*. Similarly, by definition of M and J_z, $M(J_z) = N(I_z)$, for every ω. Thus

$$E[M^*(J_z^*)|K_z] = E[N(I_z)|K_z] = E[M(J_z)|K_z]. \tag{3.15}$$

Equation (??), together with (??), yields a contradiction. □

Remark 3.6. Corollary 3.5 may be useful for the evaluation of point process models. Given a point process N and a model specifying the $\mathcal{F}^1$-conditional intensity, λ_1^*, by Corollary 3.5 the problem of assessing the fit of the model boils down to examining whether M^* is a planar Poisson process of unit rate. Much has been written on the latter problem; see e.g. Diggle (1983), Heinrich (1991), or Andersen, Borgan, Gill, and Keiding (1992).

4 Example

Rathbun (1995) describes a planar version of the self-exciting point process analyzed by Hawkes (1971). Here the $\mathcal{F}^1$-intensity may be given by:

$$\lambda_1(s,t) = f(s,t) + \int \mathbf{1}_{\{s'<s\}}\ g(s-s', |t-t'|)\ dN(s',t')$$

where f and g are deterministic, nonnegative functions from $\mathbf{R}^2$ to $\mathbf{R}$. Similarly, the $\mathcal{F}^1$-intensity of a k-dimensional version of a Hawkes process can be given by:

$$\lambda_1(t_1,...,t_k) = f(t_1,...,t_k) + \int \mathbf{1}_{\{t_1'<t_1\}}\ g(t_1-t_1', |t_2-t_2'|, ..., |t_k-t_k'|)\ dN(t_1',...,t_k')$$

where f and g are now deterministic nonnegative functions from $\mathbf{R}^k$ to $\mathbf{R}$.

Such processes generally do not satisfy the convexity condition of Merzbach and Nualart (1986) and Nair (1990). For instance, suppose N is a planar Hawkes process and f and g are decreasing functions. Then for any s, s', and $t \in \mathbf{R}_+$ such that $N(s,t) = N(s',t)$, $\lambda_1(x,y)$ is decreasing in x for $s < x < s'$ and $y \leq t$. Thus $\int_0^t \lambda_1(x,y)dy$ is decreasing in x for $s < x < s'$, which violates the convexity condition.

5 Extension to higher dimensions

In this section, Theorem 3.2 is generalized to include the case where N is a point process in $\mathbf{R}_+^k$, for $k \geq 2$. First, a few of the previous definitions must be extended.

For $z = (t_1,...,t_k)$ and $z' = (t_1',...,t_k') \in \mathbf{R}_+^k$, say $z < z'$ if $t_i < t_i'$ for each i, and say $z \leq z'$ if $t_i \leq t_i'$ for each i. A filtration $\mathcal{F}$ on $\mathbf{R}_+^k$ may be defined exactly as in Section 2. Let $\mathcal{F}^1(z)$ denote $\bigcup_{u_2,u_3,...,u_k} \mathcal{F}(t_1,u_2,u_3,...,u_k)$, where $z = (t_1,...,t_k)$. The $\mathcal{F}^1$-*predictable σ-field* $\mathcal{P}^1$, the *1-compensator* A and the $\mathcal{F}^1$- *intensity* λ of N can be defined exactly as in Section 2.

The following lemma is an extension of Lemma 3.3 of Brown, Ivanoff and Weber (1986). A slightly stronger version is given in Proposition 4.2 of Nair (1990) and proven for the three-dimensional case.

Lemma 5.1. *If N is an $\mathcal{F}^1$-adapted point process on $\mathbf{R}_+^k$ with deterministic, continuous 1-compensator A and at most one point on every hyperplane $\{t_1 = t\}$, then N is a Poisson process whose mean corresponds to A.*

Theorem 5.2. *Suppose that N is a simple $\mathcal{F}$-adapted point process on $\mathbf{R}_+^k$ with $\mathcal{F}^1$-intensity λ_1, $\mathcal{F}^2$-intensity $\lambda_2, \ldots$, and $\mathcal{F}^k$-intensity λ_k. If with probability one, for all $t_1, t_2, \ldots, t_{k-1} \in \mathbf{R}_+$,*

$$\int_0^\infty \lambda_1(\omega, t_1, t_2, t_3, ..., t_k) dt_k = \infty,$$

then there is a sequence of $\mathcal{F}^1$-predictable sets D_z, such that $M(z) := \int \mathbf{1}_{D_z} dN$ is a Poisson process on $\mathbf{R}_+^k$ with unit rate.

PROOF. Fix $z = (t_1, ..., t_k)$, and for $s_1 \leq t_1, s_2 \leq t_2, ..., s_{k-1} \leq t_{k-1}$ define $\tau_z(\omega, s_1, s_2, ..., s_{k-1})$ as $\inf\{t'_k : \int_0^{t'_k} \lambda_1(\omega, s_1, s_2, ..., s_{k-1}, s_k) ds_k > t_k\}$, letting $\tau_z(\omega, s_1, \ldots, s_{k-1}) = 0$ otherwise.

Define D_z and M by:

$$D_z = \left\{ (\omega, s_1, \ldots, s_k) \in \Omega \times \mathbf{R}_+^k : s_1 \leq t_1, \ldots, s_{k-1} \leq t_{k-1}, \int_0^{s_k} \lambda_1(\omega, s_1, \ldots, s_{k-1}, s'_k) ds'_k \leq t_k \right\},$$

$$M(z) = \int \mathbf{1}_{D_z} dN.$$

It follows by the same argument as in Theorem 3.2 that D_z is $\mathcal{F}^1$-predictable and that M is a well-defined $\mathcal{F}^1$-adapted point process. Further, from Proposition 4.1 of Nair (1990) it follows that with probability 1, N has at most one point on any hyperplane perpendicular to the t_1-axis, and therefore the same is true for M.

Let $C(z) = \int 1_{D_z} dA$, where A is the 1-compensator of N.
Now

$$A(z) = \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_k} \lambda_1(\omega, s_1, s_2, ..., s_k) ds_k ds_{k-1} ... ds_1$$

and

$$C(z) = \int_0^{t_1} \dots \int_0^{t_{k-1}} \int_0^{\tau_z(\omega, s_1, \dots, s_{k-1})} \lambda_1(\omega, s_1, s_2, \dots, s_k) ds_k \dots ds_1.$$

As in Theorem 3.2, for each $\omega \in \Omega$ and each $(s_1, \dots, s_{k-1}) \leq (t_1, \dots, t_{k-1})$,

$$\int_0^{\tau_z(\omega, s_1, \dots, s_{k-1})} \lambda_1(s_1, s_2, \dots, s_k) ds_k = t_k$$

so

$$C(z) = \int_0^{t_1} \dots \int_0^{t_{k-1}} t_k ds_{k-1} \dots ds_1 = t_1 t_2 \dots t_k.$$

Thus $C(z)$ is the 1-compensator of the unit rate Poisson process in $\mathbf{R}_+^k$.

It follows from exactly the same argument as in Theorem 3.2 that C is the 1-compensator of M. Using Lemma 5.1, the proof is complete. □

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