Path decompositions of a Brownian bridge related to the ratio of its maximum and amplitude*

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Abstract

We give two new proofs of Csáki's formula for the law of the ratio $1 - Q$ of the maximum relative to the amplitude (i.e. the maximum minus minimum) for a standard Brownian bridge. The second of these proofs is based on an absolute continuity relation between the law of the Brownian bridge restricted to the event $(Q \leq v)$ and the law of a process obtained by a Brownian scaling operation after back-to back joining of two independent three-dimensional Bessel processes, each started at v and run until it first hits 1. Variants of this construction and some properties of the joint law of Q and the amplitude are described.

Key words and phrases. Williams' decomposition, range, three-dimensional Bessel process, Brownian scaling.

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1 Introduction

In his study of asymptotic distributions arising from empirical processes in non-parametric statistics, Smirnov [25] showed that the formula

$$P(I \leq a, M \leq b) = \sum_{k=-\infty}^{\infty} \exp(-2k^2(a+b)^2) - \sum_{k=-\infty}^{\infty} \exp(-2[b+k(a+b)]^2) \tag{1}$$

for $a, b \geq 0$ defines the joint distribution of a pair of non-negative random variables (I, M). Doob [11] showed that (I, M) may be constructed as

$$I := -\inf_{0 \leq u \leq 1} b_u \quad \text{and} \quad M := \sup_{0 \leq u \leq 1} b_u$$

where $(b_u, 0 \leq u \leq 1)$ is a standard Brownian bridge. Besides the many applications of this law of (I, M) in the theory of empirical processes (for which see Shorack and Wellner [24, §2.2]), this law is of interest on account of some of its remarkable properties which can be found scattered in the probabilistic literature. To quickly recall some of these properties, the asymptotic distribution of the Kolmogorov-Smirnov statistic is that of the absolute maximum $I \vee M = \sup_{0 \leq u \leq 1} |b_u|$, which can be read from (1) as

$$P(I \vee M \leq b) = \sum_{k=-\infty}^{\infty} (-1)^k \exp(-2k^2 b^2). \tag{2}$$

As explained by Vervaat's [27] construction of a Brownian excursion from Brownian bridge, the law of the maximum of a standard Brownian excursion found by Kennedy [16] and Chung [9] is identical to the law of $I + M$, known as the *amplitude* or *range* of the bridge, whose distribution is given by the formula [12]

$$P(I + M > b) = 2 \sum_{k=1}^{\infty} (4k^2 b^2 - 1) \exp(-2k^2 b^2) \tag{3}$$

for $b \geq 0$. See also [2] for a survey of transformations related to Vervaat's construction. As observed by Chung [9], the distribution of $I \vee M$ is characterized by the Laplace transform

$$E \exp(-\tfrac{1}{2}\lambda^2 (I \vee M)^2) = \frac{\frac{\pi}{2}\lambda}{\sinh(\frac{\pi}{2}\lambda)} \tag{4}$$

while that of $I + M$ is characterized by the companion formula

$$E \exp(-\tfrac{1}{2}\lambda^2 (I + M)^2) = \left(\frac{\frac{\pi}{2}\lambda}{\sinh(\frac{\pi}{2}\lambda)} \right)^2. \tag{5}$$

Consequently, the law of $(I+M)^2$ equals the law of the sum of two independent copies of $(I \vee M)^2$. For $x \geq 0, y > 0$ let $T^{(3)}_{x,y}$ denote the first hitting time of y by a $\mathrm{BES}^{(3)}_x$ process $(R^{(3)}_{x,t}, t \geq 0)$, that is a three-dimensional Bessel process started at x, which may be constructed as $R^{(3)}_{x,t} := \sqrt{(x+B_{1,t})^2 + B^2_{2,t} + B^2_{3,t}}$ where the $(B_{i,t}, t \geq 0)$ for $i = 1, 2, 3$ are three independent standard Brownian motions started at 0. It is well known that for $y > 0$

$$E \exp(-\tfrac{1}{2}\lambda^2 T^{(3)}_{0,y}) = \frac{y\lambda}{\sinh(y\lambda)} \tag{6}$$

so the identities (4) and (5) amount to the equalities in distribution

$$(I \vee M)^2 \stackrel{d}{=} T^{(3)}_{0,\pi/2} \quad \text{and} \quad (I+M)^2 \stackrel{d}{=} T^{(3)}_{0,\pi/2} + \widehat{T}^{(3)}_{0,\pi/2} \tag{7}$$

where $\widehat{T}^{(3)}_{0,\pi/2}$ is an independent copy of $T^{(3)}_{0,\pi/2}$. As far as we know there is still no satisfying explanation in terms of Brownian paths for these remarkable identities found by Chung. For further discussion of these results, their relation to the functional equations satisfied by the Jacobi theta and Riemann theta functions, and various applications, see [5, 4, 30].

Let $Q := I/(I+M)$. Csáki [10, Theorem 2] deduced from (1) a fairly complicated expression for $P(I+M < u, Q < v)$, from which he obtained by letting $u \to \infty$ the remarkable formula [10, (2.12)]

$$P(Q \leq v) = 2v^2(1-v) \sum_{n=1}^{\infty} \frac{1}{n^2 - v^2} = (1-v)(1 - \pi v \cot(\pi v)) \tag{8}$$

for $0 < v < 1$. Section 2 of this paper presents a novel approach to Csáki's formula (8) via the alternative expression

$$P(Q \leq v) = 2v^2(1-v) \int_0^{\infty} d\lambda \left(\frac{\sinh(v\lambda)}{v \sinh(\lambda)} \right)^2. \tag{9}$$

By (6), for $T^{(3)}_{v,1}$ the hitting time of 1 by a $\mathrm{BES}^{(3)}_v$ process, there is the standard formula

$$E \exp(-\tfrac{1}{2}\lambda^2 T^{(3)}_{v,1}) = \frac{\sinh(v\lambda)}{v \sinh(\lambda)} \tag{10}$$

so if we let $\widehat{T}^{(3)}_{v,1}$ denote an independent copy of $T^{(3)}_{v,1}$, and set

$$T^*_v := T^{(3)}_{v,1} + \widehat{T}^{(3)}_{v,1}$$

then

$$E\exp(-\tfrac{1}{2}\lambda^2 T_v^*) = \left(\frac{\sinh(v\lambda)}{v\sinh(\lambda)}\right)^2. \tag{11}$$

In Section 3, the appearance of this quantity as the integrand in (9) is explained in terms of the path decomposition at the maximum for the Brownian bridge, deduced as in Pitman-Yor [20] from Williams' [28] path decomposition at the maximum for a one-dimensional diffusion. The path decomposition of the bridge at its maximum allows the law of the bridge restricted to the event $(Q \le v)$ to be constructed by a random Brownian scaling operation from a back-to-back joining of the paths of two independent $\mathrm{BES}_v^{(3)}$ processes run until their first hits of 1. In Section 4 we deduce some corollaries of this result involving the joint law of M and Q. Section 5 presents a more refined result, which gives an explicit description of the law of the bridge conditioned on $Q = q$ for an arbitrary $q \in [0, 1]$. We note in particular that in the limiting case $q = 0$ this conditional distribution on $C[0, 1]$ is absolutely continuous with respect to the law of a standard Brownian excursion, with a density factor at $\omega \in C[0, 1]$ that is proportional to $(\sup_{0\le u\le 1} \omega_u)^2$. In Section 6 we present some further identities involving the local time of the bridge at 0 up to time 1. Finally, Section 7 records some basic properties of the distribution of Q determined by Csáki's formula (8).

2 A derivation of Csáki's formula

Let $|N|$ denote the absolute value of a standard Gaussian variable N, so

$$P(|N| \le x) = \int_0^x \sqrt{\tfrac{2}{\pi}} e^{-\frac{1}{2}y^2}\, dy$$

and assume that N is independent of the bridge $(b_t, 0 \le t \le 1)$. Our starting point is the formula

$$P(|N|I \le x, |N|M \le y) = \frac{2}{\coth x + \coth y} \tag{12}$$

which we have discussed already in [23, Ex. (4.24) of Chapter XII]. See also [8, 22]. As shown by Perman and Wellner [18], the Smirnov-Doob formula (1) can be deduced from (12) by inversion of Laplace transforms. But since

$$Q := \frac{I}{I + M} = \frac{|N|I}{|N|I + |N|M} \tag{13}$$

we can proceed directly from (12) to the distribution of Q, without consideration of Laplace transforms. Easily from (12), for $x, y \geq 0$

$$P(|N|I \leq x, |N|M \in dy) = \frac{2\sinh^2(x)\,dy}{\sinh^2(x+y)} \tag{14}$$

which combined with (13) gives

$$P(Q \leq v) = P\left(|N|I \leq \frac{v}{(1-v)}|N|M\right) \tag{15}$$

$$= \int_0^\infty dy\,\frac{2\sinh^2\left(\frac{vy}{1-v}\right)}{\sinh^2\left(\frac{y}{1-v}\right)} \tag{16}$$

$$= (1-v)\int_0^\infty d\lambda\,\frac{2\sinh^2(v\lambda)}{\sinh^2(\lambda)} \tag{17}$$

so we have arrived at formula (9). To complete the proof of Csáki's formula (8), it only remains to check the identity

$$\int_0^\infty d\lambda\,\frac{2\sinh^2(v\lambda)}{\sinh^2(\lambda)} = 1 - \pi v\cot(\pi v). \tag{18}$$

But after expanding

$$2\sinh^2(v\lambda) = \cosh(2v\lambda) - 1 = \sum_{n=1}^{\infty}\frac{(2v)^{2n}}{(2n)!}\,\lambda^{2n}$$

the identity (18) follows easily from the classical identities [13, 3.523.2]

$$\int_0^\infty d\lambda\,\frac{\lambda^{2n}}{\sinh^2(\lambda)} = \pi^{2n}|B_{2n}| \qquad (n = 1, 2, \ldots) \tag{19}$$

where B_m is the mth Bernoulli number, and [13, 1.411.7]

$$\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}\,x^{2n} = 1 - x\cot x \qquad (|x| < \pi). \tag{20}$$

3 Path decomposition at the maximum

We start by formulating the path decomposition of the Brownian bridge at its maximum in terms of the following construction, which we adapt from [28, 29, 19, 5, 20]. See also [21] for variations of this construction and [14, 15, 26] for other decompositions of the Brownian path involving the range process and $\mathrm{BES}^{(3)}$ pieces.

Construction 1 Given two continuous path processes with random finite lifetimes, each with initial value 0 and final value z, say $R := (R(t), 0 \le t \le \eta)$ and $(\widehat{R} := (\widehat{R}(t), 0 \le t \le \widehat{\eta})$ with $R(\eta) = \widehat{R}(\widehat{\eta}) = z$, construct a random element r of $C[0,1]$, say

$$r := (r(u), 0 \le u \le 1) := \mathrm{BRIDGE}\left[(R(t), 0 \le t \le \eta); (\widehat{R}(t), 0 \le t \le \widehat{\eta})\right]$$

with $r(0) = r(1) = 0$ by first pasting R and $\widehat{R}$ back to back and then transforming the resulting path by Brownian scaling to have lifetime 1; that is

$$r(u) := \begin{cases} \zeta^{-1/2} R(u\zeta) & \text{if } 0 \le u \le V \\ \zeta^{-1/2} \widehat{R}((1-u)\zeta) & \text{if } V \le u \le 1 \end{cases} \tag{21}$$

where $\zeta := \eta + \widehat{\eta}$ and $V := \eta/\zeta$.

In the following applications, η and $\widehat{\eta}$ will be the first hitting times of some level $z > 0$ by the processes R and $\widehat{R}$ respectively. Then V is evidently the a.s. unique time at which r attains its maximum level, so V is a measurable function of r with

$$\sup_{0 \le u \le 1} r(u) = r(V) = z\zeta^{-1/2}$$

and R and $\widehat{R}$ can then be recovered from r via the formulae

$$\zeta = z^2/r^2(V)$$

$$(R(t), 0 \le t \le \eta) = (zr(t/\zeta)/r(V), 0 \le t \le V\zeta)$$

$$(\widehat{R}(t), 0 \le t \le \widehat{\eta}) = (zr(1 - t/\zeta)/r(V), 0 \le t \le (1-V)\zeta).$$

So the joint distribution of $(R, \widehat{R})$ determines the distribution of $r := \mathrm{BRIDGE}\left[R; \widehat{R}\right]$, and vice versa.

Theorem 2 *Let* $(b_u, 0 \le u \le 1)$ *be a standard Brownian bridge, and let*

$$(b^*_u, 0 \le u \le 1) := \mathrm{BRIDGE}\left[(B_t, 0 \le t \le \sigma_1); (\hat{B}_t, 0 \le t \le \hat{\sigma}_1)\right] \tag{22}$$

where $(B_t, 0 \le t \le \sigma_1)$ *and* $(\hat{B}_t, 0 \le t \le \hat{\sigma}_1)$ *are two independent copies of a standard Brownian motion started at* 0 *and run until its first hitting time of* 1*. Then for every non-negative measurable function* F *defined on the path space* $C[0,1]$ *there is the identity*

$$E\left[F(b_u, 0 \le u \le 1)\right] = \sqrt{2\pi}\, E\left[F(b^*_u, 0 \le u \le 1) M^*\right] \tag{23}$$

where

$$M^* := \sup_{0 \le u \le 1} b^*_u = 1/\sqrt{\sigma_1 + \hat{\sigma}_1}. \tag{24}$$

Proof. Copy the proof of [20, Theorem 3.1] in dimension 1, with the one-dimensional Bessel process $(|B_t|, t \ge 0)$ replaced by $(B_t, t \ge 0)$. □

Corollary 3 *Fix* $0 < v < 1$ *and let*

$$(\tilde{b}_{v,u}, 0 \le u \le 1) := \mathrm{BRIDGE}\left[(R^{(3)}_{v,t} - v, 0 \le t \le T^{(3)}_{v,1}); (\hat{R}^{(3)}_{v,t} - v, 0 \le t \le \hat{T}^{(3)}_{v,1})\right] \tag{25}$$

where $(R^{(3)}_{v,t}, 0 \le t \le T^{(3)}_{v,1})$ *and* $(\hat{R}^{(3)}_{v,t}, 0 \le t \le \hat{T}^{(3)}_{v,1})$ *are two independent copies of a* $BES^{(3)}_v$ *process run until its first hitting time of* 1*. Then for every non-negative measurable function* F *defined on the path space* $C[0,1]$ *there is the identity*

$$E\left[F(b_u, 0 \le u \le 1) 1(Q \le v)\right] = \sqrt{2\pi}\, v^2 E\left[F(\tilde{b}_{v,u}, 0 \le u \le 1) \widetilde{M}_v\right] \tag{26}$$

where

$$\widetilde{M}_v := \sup_{0 \le u \le 1} \tilde{b}_{v,u} = \frac{(1-v)}{\sqrt{T^*_v}} \quad \text{with} \quad T^*_v := T^{(3)}_{v,1} + \hat{T}^{(3)}_{v,1}. \tag{27}$$

Proof. In (23) replace $F(\cdots)$ by

$$F(\cdots)1(Q \le v) = F(\cdots)1(I/M \le a) \quad \text{where} \quad v = a/(a+1), a = v/(1-v)$$

to see that

$$E\left[F(b_t, 0 \le u \le 1) 1(Q \le v)\right] = \sqrt{2\pi} E\left[F(b^*_u, 0 \le u \le 1) M^* 1(G_a)\right] \tag{28}$$

for G_a the event

$$G_a := (I_{\sigma_1} > -a) \cap (\hat{I}_{\hat{\sigma}_1} > -a)$$

where $I_t := \inf_{0 \le u \le t} B_u$ and hats indicate corresponding variables defined in terms of the other independent Brownian motion. Since $P(G_a) = (a/(a+1))^2 = v^2$, formula (28) can be recast as

$$E\left[F(b_t, 0 \le u \le 1) 1(Q \le v)\right] = \sqrt{2\pi}\, v^2\, E\left[F(b^*_u, 0 \le u \le 1) M^* \,|\, G_a\right]. \tag{29}$$

But conditionally on G_a the processes $(B_t, 0 \le t \le \sigma_1)$ and $(\widehat{B}_t, 0 \le t \le \widehat{\sigma}_1)$ are two independent copies of Brownian motion started at 0 and run until its hitting time of 1, with conditioning to hit 1 before $-a$. By mapping the interval $[-a, 1]$ linearly to $[0, 1]$, and scaling time by a factor of $(a+1)^2 = 1/(1-v)^2$, these two processes can be constructed from two independent copies of Brownian motion started at v and run until its hitting time of 1, with conditioning to hit 1 before 0. As shown by Williams [28], such a conditioned Brownian motion is a copy of $(R^{(3)}_{v,t}, 0 \le t \le T^{(3)}_{v,1})$. Thus the processes $(B_t, 0 \le t \le \sigma_1)$ and $(\widehat{B}_t, 0 \le t \le \widehat{\sigma}_1)$ given G_a are distributed like two independent copies of the process

$$\left(\frac{R^{(3)}_{v, t(1-v)^2} - v}{1 - v}, 0 \le t \le \frac{T^{(3)}_{v,1}}{(1-v)^2} \right) \tag{30}$$

Thus (29) holds with the process $(b^*_t, 0 \le t \le 1)$ conditioned on G_a replaced by $(\widetilde{b}_t, 0 \le t \le 1)$ defined as in (25), and with the density factor M^* in (29) replaced by the corresponding quantity defined in terms of the $\mathrm{BES}^{(3)}_v$ processes, that is

$$\widetilde{M}_v := \frac{1}{\sqrt{(T^*_v)/(1-v)^2}} = \frac{(1-v)}{\sqrt{T^*_v}},$$

and these substitutions in (29) yield (26). □

As a check on formula (26), we note that the previous formula (23) is recovered from (26) in the limit as $v \uparrow 1$. To see this, observe that as $v \uparrow 1$ the distribution of the process in (30) converges to that of $(B_t, 0 \le t \le \sigma_1)$, and hence the distribution of the process $(\widetilde{b}_{v,u}, 0 \le u \le 1)$ converges to that of $(b^*_u, 0 \le u \le 1)$. For a discussion of the limiting case of (26) as $v \downarrow 0$, see the end of Section 5.

4 Some consequences of the path decomposition

If in (26) we take $F(b_u, 0 \leq u \leq 1) = M^{-1} f(M)$ with $M := \sup_{0 \leq u \leq 1} b_u$ as before, and f an arbitrary non-negative Borel function, then we deduce from (26) that

$$E\left(M^{-1} f(M) 1(Q \leq v)\right) = \sqrt{2\pi}\, v^2 E\left[f\left(\frac{1-v}{\sqrt{T_v^*}}\right)\right] \tag{31}$$

where the distribution of T_v^* is determined by the Laplace transform (11). In particular, for arbitrary real r

$$E\left(M^r 1(Q \leq v)\right) = \sqrt{2\pi}\, v^2 (1-v)^{r+1} E\left((T_v^*)^{-(r+1)/2}\right). \tag{32}$$

For any non-negative random variable X there is the formula

$$E(X^{-p}) = \frac{2^{1-p}}{\Gamma(p)} \int_0^\infty d\lambda\, \lambda^{2p-1} E \exp(-\tfrac{1}{2}\lambda^2 X) \qquad (p > 0) \tag{33}$$

obtained by application of Fubini's theorem. So (32) combined with (11) yields

$$E\left(M^r 1(Q \leq v)\right) = \sqrt{2\pi}\, (1-v)^{r+1} \frac{2^{\frac{1-r}{2}}}{\Gamma(\frac{r+1}{2})} \int_0^\infty d\lambda\, \lambda^r \left(\frac{\sinh(v\lambda)}{\sinh(\lambda)}\right)^2 \qquad (r > -1). \tag{34}$$

This formula determines the distribution of M restricted to the event $(Q \leq v)$ by a Mellin transform. In the special case $r = 0$ we recover from (34) the alternative form (9) of Csáki's formula (8).

By another application of formulae (31) and (11), we deduce the following characterization of the law of M restricted to the event $(Q \leq v)$: for all real ξ and $0 < v < 1$

$$E\left[\frac{1}{M} \exp\left(-\frac{\xi^2}{2M^2}\right) 1(Q \leq v)\right] = \sqrt{2\pi}\, \frac{\sinh^2(\xi v/\bar{v})}{\sinh^2(\xi/\bar{v})} \quad \text{where} \quad \bar{v} := (1-v). \tag{35}$$

5 Conditioning the bridge on Q

Formulae for various conditional expectations given $Q = v$ are obtained by differentiating formulae of the previous section with respect to v. For instance, in the special case $r = -1$ formula (32) simplifies to give for $0 \leq v \leq 1$

$$E[M^{-1} 1(Q \leq v)] = \sqrt{2\pi} v^2 \tag{36}$$

and hence by differentiation

$$E(M^{-1}\,|\,Q=v) = 2\sqrt{2\pi}\,v/f_Q(v) \tag{37}$$

where, by application of Csáki's formula (8),

$$f_Q(v) := P(Q\in dv)/dv = \frac{d}{dv}(1-v)(1-\pi v\cot(\pi v)). \tag{38}$$

See Section 8 for further discussion of this density. By differentiation of formula (35) we obtain for $0<v<1$, with $\bar{v}:=1-v$,

$$E\left[\frac{1}{M}\exp\left(-\frac{\xi^2}{2M^2}\right)1(Q\in dv)\right] = \frac{2\sqrt{2\pi}\,\xi}{\bar{v}^2}\,\frac{\sinh(\xi v/\bar{v})\sinh(\xi)}{\sinh^3(\xi/\bar{v})}\,dv \tag{39}$$

If we apply this formula with $\lambda:=\xi/\bar{v}$ and v replaced by q then in terms of the amplitude

$$A := I+M = M/(1-Q)$$

we deduce the simpler formula

$$E\left[\frac{1}{A}\exp\left(-\frac{\lambda^2}{2A^2}\right)\middle|\, Q=q\right] = \frac{2\sqrt{2\pi}\,\lambda\sinh(\lambda q)\sinh(\lambda(1-q))}{f_Q(q)\sinh^3(\lambda)} \tag{40}$$

for $0<q<1$. In view of (6) and (10) this can be interpreted as follows. Let

$$T_q := T^{(3)}_{0,1}+T^{(3)}_{q,1}+T^{(3)}_{1-q,1}$$

where $T^{(3)}_{x,y}$ is as before the first hitting time of y by a $\mathrm{BES}^{(3)}_x$ process, and we now assume that the three random times $T^{(3)}_{0,1}$, $T^{(3)}_{q,1}$, and $T^{(3)}_{1-q,1}$ are independent. Then from (6) and (10) we have

$$E\exp(-\tfrac{1}{2}\lambda^2 T_q) = \frac{\lambda\sinh(\lambda q)\sinh(\lambda(1-q))}{q(1-q)\sinh^3(\lambda)}. \tag{41}$$

Let

$$A_q := 1/\sqrt{T_q}. \tag{42}$$

Then (41) allows (40) to be rewritten

$$E\left[\frac{1}{A}\exp\left(-\frac{\lambda^2}{2A^2}\right)\middle|\, Q=q\right] = \frac{2\sqrt{2\pi}q(1-q)}{f_Q(q)}\,E\left[\exp\left(-\frac{\lambda^2}{2A_q^2}\right)\right]. \tag{43}$$

It now follows by uniqueness of Laplace transforms that for an arbitrary non-negative Borel function g and $0 < q < 1$ there is the identity

$$E\left[g(A) \mid Q=q\right] = \frac{2\sqrt{2\pi}\, q(1-q)}{f_Q(q)} E(A_q g(A_q)). \tag{44}$$

That is to say, the conditional density of A at a given $Q = q$ is identical to $a f_{A_q}(a)/E(A_q)$, where f_{A_q} is the density of $A_q := 1/\sqrt{T_q}$. In particular, by (44) for $g = 1$,

$$E(A_q) = \frac{1}{2\sqrt{2\pi}} \frac{f_Q(q)}{q(1-q)}. \tag{45}$$

Formula (61) gives bounds which imply that $E(A_q)$ lies in the interval $(0.19, 0.22)$ for all $q \in (0,1)$. In view of (41), (42) and (33) for $p = 1/2$, we see that (45) amounts to the identity

$$\int_0^\infty d\lambda\, \frac{\lambda \sinh(\lambda q)\sinh(\lambda(1-q))}{\sinh^3(\lambda)} = \frac{f_Q(q)}{4}. \tag{46}$$

This identity can also be deduced by integration of (40) with respect to $d\lambda$. Since

$$4\int_0^v dq\, \lambda \sinh(\lambda q)\sinh(\lambda(1-q)) = 2\lambda v\cosh(\lambda) - \sinh(\lambda) + \sinh(\lambda - 2v\lambda) \tag{47}$$

the identity (46) is in turn equivalent to

$$\int_0^\infty d\lambda\, \frac{(2\lambda v\cosh(\lambda) - \sinh(\lambda) + \sinh(\lambda - 2v\lambda))}{\sinh^3(\lambda)} = P(Q \le v) \tag{48}$$

as given by Csáki's formula (8). We were able to confirm this by symbolic integration using *Mathematica.*

The above discussion invites an interpretation in terms of a path decomposition of the bridge conditioned on $Q = q$. Such an interpretation is provided by the following corollary of Theorem 2, which extends the previous formula (44) from an identity of one-dimensional distributions to an identity of distributions on the path space $C[0,1]$.

Fix $q \in (0,1)$. Take three independent $\mathrm{BES}^{(3)}$ processes, with starting levels $0, q$ and $1-q$, say R_0, R_q and $\widehat{R}_{1-q}$, whose hitting times of 1 are $T^{(3)}_{0,1}$, $T^{(3)}_{q,1}$ and $\widehat{T}^{(3)}_{1-q,1}$. Define a continuous path $S := (S(w), 0 \le w \le T_q)$, starting at q at time 0, and ending at q at time $T_q := T^{(3)}_{0,1} + T^{(3)}_{q,1} + T^{(3)}_{1-q,1}$, by concatenation of the three paths

$$(R_q(t), 0 \le t \le T^{(3)}_{q,1}),$$

$$(R_0(T^{(3)}_{0,1} - u), 0 \le u \le T^{(3)}_{0,1}),$$
$$(1 - R_{1-q}(T^{(3)}_{1-q,1} - v), 0 \le v \le T^{(3)}_{1-q,1}).$$

Let $(b^\dagger_{q,u}, 0 \le u \le 1)$ be the process derived from S by the Brownian scaling operation $b^\dagger_{q,u} := (S(uT_q) - q)/\sqrt{T_q}$. So by construction, $(b^\dagger_{q,u}, 0 \le u \le 1)$ is a process starting and ending at 0 whose amplitude is $A_q := 1/\sqrt{T_q}$ as above, with the feature that the process attains its maximum value before its minimum.

Corollary 4 *Let $\rho_{\min}$ denote the a.s. unique time that the Brownian bridge $(b_u, 0 \le u \le 1)$ attains its minimum on $[0,1]$, and $\rho_{\max}$ the corresponding time for the maximum. Then for every non-negative measurable function F defined on the path space $C[0,1]$ there is the identity*

$$E\left[F(b_u, 0 \le u \le 1) \,|\, \rho_{\max} < \rho_{\min}, Q = q\right] = \frac{2\sqrt{2\pi}q(1-q)}{f_Q(q)} E\left[F(b^\dagger_{q,u}, 0 \le u \le 1)A_q\right] \tag{49}$$

where $A_q := 1/\sqrt{T_q}$ is the amplitude of $(b^\dagger_{q,u}, 0 \le u \le 1)$.

Proof. By application of (23), and the definition of conditional expectations, we deduce that for $0 < q < 1$

$$E\left[F(b_u, 0 \le u \le 1) \,|\, \rho_{\max} < \rho_{\min}, Q = q\right] = \frac{E\left[F(b^*_u, 0 \le u \le 1)M^* \,|\, \rho^*_{\max} < \rho^*_{\min}, Q^* = q\right]}{E\left[M^* \,|\, \rho^*_{\max} < \rho^*_{\min}, Q^* = q\right]}$$

where M^*, $\rho^*_{\max}$, $\rho^*_{\min}$ and Q^* are $M, \rho_{\max}$, $\rho_{\min}$ and Q evaluated for $(b^*_u, 0 \le u \le 1)$ instead of $(b_u, 0 \le u \le 1)$. In particular, by construction $M^* = 1/\sqrt{\sigma_1 + \widehat{\sigma}_1}$. Now, from the construction of $(b^*_u, 0 \le u \le 1)$, we see that the event $(\rho^*_{\max} < \rho^*_{\min}, Q^* = q)$ is identical to the event $(I_{\sigma_1} > c, \widehat{I}_{\widehat{\sigma}_1} = c)$ where $c/(c+1) = q, c = q/(1-q)$. With this conditioning, the process $(B_u, 0 \le u \le \sigma_1)$ becomes a Brownian motion run until it first reaches 1, conditioned to reach 1 before reaching $-c$, while the process $(\widehat{B}_u, 0 \le u \le \widehat{\sigma}_1)$ is a Brownian motion run until it first reaches 1 and conditioned on $\inf_{0 \le u \le \widehat{\sigma}_1} B_u = -c$. According to Williams' path decomposition at the minimum [28], the latter process can be constructed by concatenation of two $BES^{(3)}$ pieces. After rescaling as in the proof of Corollary 3 these two fragments are represented by the second two paths in the concatenation of three paths which defines the process S, and the argument is completed similarly to the proof of Corollary 3. □

Note that due to the invariance of the bridge under time reversal, the event $(\rho_{\max} < \rho_{\min})$ appearing above is an event of probability 1/2 that is independent of the pair

(Q, A). Corollary 4 combined with this remark provides an explicit description of the unique family of conditional distributions for $(b_u, 0 \le u \le 1)$ given $Q = q$ that is weakly continuous in q for $q \in [0,1]$. In particular, the law of $(b_u, 0 \le u \le 1)$ given $Q = 0$ is obtained either by letting $q \downarrow 0$ in Corollary 4, or by conditioning on $(Q \le v)$ and letting $v \downarrow 0$ in Corollary 3. (See formulae (58) and (59) for the required asymptotics of $f_Q(v)$ and $P(Q \le v)$ as $v \downarrow 0$). Let $(\tilde{b}_{0,u}, 0 \le u \le 1)$ be the process defined by formula (25) for $v = 0$. That is, $(\tilde{b}_{0,u}, 0 \le u \le 1)$ is constructed by putting back-to-back two independent copies of a $\mathrm{BES}_0^{(3)}$ run until its first hit of 1, then Brownian scaling to obtain lifetime 1. Then formula (26) implies that

$$E\left[F(b_u, 0 \le u \le 1) \,|\, Q = 0\right] = \frac{3\sqrt{2\pi}}{\pi^2} E\left[F(\tilde{b}_{0,u}, 0 \le u \le 1)\widetilde{M}_0\right] \tag{50}$$

where

$$\widetilde{M}_0 := \sup_{0 \le u \le 1} \tilde{b}_{0,u} = \frac{1}{\sqrt{T_0^*}} \quad \text{with} \quad T_0^* := T_{0,1}^{(3)} + \hat{T}_{0,1}^{(3)}. \tag{51}$$

It is known [7] that the law of $(b_u, 0 \le u \le 1)$ given $I = 0$, defined similarly as a weak limit, is the law of a standard Brownian excursion (or $\mathrm{BES}^{(3)}$ bridge), as determined by [20, Theorem 3.1 with $\delta = 3$],

$$E\left[F(b_u, 0 \le u \le 1) \,|\, I = 0\right] = \sqrt{\frac{\pi}{2}} E\left[F(\tilde{b}_{0,u}, 0 \le t \le 1)(\widetilde{M}_0)^{-1}\right]. \tag{52}$$

Thus the limit in distribution as $v \downarrow 0$ of $(b_u, 0 \le u \le 1)$ given $(Q \le v)$, as determined by (50), is not the same as the limit in distribution as $v \downarrow 0$ of $(b_u, 0 \le u \le 1)$ given $(I \le v)$, as determined by (52), despite the identity of the events $(Q = 0)$ and $(I = 0)$. See Billingsley [6, p. 441] for similar variations of the classical Borel paradox. As a check on the constants of integration, it is known [5] that the mean squared maximum of a Brownian excursion is $\pi^2/6$. Thus (52) for $F(\cdots) = M^2$ gives

$$\frac{\pi^2}{6} = E(M^2 \,|\, I = 0) = \sqrt{\frac{\pi}{2}} E(\widetilde{M}_0)$$

in agreement with (50) for $F(\cdots) = 1$.

6 Some Further Identities

The formula (12) which we used as our starting point was derived in [23] as a consequence of the following trivariate identity, which characterizes the joint law of (I, M, L) where

$$L := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^1 dt \, 1(|b_t| \le \varepsilon)$$

is the *local time at* 0 *of the bridge up to time* 1:

$$P(|N|I \le x, |N|M \le y, |N|L \in d\ell) = \exp\left(-\frac{\ell}{2}(\coth x + \coth y)\right) d\ell \tag{53}$$

We note that Corollary 3 could be applied to give another characterization of the law of (I, M, L).

Let $(\tau_\ell, \ell \ge 0)$ denote the usual local time process at zero of a standard Brownian motion $(B_t, t \ge 0)$. As shown in [3], the law of the *pseudo-bridge* $(b_t^\#, 0 \le t \le 1)$ defined by

$$b_t^\# := B_{t\tau_1}/\sqrt{\tau_1}$$

is absolutely continuous with respect to that of the bridge, with density $(\sqrt{\pi/2}L)^{-1}$. Equivalently, for every non-negative measurable function F defined on the path space $C[0,1]$ there is the identity

$$E\left[F(b_t, 0 \le t \le 1)\right] = \sqrt{\tfrac{\pi}{2}}\, E\left[F(b_t^\#, 0 \le t \le 1)L^\#\right] \tag{54}$$

where $L^\# = 1/\sqrt{\tau_1}$ is the local time at 0 of $(b_t^\#, 0 \le t \le 1)$ up to time 1. In terms of the Brownian motion (B_t), define

$$I_t := -\inf_{0\le u\le t} B_u; \quad M_t := \sup_{0\le u\le t} B_u; \quad A_t := I_t + M_t; \quad Q_t := \frac{I_t}{A_t}.$$

It was shown in [21] that Q_{τ_1} has uniform distribution on $(0,1)$. In view of (54) and Csáki's formula (8), this implies

$$E\left[L^{-1}g(Q)\right] = \sqrt{\frac{\pi}{2}} \int_0^1 dv\, g(v)$$

for all non-negative Borel functions g. This formula can also be obtained quite easily from Theorem 2. From this formula we deduce that

$$E[L^{-1}\,|\,Q = v] = \sqrt{\frac{\pi}{2}}\frac{1}{f_Q(v)}. \tag{55}$$

Compared with (37), this gives the curious formula

$$E[M^{-1}\,|\,Q = v] = 4vE[L^{-1}\,|\,Q = v]. \tag{56}$$

As shown by Lévy[17], the random variables M and $2L$ have identical Rayleigh distributions, with $P(M > x) = P(2L > x) = \exp(-2x^2)$ for $x > 0$. The conditional distribution of M given $Q = v$, which is determined by (35), could also be described by a series density derived from (1). It does not seem easy to describe the conditional law of L given $Q = q$ so explicitly, though the density of $|N|L$ on the event $(Q \le v)$ can be read from (53), and this could be used to give integral expressions for conditional moments of L given $Q \le v$ or $Q = v$.

7 The distribution of Q

We record in this section some properties of the distribution of Q which follow from Csáki's formula (8) for $P(Q \le v)$. By differentiation of (8), the density at $q \in (0,1)$ is

$$f_Q(q) = \frac{\pi^2 q(1-q)}{\sin^2 \pi q} + (2q-1)\pi \cot \pi q - 1 \tag{57}$$

It is easily checked using (57) that

$$f_Q(q) = f_Q(1-q) \sim \frac{2\pi^2}{3} q \quad \text{as} \quad q \downarrow 0 \tag{58}$$

where the first equality is obvious from the symmetry of Brownian bridge with respect to a sign change. Easily from (58)

$$P(Q \le q) = P(Q \ge 1-q) \sim \frac{\pi^2}{3} q^2 \quad \text{as} \quad q \downarrow 0 \tag{59}$$

This distribution of Q is close in most respects to the beta(2,2) distribution with density $6q(1-q)$. Both densities are concave and symmetric about $1/2$. The beta(2,2) distribution is slightly more peaked, with modal density $3/2 = 1.5$ at $q = 1/2$, whereas

$$f_Q(1/2) = \frac{\pi^2}{4} - 1 = 1.4674\ldots. \tag{60}$$

The density of the law of Q relative to the beta(2,2) law is subject to the bounds

$$0.978 \approx \frac{\pi^2 - 4}{6} \le \frac{f_Q(q)}{6q(1-q)} \le \frac{\pi^2}{9} \approx 1.097 \tag{61}$$

where the lower bound is attained at $1/2$ and the upper bound is sharp at $0+$ and $1-$. The total variation distance between these two densities was found by numerical integration using *Mathematica* to be

$$\int_0^1 dq\, |f_Q(q) - 6q(1-q)| \approx 0.019 \tag{62}$$

For $n > 0$ the nth moment $E(Q^n) = \int_0^1 dq\, q^n f_Q(q)$ can be evaluated by integration by parts as follows:

$$E(Q^n) = \int_0^1 dv\,(1 - P(Q \leq v))nv^{n-1} = \frac{n}{n+1} + n\pi \int_0^1 dv v^n (1-v)\cot(\pi v). \tag{63}$$

For $m = 1, 2, \ldots$ there is the classical identity [1, 23.2.17]

$$\int_0^1 dv\, B_{2m+1}(v)\cot(\pi v) = 2(2m+1)!(-1)^{m+1}\frac{\zeta(2m+1)}{(2\pi)^{2m+1}} \tag{64}$$

where $B_n(v)$ is the nth Bernoulli polynomial, which is of degree n with rational coefficients, and $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Also, by symmetry,

$$E[(Q - 1/2)^{2m-1}] = 0. \tag{65}$$

It follows that for $n = 1, 2, \ldots$

$$E(Q^n) = \frac{n}{n+1} + \sum_{m=1}^{\lfloor n/2 \rfloor} a_{n,m} \frac{\zeta(2m+1)}{\pi^{2m}} \tag{66}$$

for some rational coefficients $a_{n,m}$ determined by (63), (64) and (65). For instance

$$E(Q) = \frac{1}{2}; \quad E(Q^2) = \frac{2}{3} - \frac{3\zeta(3)}{\pi^2}; \quad E(Q^3) = \frac{3}{4} - \frac{9\zeta(3)}{2\pi^2}; \tag{67}$$

$$E(Q^4) = \frac{4}{5} - \frac{8\zeta(3)}{\pi^2} + \frac{30\zeta(5)}{\pi^4}; \quad E(Q^5) = \frac{5}{6} - \frac{25\zeta(3)}{2\pi^2} + \frac{75\zeta(5)}{\pi^4}. \tag{68}$$

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