The Feynman-Kac formula and decomposition of Brownian paths *

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1 Introduction

The formula of Feynman-Kac [12, 20, 21] is a central result in the modern theory of Brownian motion and diffusion processes. To state the basic Feynman-Kac formula for one-dimensional Brownian motion in the form presented in M. Kac [20] let $q : \mathbb{R} \to \mathbb{R}_+$ be a Borel function, and let $f : \mathbb{R} \to \mathbb{R}_+$ be a locally bounded Borel function. Then for $k > 0$ and $\lambda > 0$

$$\int_0^\infty dt\, e^{-\frac{k^2}{2}t} E\left[q(B_t)\exp\left(-\lambda \int_0^t f(B_s)ds\right)\right] = \int_{-\infty}^\infty dx\, q(x) U^{\lambda f}(k,x) \qquad (1.1)$$

where E is the expectation operator with respect to a probability distribution P governing $(B_t, t \geq 0)$ as a standard one-dimensional Brownian motion started at

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0, and $U(x) = U^{\lambda f}(k,x)$ is the unique solution of the differential equation

$$\frac{1}{2}U''(x) = \left(\frac{k^2}{2} + \lambda f(x)\right) U(x) \qquad (x \neq 0) \tag{1.2}$$

subject to the requirements that $U'(x)$ exists for $x \neq 0$ and is uniformly bounded, that U vanishes at $\pm\infty$, and that

$$U'(0+) - U'(0-) = -2 \tag{1.3}$$

In his 1949 article, M. Kac proved the Feynman-Kac formula by approximating Brownian motion with simple random walks. The exposition of the present paper is intended for a reader acquainted with the modern theory of Brownian motion based on martingale calculus and excursion theory, as presented for example in Revuz-Yor [33] or Rogers-Williams [34, 35]. For a lighter treatment of some of this background material, see the Caracas lecture notes [48].

Our main purpose here is to present connections between the Feynman-Kac formula, related Sturm-Liouville equations, and various decompositions of Brownian paths into independent components. Path decompositions are discussed in Section 2 of this paper. In Section 3 we show how some refinements of the Feynman-Kac formula may be understood in terms of a decomposition of the Brownian path at the time of the last visit to zero before time θ where θ is an exponentially distributed random time independent of the Brownian motion. We also show how D.Williams' decomposition at the maximum of the generic excursion under Itô's measure translates in terms of solutions of a Sturm-Liouville equation. Finally, Section 4 is devoted to explicit computations of the laws of

$$A_t^f := \int_0^t ds\, f(B_s)$$

for specific functions f, among which $f(x) = \lambda |x|^\alpha$, for some special values of α, and $f(x) = \lambda \coth(\mu x)$, borrowed from the recent thesis of Alili [2].

While we have chosen here to stay fairly close to Kac's presentation of the Feynman-Kac formula in terms of a one-dimensional Brownian motion, a substantial literature has developed around extensions of the Feynman-Kac formula to various other contexts. See for instance Kesten's paper [22] for a survey of such developments, and [29, 43, 44] for some other recent studies. Simon [38], Chung-Zhao [11], Nagasawa [26], and Aebi [1] provide connections with the theory of Schrödinger semigroups. For further examples involving the techniques of this paper, and a wealth of other formulae related to Brownian motion and diffusions, see the forthcoming handbook of Brownian motion by Borodin-Salminen [7].

2 Some decompositions of Brownian paths

For $a \in \mathbb{R}$ let P_a denote a probability distribution governing $(B_t, t \geq 0)$ a one-dimensional Brownian motion (BM) started at a. For $a = 0$ we write simply

P instead of P_0. For $a \geq 0$ and $\delta \geq 0$ let $P_a^{(\delta)}$ govern a process $(R_t, t \geq 0)$ as a BES(δ), that is a δ-dimensional Bessel process, started at a. We shall be particularly concerned with the BES(3) *process*, which is obtained by conditioning a BM in $(0, \infty)$ to never reach 0, and the BES(3) *bridge* obtained by conditioning a BES(3) process started at 0 at time 0 to return to 0 at time 1. See [33] for background, and further discussion of this fundamental relationship between BM and BES(3), and [36, 13, 32] for further discussion of diffusion bridges.

Let E, E_a and E_a^δ denote the expectation operators derived from P, P_a and P_a^δ. While we find it convenient to use the different notations $(B_t, t \geq 0)$ for a BM and $(R_t, t \geq 0)$ for a Bessel process, we shall occasionally regard P, P_a and P_a^δ as probability distributions on the canonical path space $C[0, \infty)$, in which case $(X_t, t \geq 0)$ may be used to denote the coordinate process on $C[0, \infty)$ which might be governed by any of these laws.

2.1 A decomposition of $(B_u; 0 \leq u \leq d_t)$

Given a Brownian motion $(B_u;\ u \geq 0)$, and an arbitrary fixed time $t > 0$, let $d_t := \inf\{u : u > t, B_u = 0\}$ and $g_t := \sup\{u : u \leq t, B_u = 0\}$.

a. The process $\left(b_u := \frac{1}{\sqrt{g_t}} B_{u g_t}; 0 \leq u \leq 1\right)$ is a *standard Brownian bridge* which is independent of the σ-field generated by the variable g_t and the process $(B_{g_t+u}; u \geq 0)$.

b. The *standard Brownian excursion*, $\left(\frac{1}{\sqrt{d_t - g_t}} \left|B_{g_t + u(d_t - g_t)}\right|; 0 \leq u \leq 1\right)$ is a BES(3) bridge which is independent of $\sigma\{B_u,\ 0 \leq u \leq g_t, u \geq d_t\}$.

c. The process

$$\left(m_u = \frac{1}{\sqrt{t - g_t}} \left|B_{g_t + u(t - g_t)}\right|; 0 \leq u \leq 1\right),$$

known as a *Brownian meander* [10], has a law which does not depend on t, and the meander is independent of $\sigma\{B_u,\ 0 \leq u \leq g_t\}$. Details of this decomposition can be found in Revuz-Yor [33] and Yor [48]. See also Bertoin-Pitman [4] for a survey of various transformations relating the bridge, excursion and meander derived as above from Brownian motion.

d. Finally, as a consequence of the strong Markov property of BM, the process $(B_{d_t+u}, u \geq 0)$ is a BM independent of $(B_u, 0 \leq u \leq d_t)$.

2.2 A decomposition of $(B_u; 0 \leq u \leq \theta_k)$ at g_{θ_k} for an independent exponential time θ_k

Let θ_k be a random time whose distribution is exponential with rate $k^2/2$, that is

$$P(\theta_k \in dt) = \frac{k^2}{2} \exp(-\frac{k^2 t}{2})\, dt \qquad (t > 0),$$

and assume that θ_k and $(B_t; t \geq 0)$ are independent. Then the left-hand side of the Feynman-Kac formula (1.1) equals

$$\frac{2}{k^2} E\left[q(B_{\theta_k}) \exp\left(-A^{\lambda f}_{\theta_k}\right)\right]$$

Since the well known formula for the resolvent of BM [17] implies that

$$P(B_{\theta_k} \in dx) = \frac{k}{2} e^{-k|x|} dx \qquad (x \in I\!R)$$

it is clear that the function $U^{\lambda f}(k,x)$ appearing on the right-hand side of (1.1) is

$$U^{\lambda f}(k,x) = \frac{e^{-k|x|}}{k} E\left[\exp\left(-A^{\lambda f}_{\theta_k}\right) \Big| B_{\theta_k} = x\right] \tag{2.1}$$

Our aim now is to explain Kac's characterization of $U^{\lambda f}(k,x)$ as the solution of a Sturm-Liouville problem in terms of a decomposition of the path $(B_u; 0 \leq u \leq \theta_k)$ at time g_{θ_k}, the last zero of B before the independent exponential time θ_k. By a last-exit decomposition (see e.g. [46] chap. 3, p. 35 or [48] lecture 5, p. 73) the processes $(B_u;\ 0 \leq u \leq g_{\theta_k})$ and $(B_{\theta_k - u};\ 0 \leq u \leq \theta_k - g_{\theta_k})$ are independent. As a consequence, if $(\ell_t, t \geq 0)$ denotes the local time of B at 0, (see e.g. [33, Ch. VI]) and if $\tau_s = \inf\{t : \ell_t > s\}$ is the inverse local time, then ℓ_{θ_k} is independent of B_{θ_k} with

$$P(\ell_{\theta_k} \in d\ell) = k \exp(-k\ell)\, d\ell \qquad (\ell \geq 0).$$

Therefore, for every pair of Brownian functionals F and G, taking values in $I\!R_+$, we have

$$E\left[F(B_u; 0 \leq u \leq g_{\theta_k})\, G(B_{\theta_k - u}; 0 \leq u \leq \theta_k - g_{\theta_k})\, | \ell_{\theta_k} = \ell\,, B_{\theta_k} = x\right] = \tag{2.2}$$

$$\left(e^{k\ell} E\left[\exp\left(-\frac{k^2 \tau_\ell}{2}\right)\ F(B_u; 0 \leq u \leq \tau_\ell)\right]\right)\left(e^{k|x|}\, E_x\left[\exp\left(-\frac{k^2}{2} T_0\right)\ G(B_u, 0 \leq u \leq T_0)\right]\right)$$

where $T_0 = \inf\{t : B_t = 0\}$ and consequently

$$E\left[F(B_u; 0 \leq u \leq g_{\theta_k})\right] = k \int_0^\infty d\ell\, E\left[\exp\left(-\frac{k^2 \tau_\ell}{2}\right)\ F(B_u; 0 \leq u \leq \tau_\ell)\right] \tag{2.3}$$

$$E\left[G(B_{\theta_k - u}; 0 \leq u \leq \theta_k - g_{\theta_k})\right] = \frac{k}{2} \int_{-\infty}^\infty da\, E_a\left[\exp\left(-\frac{k^2}{2} T_0\right)\ G(B_u; 0 \leq u \leq T_0)\right].$$

For example,

$$E\left[\exp\left(-A^f_{g_{\theta_k}}\right)\right] = k \int_0^\infty d\ell\, E\left[\exp - \left(\frac{k^2}{2}\tau_\ell + A^f_{\tau_\ell}\right)\right] \tag{2.4}$$

whereas

$$E\left[\exp - \left(A^f_{\theta_k} - A^f_{g_{\theta_k}}\right)\right] = \frac{k}{2} \int_{-\infty}^\infty da\, E_a\left[\exp - \left(\frac{k^2}{2} T_0 + A^f_{T_0}\right)\right] \tag{2.5}$$

and

$$E\left[\exp\left(-A^f_{\theta_k}\right)\right] = E\left[\exp\left(-A^f_{g_{\theta_k}}\right)\right]\ E\left[\exp -\left(A^f_{\theta_k} - A^f_{g_{\theta_k}}\right)\right]. \tag{2.6}$$

We explain in the next sub-section how the basic formulae (2.2) and (2.3) can be expressed in terms of excursion theory. Then in Section 3 we show how the right-hand sides of (2.4) and (2.5) can each be evaluated in terms of solutions of appropriate Sturm-Liouville equations. These evaluations then combine via (2.6) to yield Kac's description of the function $U^{\lambda f}(k,x)$ appearing in (1.1) and (2.1).

2.3 Integral representations of Brownian laws

We now find it convenient to suppose that the laws P_a governing BM and $P_a^{(3)}$ governing BES(3) are defined on the canonical space $C[0,\infty)$, with coordinate process $(X_t, t\geq 0)$. For $m \geq 0$ let $P_a^{T_m}$ and $(P_a^{(3)})^{T_m}$ denote the previous laws derived from P_a and $P_a^{(3)}$ by killing the process $(X_t, t \geq 0)$ at time $T_m = \inf\{t : X_t = m\}$. Formulae (2.2) and (2.3) are presented more concisely by the following master formula, in which the symbol $\circ$ stands for the operation of concatenation of two independent trajectories. (See [48] lecture 5, par. 6, or [33], page 481, Exercise (4.18)).

$$\int_0^\infty dt\, P^t = \left(\int_0^\infty d\ell\, P^{\tau_\ell}\right) \circ \left(\int_{-\infty}^\infty da\ {}^r(P_a^{T_0})\right) \tag{2.7}$$

where P^t is the law of $(X_u, 0\leq u \leq t)$ under P, i.e., the measure defined on the space of continuous functions $\{\omega(u), 0 \leq u \leq \zeta_\omega\}$. Thus

$$\left(\int_0^\infty dt\, P^t\right)[F(X_u, u\leq \zeta)h(\zeta)] = \int_0^\infty dt\, h(t)\, E\left[F(X_u, u \leq t)\right]$$

and

$$\left(\int_0^\infty d\ell\, P^{\tau_\ell}\right)[F(X_u, u\leq \tau_\ell)h(\tau_\ell)] = \int_0^\infty d\ell\, E\left[F(X_u, u \leq \tau_\ell)\, h(\tau_\ell)\right].$$

On the other hand, ${}^r(P_a^{T_0})$ is the law $P_a^{T_0}$ with reversed time. To be precise, ${}^r(P_a^{T_0}) = (P_0^{(3)})^{\gamma^a}$ is the law of a BES(3) process starting from 0 and ending at $\gamma^a = \sup\{t\geq 0 : R_t = a\}$. The formula (2.7) is taken from Biane-Yor [6], where it is proved by excursion theory. See also Biane [5] for a systematic study of such integral representations and their applications, Leuridan [25] for a more elementary approach, and van der Hofstad et al. [40] for some closely related presentations.

2.4 The agreement between two descriptions of Brownian excursions

Two important integral representations of $\mathbf{n}_+$, the Itô measure of positive Brownian excursions, are as follows:

$$\mathbf{n}_+ = \frac{1}{2}\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\, \Pi^v \tag{2.8}$$

and

$$\mathbf{n}_+ = \frac{1}{2}\int_0^\infty \frac{dm}{m^2}\left(P_0^{(3)}\right)^{T_m} \circ \left({}^r\left(P_0^{(3)}\right)^{T_m}\right) \tag{2.9}$$

where Π^v is the law of the BES(3) bridge of length v (see [6] par. 6.1, [33] chap. 12, [42, 32]). The comparison of (2.8) and (2.9) yields the so-called *agreement formula*

$$\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\,\Pi^v = \int_0^\infty \frac{dm}{m^2}\left(P_0^{(3)}\right)^{T_m} \circ \left({}^r\left(P_0^{(3)}\right)^{T_m}\right) \tag{2.10}$$

which is discussed in some generality in [32]. According to the basic formula of excursion theory, for any Borel function $f : \mathbb{R} \to \mathbb{R}_+$,

$$E[\exp(-A^f_{\tau_\ell})] = \exp\left(-\,\ell \int \mathbf{n}(d\varepsilon)\,\left(1-\exp(-A^f_V(\varepsilon))\right)\right) \tag{2.11}$$

where $\mathbf{n} = \mathbf{n}^+ + \mathbf{n}^-$ is the Itô measure of Brownian excursions ($\mathbf{n}^-$ is the image of $\mathbf{n}^+$ by $\varepsilon \to -\varepsilon$) and V is the length of the excursion.

3 Sturm-Liouville equations

In order to obtain explicit formulae for (2.4) and (2.5), we first present probabilistic interpretations of the fundamental solutions of the Sturm-Liouville equation

$$\frac{1}{2}F''(x) = m(x)F(x) \tag{3.1}$$

for an unknown function $F : \mathbb{R}_+ \to \mathbb{R}$, where $m : \mathbb{R}_+ \to \mathbb{R}_+$ is a known non-negative and locally integrable Borel function. The function m determines two fundamental solutions of (3.1). These are

i) $F(x) = \Phi(x)$ say, the non-increasing solution of (3.1) such that $F(0) = 1$;

ii) $F(x) = \Psi(x)$ say, the solution of (3.1) such that $F(0) = 0$, $F'(0+) = 1$.

The following probabilistic interpretations of Φ and Ψ have been known for some time as applications of stochastic calculus. See for instance [18],[48, p. 76]. First of all,

$$\Phi(a) = E_a\left[\exp\left(-\int_0^{T_0} m(B_s)\,ds\right)\right] \quad (0 \le a) \tag{3.2}$$

which by application of the strong Markov property of B implies

$$\frac{\Phi(a)}{\Phi(b)} = E_a\left[\exp\left(-\int_0^{T_b} m(B_s)\,ds\right)\right] \qquad (0 \le b \le a). \tag{3.3}$$

Next,

$$\frac{b\Psi(a)}{a\Psi(b)} = E_a^{(3)}\left[\exp\left(-\int_0^{T_b} m(R_s)\,ds\right)\right] \qquad (0 < a < b). \tag{3.4}$$

Equivalently, by the Doob h-transform relation between Brownian motion and BES(3) for $h(x) = x$ (see e.g. [33] p. 431)

$$\frac{\Psi(a)}{\Psi(b)} = E_a\left[1_{\{T_b<T_0\}} \exp\left(-\int_0^{T_b} m(B_s)\,ds\right)\right] \qquad (0<a<b) \tag{3.5}$$

For completeness, we record also the following companion to (3.5), which follows easily from (3.3) and (3.5) by application of the strong Markov property of B at time T_b:

$$\Phi(a) - \frac{\Psi(a)}{\Psi(b)}\Phi(b) = E_a\left[1_{\{T_0<T_b\}} \exp\left(-\int_0^{T_0} m(B_s)\,ds\right)\right] \qquad (0<a\le b) \tag{3.6}$$

In the next two subsections we show how the above formulae may be applied with $m(x) = \frac{k^2}{2} + \lambda f_+(x)$ and $m(x) = \frac{k^2}{2} + \lambda f_-(x)$ where f_+ is the restriction of f to $\mathbb{R}_+$ and $f_-(x) = f(-x)$, $x \ge 0$ for a generic $f : \mathbb{R} \to \mathbb{R}_+$. For simplicity, we take $\lambda = 1$. Replace f by λf to introduce the parameter λ.

3.1 The Brownian path from g_{θ_k} to θ_k

The following Theorem provides an evaluation of the right-hand side of (2.5).

Theorem 3.1 *Let $f_+ : \mathbb{R}_+ \to \mathbb{R}_+$ be a locally bounded function, and let*

$$\Phi^{f_+}(k,a) := E_a\left[\exp -\left(\frac{k^2}{2}T_0 + \int_0^{T_0} ds\, f_+(B_s)\right)\right] \qquad (a \ge 0). \tag{3.7}$$

The function $u(a) = \Phi^{f_+}(k,a)$ is the unique bounded solution of the Sturm-Liouville equation

$$\frac{1}{2}u'' = \left(\frac{k^2}{2} + f_+\right)u\,; \quad u(0) = 1\,. \tag{3.8}$$

Moreover

$$\Phi^{f_+}(k,a) = \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \exp\left(-\frac{k^2 t}{2}\right) H^{f_+}(t,a)\,, \tag{3.9}$$

where

$$H^{f_+}(t,a) = a\,\exp\left(-\frac{a^2}{2t}\right) E_0^{(3)}\left[\exp\left(-\int_0^t du\, f_+(R_u)\right) \Big| R_t = a\right]. \tag{3.10}$$

Proof: The first part of the theorem follows from (3.2). The right-hand-side of (3.7) may be written as

$$\int_0^\infty dt\,\frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \exp\left(-\frac{k^2 t}{2}\right) E_a\left[\exp -\int_0^{T_0} ds\, f_+(B_s)\Big| T_0 = t\right].$$

Using D. Williams' time reversal result, and conditioning with respect to $\gamma_a = \sup\{t : R_t = a\}$, we obtain

$$E_a\left[\exp -\int_0^{T_0} ds\, f_+(B_s)\Big| T_0 = t\right] = E_0^{(3)}\left[\exp -\int_0^t ds\, f_+(R_s)\Big| R_t = a\right].$$

and hence the representation (3.10) of H^{f_+}. See [48, p. 52]. □

Corollary 3.2 *With the previous notation, we obtain the following expression for $f : \mathbb{R} \to \mathbb{R}_+$ a locally bounded function:*

$$E\left[\exp - \left(A^f_{\theta_k} - A^f_{g_{\theta_k}}\right)\right] = \frac{k}{2}\int_0^\infty da\,\left(\Phi^{f+}(k,a) + \Phi^{f-}(k,a)\right)$$

$$= \frac{k}{2}\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\,\exp\left(-\frac{k^2 t}{2}\right)\int_0^\infty da\,\left(H^{f+}(t,a) + H^{f-}(t,a)\right)$$

3.2 The Brownian path before g_{θ_k}

The next theorem provides an evaluation of the right-hand side of (2.4). We keep the notation of Theorem 3.1.

Theorem 3.3 *The limit $K^{f+}(t) := \lim_{a\to 0+} \dfrac{H^{f+}(t,a)}{a}$ exists and satisfies the equality*

$$K^{f+}(t) = E_0^{(3)}\left[\exp - \int_0^t du\, f_+(R_u)\,|R_t = 0\right]$$

Furthermore, this function K^{f+} is characterized via the following transform:

$$-\,(\Phi^{f+})'(k,0+) = \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\left(1 - \exp(-\frac{k^2 t}{2})\,K^{f+}(t)\right) \qquad (3.11)$$

where

$$(\Phi^{f+})'(k,0+) = \left.\frac{\partial}{\partial a}\right|_{a=0+}\Phi^{f+}(k,a)$$

Remark: As a check, recall that $\Phi^{f+}(k,a)$ is a decreasing function of a, hence, both sides of (3.11) take values in $\mathbb{R}_+$.
Proof: We divide by a the two sides of the equality

$$\Phi^{f+}(k,a) - 1 = (e^{-ka} - 1) + (\Phi^{f+}(k,a) - e^{-ka})\,.$$

Then, using

$$\frac{1}{a}\,e^{-ka} = \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\,\exp\left[-\frac{1}{2}\left(k^2 t + \frac{a^2}{t}\right)\right]\,,$$

we obtain from (3.9) and (3.10) that $-(\Phi^{f+})'(k,0+)$ equals

$$k + \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\exp(-\frac{k^2 t}{2})\left(1 - \,E_0^{(3)}\left[\exp(-\int_0^t ds\, f_+(R_s))\middle| R_t = 0\right]\right)\,.$$

We also note that $k = \displaystyle\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\left(1 - \exp(-\frac{k^2 t}{2})\right)$, which leads to the form (3.11) of the right derivative of the function $\Phi^{f+}(k,a)$ at $a = 0$. $\square$

Corollary 3.4 *With the previous notation, we obtain*

$$E\left[\exp-\left(\frac{k^2}{2}\tau_\ell+A^f_{\tau_\ell}\right)\right]=\exp\left(\frac{\ell}{2}\left[(\Phi^{f+})'(k,0+)+(\Phi^{f-})'(k,0+)\right]\right)$$

As a consequence, we have

$$E\left[\exp\left(-A^f_{g_{\theta_k}}\right)\right]=\frac{2k}{-(\Phi^{f+})'(k,0+)-(\Phi^{f-})'(k,0+)}$$

Proof: This follows from Tanaka's formula and optional sampling theorem. See e.g. [18], or the proof of the second Ray-Knight theorem in [33], Chap. XI, theorem 2.3. Another proof is obtained from (2.11) and the observation that $\mathbf{n}=\lim_{a\downarrow 0}\frac{P_a}{a}$ in a suitable sense. See Pitman-Yor [30] where this result is presented for a general diffusion, with $\frac{P_a}{s(a)}$ instead of $\frac{P_a}{a}$ for s the scale function of the diffusion. □

Corollary 3.5

$$E\left[\exp\left(-A^f_{\theta_k}\right)\right]=\frac{k^2\int_0^\infty da\,(\Phi^{f+}(k,a)+\Phi^{f-}(k,a))}{-(\Phi^{f+})'(k,0+)-(\Phi^{f-})'(k,0+)} \tag{3.12}$$

Remark: Replace f by λf for $\lambda>0$ to see that this formula determines the Laplace transform of $A^f_{\theta_k}$, hence the distribution of $A^f_{\theta_k}$, in terms of the functions $\Phi^{\lambda f+}$ and $\Phi^{\lambda f-}$ for $\lambda>0$. The formula of the corollary extends to

$$E\left[q(B_{\theta_k})\,\exp\left(-A^f_{\theta_k}\right)\right]=\frac{k^2\int_0^\infty da\,(q(a)\Phi^{f+}(k,a)+q(-a)\,\Phi^{f-}(k,a))}{-(\Phi^{f+})'(k,0+)-(\Phi^{f-})'(k,0+)} \tag{3.13}$$

and

$$E\left[q(B_{\theta_k})\,p(\ell_{\theta_k})\,\exp\left(-A^f_{\theta_k}\right)\right]=$$
$$\frac{k^2}{2}\int_0^\infty d\ell\,p(\ell)\exp\left(\frac{\ell}{2}\left((\Phi^{f+})'(k,0+)+(\Phi^{f-})'(k,0+)\right)\right)$$
$$\times\int_0^\infty da\,(q(a)\Phi^{f+}(k,a)+q(-a)\,\Phi^{f-}(k,a))$$

where q and p are positive functions.

From (3.13) and the discussion below (2.6) it is clear that the function $U^{\lambda f}(k,x)$ in (1.1) and (2.1) is expressed in present notation as

$$U^{\lambda f}(k,x)=\frac{2\left(\Phi^{\lambda f+}(k,x)1(x>0)+\Phi^{\lambda f-}(k,-x)1(x<0)\right)}{-(\Phi^{\lambda f+})'(k,0+)-(\Phi^{\lambda f-})'(k,0+)} \tag{3.14}$$

It follows also that $U(x):=U^{\lambda f}(k,x)$ satisfies the Sturm-Liouville equation (1.2) with the boundary conditions described around (1.3), in particular $U'(0+)-U'(0-)=-2$. That the U so obtained is unique requires a further analytic argument. Note how the presentation of Kac's basic function in (3.14) as a ratio reflects the path decomposition at time g_{θ_k}.

3.3 Relations with the agreement formula

We now examine how the agreement formula (2.10) translates in terms of solutions of certain Sturm-Liouville equations.

Theorem 3.6 *Let $f_+ : \mathbb{R}_+ \to \mathbb{R}_+$ be a locally bounded Borel function. Keep the notation $\Phi^{f_+}(k,x) = E_x\left[\exp -\left(\frac{k^2}{2}T_0 + A^{f_+}_{T_0}\right)\right]$ and define another function $\Psi^{f_+}(k,x)$ by*

$$\frac{x}{\Psi^{f_+}(k,x)} := E_0^{(3)}\left[\exp -\left(\frac{k^2}{2}T_x + A^{f_+}_{T_x}\right)\right]$$

Then
i) $\Psi = \Psi^{f_+}(k,x)$ is the unique solution of

$$\frac{1}{2}\Psi''(x) = \left(\frac{k^2}{2} + f_+(x)\right)\Psi(x)\,;\ \Psi(0)=0;\ \Psi'(0+)=1\,;\ x \le \Psi(x)$$

ii) The function $\Psi^{f_+}(k,x)$ is an increasing function of x, and the following relation holds:

$$-\,(\Phi^{f_+})'(k,0+) = \int_0^\infty dx\left(\frac{1}{x^2} - \frac{1}{(\Psi^{f_+}(k,x))^2}\right) \tag{3.15}$$

Proof: Part i) follows from (3.4). Turning to ii), the left and right hand sides of (3.15) are the left and right hand sides of the agreement formula (2.10) applied to the functional

$$1 - \exp\left(-\frac{k^2}{2}\zeta(\omega) - \int_0^{\zeta(\omega)} ds\, f_+(\omega(s))\right)$$

where $\zeta(\omega)$ is the lifetime of the generic excursion $(\omega(s), 0 \le s \le \zeta(\omega))$. $\square$

As an alternative approach, here is an elementary proof of (3.15). As observed in [31]

$$\frac{1}{\Psi^2(m)} = -\frac{d}{dm}\left(\int_0^m \frac{dy}{\Phi^2(y)}\right)^{-1} \tag{3.16}$$

Hence, the right hand side of (3.15) is equal to the limit, when ϵ tends to 0 of

$$I_\epsilon := \int_\epsilon^\infty dm\left(\frac{1}{m^2} + \frac{d}{dm}\left(\int_0^m \frac{dy}{\Phi^2(y)}\right)^{-1}\right) = \frac{1}{\epsilon} - \left(\int_0^\epsilon \frac{dy}{\Phi^2(y)}\right)^{-1} = \frac{\int_0^\epsilon dy\left(\frac{1}{\Phi^2(y)} - 1\right)}{\epsilon\int_0^\epsilon \frac{dy}{\Phi^2(y)}}$$

Since $\Phi(0)=1$, it follows that as $\epsilon \downarrow 0$

$$I_\epsilon \simeq \left(\frac{1}{\epsilon^2}\right)\int_0^\epsilon dy(1-\Phi^2(y)) \simeq \frac{2}{\epsilon^2}\int_0^\epsilon dy(1-\Phi(y)) \simeq -\Phi'(0+) \qquad \square$$

We now comment on some variations of the identity (3.16). Since both Φ and Ψ are non-negative, the identity (3.16) is equivalent to

$$\Psi(m) = \Phi(m) \int_0^m \frac{dy}{\Phi^2(y)} \tag{3.17}$$

which in turn is a variation of the Wronskian identity

$$\Psi'\Phi - \Phi'\Psi = 1. \tag{3.18}$$

As a complement to (3.16) and the discussion of Φ and Ψ given at the beginning of this section, we deduce that for a C^1 function $\theta : I\!R_+ \to I\!R_+$, with compact support,

$$\int_0^\infty dx\, \theta(x) \left(\frac{1}{x^2} - \frac{1}{\Psi^2(x)} \right) = -\theta(0)\Psi'(0) + \int_0^\infty dx\, \theta'(x) \left(\frac{1}{x} - \left(\int_0^x \frac{dy}{\Phi^2(y)} \right)^{-1} \right) \tag{3.19}$$

Writing $\theta(x) - \theta(0) = \int_0^x dy\, \theta'(y)$, and dividing both sides of (3.19) by 2, we deduce from (2.9) and (2.10) that

$$\frac{1}{2} \left(\frac{1}{x} - \left(\int_0^x \frac{dy}{\Phi^2(y)} \right)^{-1} \right) = \mathbf{n}_+(M \geq x) - \int_{\{M \geq x\}} \mathbf{n}_+(d\varepsilon) \exp\left(- \int_0^V ds\, f(\varepsilon_s) \right)$$

where $M = \sup_{s \geq 0} \varepsilon(s)$, and $V = \inf\{s > 0 : \varepsilon(s) = 0\}$. Equivalently,

$$\left(2 \int_0^x \frac{dy}{\Phi^2(y)} \right)^{-1} = \int_{\{M \geq x\}} \mathbf{n}_+(d\varepsilon) \exp\left(- \int_0^V ds\, f(\varepsilon_s) \right) \tag{3.20}$$

4 Examples

Example 0: $f_\pm(x) = \dfrac{\lambda_\pm^2}{2}$

We start with the constant function $f_\pm(x) = \dfrac{\lambda^2}{2}$. Then, we have $\Phi^{f_\pm}(k,a) = \exp(-\nu a)$, where $\nu = (k^2 + \lambda^2)^{1/2}$. We now check formula (3.12) in this case: easily, on the left-hand side, we find

$$E[\exp(-\frac{\lambda^2}{2}\theta_k)] = \frac{k^2}{\nu^2} \tag{4.1}$$

On the other hand, $-(\Phi^{f_+})'(k, 0+) - (\Phi^{f_-})'(k, 0+) = 2\nu$, whereas the integral $\displaystyle\int_0^\infty da(\Phi^{f_+}(k,a) + \Phi^{f_-}(k,a))$ is equal to $(2/\nu)$. Finally, both sides of (3.12) are equal to (k^2/ν^2).

A more interesting case, which following Kac [20] has been presented many times in the literature [11, 23, 17, 33, 41], arises when $f_+(x) = \frac{\lambda_+^2}{2}$, $f_-(x) = \frac{\lambda_-^2}{2}$, with $\lambda_+ \neq \lambda_-$. The previous (trivial) formula (4.1) is then modified as follows:

$$E\left[\exp -\left(\frac{\lambda_+^2}{2}\int_0^{\theta_k} ds\, 1_{\{B_s>0\}} + \frac{\lambda_-^2}{2}\int_0^{\theta_k} ds\, 1_{\{B_s<0\}}\right)\right] = \frac{k^2}{\nu_+\nu_-} \tag{4.2}$$

where $\nu_\pm = (k^2 + \lambda_\pm^2)^{1/2}$. Consequently, the random times $\int_0^{\theta_k} ds\, 1_{\{B_s>0\}}$ and $\int_0^{\theta_k} ds\, 1_{\{B_s<0\}}$ are independent with a common gamma distribution with shape parameter 1/2. Lévy's arc sine law for the distribution of $\int_0^1 ds\, 1_{\{B_s>0\}}$ is a well known consequence.

In the case $f_\pm(x) = \frac{\lambda^2}{2}$, it is well known that $\frac{1}{\Psi(m)} = \frac{\nu}{\sinh(m\nu)}$. Hence, the equality (3.15) then reduces to

$$\nu = \int_0^\infty dm \left(\frac{1}{m^2} - \left(\frac{\nu}{\sinh(m\nu)}\right)^2\right). \tag{4.3}$$

With the change of variables : $m = x/\nu$, the right-hand-side of (4.3) is equal to $\int_0^\infty dx \left(\frac{\nu}{x^2} - \frac{\nu}{(\sinh x)^2}\right)$, so that (4.3) reduces to

$$\int_0^\infty dx \left(\frac{1}{x^2} - \frac{1}{(\sinh x)^2}\right) = 1 \tag{4.4}$$

In fact, the following expression of the Fourier transform of $\left(\frac{1}{x^2} - \frac{1}{(\sinh x)^2}, x \in I\!R\right)$ is well known to be

$$\int_{-\infty}^\infty dx \exp(i\xi x)\left(\frac{1}{x^2} - \frac{1}{(\sinh x)^2}\right) = \pi\left(\xi \coth\frac{\pi\xi}{2} - |\xi|\right) \tag{4.5}$$

which easily yields (4.4), by letting $\xi \to 0$.

Example 1: $f(x) = \lambda|x|,\ \lambda > 0$

One of the first applications, if not the first, of the Feynman-Kac formula, has been to compute the law of $\int_0^t du\, |B_u|$ (see Cameron-Martin [9] and Kac [19]). A number of variants of this computation, where B is replaced by a 1-dimensional or 3-dimensional Bessel bridge have been made [37, 39], and continue to be of interest. See e.g. Perman-Wellner [28], which contains a general discussion similar

to our Sections 2 and 3.
In order to exploit the previous results, we introduce the Airy function Ai which is the unique bounded solution of

$$u'' = xu,\ u(0) = 1$$

and is defined as

$$(Ai)(x) := \frac{1}{\pi}\left(\frac{x}{3}\right)^{1/2} K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

where K_ν is the usual modified Bessel function with index ν. The solution of

$$u'' = (k^2 + \lambda x)u\,,\, u(0) = 1$$

can be written as

$$\frac{(Ai)[(x\lambda + k^2)/\lambda^{2/3}]}{(Ai)[k^2/\lambda^{2/3}]}\,.$$

Some easy computations lead to

$$E(\exp -\lambda \int_0^{g_{\theta_k}} du|B_u|) = \frac{K_{1/3}}{K_{2/3}}(\mu) \tag{4.6}$$

$$E(\exp -\lambda \int_{g_{\theta_k}}^{\theta_k} du|B_u|) = \frac{1}{K_{1/3}(\mu)} \int_\mu^\infty dv\, K_{1/3}(v) \tag{4.7}$$

$$E(\exp -\lambda \int_0^{\theta_k} du|B_u|) = \frac{1}{K_{2/3}(\mu)} \int_\mu^\infty dv\, K_{1/3}(v) \tag{4.8}$$

with $\mu = \dfrac{k^3}{3\lambda}$.
The quantity $E(\exp -\lambda \int_0^t du\,|B_u|)$ is computed in Kac [19], the density and the moments of $\int_0^1 du\,|B_u|$ appear in Takács [39] and are related with the Airy function and its zeros. Getoor and Sharpe [14] provide the computation of $E(\exp -\lambda \int_{g_{\theta_k}}^{d_{\theta_k}} du|B_u|)$ which can be obtained also from the previous results, together with the Markov property, which yields:

$$E(\exp[-\lambda \int_t^{d_t} du|B_u| - \alpha|B_t|]) = E\left(\exp(-\alpha|B_t|)\, E_{|B_t|}(\exp - \int_0^{T_0} du|B_u|)\right).$$

Using the equality $g_{\theta_k} \overset{law}{=} \theta_k - g_{\theta_k} \overset{law}{=} (N/k)^2$ where N is a standard $N(0,1)$ variable and the scaling property of Brownian motion, we obtain

$$E(\exp -\frac{\lambda}{k^3}|N|^3 \int_0^1 du|b_u|) = \frac{K_{1/3}}{K_{2/3}}(\mu)$$

$$E(\exp -\frac{\lambda}{k^3}|N|^3 \int_0^1 du\, m_u) = \frac{1}{K_{1/3}(\mu)} \int_\mu^\infty dv\, K_{1/3}(v)$$

where b is a Brownian bridge and m a Brownian meander. These formulae can be found in Biane-Yor [6], Groeneboom [15], Shepp [37], Perman-Wellner [28].

Example 2: $f(x) = i\lambda x$

to We now exploit the Gaussian character of $(B_t, t \geq 0)$. Let $\theta := \theta_1$ be an exponential time with parameter $1/2$. The equality

$$E[\exp i(\lambda \int_0^\theta du\, B_u + \mu B_\theta)] = \\ E[\exp i(\lambda \int_0^{g_\theta} du\, B_u)]\, E[\exp i(\lambda \int_{g_\theta}^\theta du\, B_u + \mu(B_\theta - B_{g_\theta}))]$$

can be transformed in

$$E(\exp i(\lambda\theta^{3/2} \int_0^1 du\, B_u + \mu\theta^{1/2} B_1)) = \\ E[\exp(i\lambda |N|^3 \int_0^1 du\, b_u)]\, E(\exp i(\lambda \tilde{N}^3 \int_0^1 du\, m_u + \tilde{N}\mu m_1)) \tag{4.9}$$

where $N \stackrel{law}{=} \tilde{N} \stackrel{law}{=} \epsilon(g_\theta)^{1/2} \stackrel{law}{=} \epsilon(\theta - g_\theta)^{1/2}$, with N and $\tilde{N}$ standard $N(0,1)$ variables, ϵ a symmetric Bernoulli variable independent of the pair (θ, g_θ), $(b_u; u \leq 1)$ a standard Brownian bridge, independent of N, and $(m_u; u \leq 1)$ a Brownian meander, independent of $\tilde{N}$. Two among the three expectations found in (4.9) can be evaluated thanks to the following elementary

Lemma 4.1 *For every $(\lambda, \mu) \in I\!R$, one has*

$$E\left[\left(\lambda \int_0^1 du\, B_u + \mu B_1\right)^2\right] = \mu^2 + \lambda\mu + \frac{\lambda^2}{3}$$

Moreover,

$$E\left[\left(\int_0^1 ds\, b_s\right)^2\right] = \frac{1}{12}$$

Proof: The last equality in this lemma follows from the obvious remark

$$\int_0^1 du\, B_u = \int_0^1 du\,(B_u - uB_1) + \int_0^1 du\, uB_1 \stackrel{law}{=} \int_0^1 du\, b_u + \frac{1}{2} B_1$$

where $(b_u := B_u - uB_1, 0 \leq u \leq 1)$ is independent of B_1. □

Plugging the results of the lemma into (4.9), one obtains

$$E\left[\exp i\left(\lambda\theta^{3/2} \int_0^1 du\, B_u + \mu\theta^{1/2} B_1\right)\right] = E\left[\exp -\frac{1}{2}\left(\mu^2\theta + \lambda\mu\theta^2 + \frac{\lambda^2\theta^3}{3}\right)\right]$$

whereas

$$E\left[\exp\left(i\lambda |N|^3 \int_0^1 du\, b_u\right)\right] = E\left[\exp\left(-\frac{\lambda^2 N^6}{24}\right)\right]$$

and, in an obvious way

$$E\left[\exp i\left(\lambda \int_0^t du\, B_u + \mu B_t\right)\right] = \exp -\frac{1}{2}\left(\mu^2 t + \lambda\mu t^2 + \frac{\lambda^2 t^3}{3}\right).$$

Hence, we deduce from (4.9) the formula

$$E\left[\exp i\left(\lambda\tilde{N}^3\int_0^1 du\, m_u + \mu\tilde{N}m_1\right)\right] = \frac{E\left[\exp -\frac{1}{2}\left(\mu^2\theta + \lambda\mu\theta^2 + \frac{\lambda^2\theta^3}{3}\right)\right]}{E\left[\exp\left(-\frac{\lambda^2 N^6}{24}\right)\right]} \tag{4.10}$$

Example 3: $f(x) = \frac{\lambda^2}{2}x^2$,

This is an extremely important example, closely related to Ornstein-Uhlenbeck processes and P. Lévy's area formula [27], which may be tackled by using several different techniques. We refer to Ikeda et al. [16] for a number of extensions and references to recent developments of Lévy's area formula.

a. Feynman-Kac formula

The bounded solution of

$$u'' = (k^2 + \lambda^2 x^2)u\,,\ u(0) = 1$$

is

$$u(x) = \frac{1}{2^{\nu/2}\Gamma(1/2)}\Gamma\left(\frac{1-\nu}{2}\right) D_\nu(x\sqrt{2\lambda})$$

where D_ν is a parabolic cylinder function (See Lebedev [24] 10.2.1, 10.2.18) and $\nu = -\frac{1}{2}(\frac{k^2}{\lambda} + 1)$. We obtain, using various results on functions D_ν found in Lebedev

$$E\left[\exp -\frac{\lambda^2}{2}\int_0^{\theta_k} du\, B_u^2\right] = \frac{k^2}{2}\int_0^\infty dt\, \exp\left(-\frac{k^2}{2}t\right)\ \frac{1}{(\cosh t\lambda)^{1/2}}$$

which, from the uniqueness of the Laplace transform gives

$$E\left[\exp -\frac{\lambda^2}{2}\int_0^t du\, B_u^2\right] = \frac{1}{(\cosh t\lambda)^{1/2}}\,, \tag{4.11}$$

a formula which traces back to Cameron-Martin [9]. See also Kac [20], Breiman [8].

b. Direct computation

It is well known that the previous result can be obtained as a consequence of Girsanov's transformation (see e.g. [46], chap. 2), working up to a fixed time t. More generally, we obtain

$$E\left[\exp -\left(\nu B_t^2 + \frac{\lambda^2}{2}\int_0^t du\, B_u^2\right)\right] = \left(\cosh(t\lambda) + \frac{2\nu}{\lambda}\sinh(t\lambda)\right)^{-1/2} \tag{4.12}$$

which leads in particular to (4.11) and, if the Brownian motion starts from a, to

$$E_a\Big[\exp -\Big(\nu B_t^2 + \frac{\lambda^2}{2}\int_0^t du\, B_u^2\Big)\Big] = \frac{\exp\left(-\dfrac{a^2\lambda}{2}\,\dfrac{1+\frac{2\nu}{\lambda}\coth\lambda t}{\coth(\lambda t)+\frac{2\nu}{\lambda}}\right)}{\left(\cosh(t\lambda)+\dfrac{2\nu}{\lambda}\sinh(t\lambda)\right)^{1/2}}.$$

If $(R_t, t \ge 0)$ denotes a δ-dimensional Bessel process starting from a, whose law we denote by $P_a^{(\delta)}$, one obtains, using formula (4.12) and the additivity of squared Bessel processes

$$E_a^{(\delta)}\left[\exp -\frac{\lambda^2}{2}\int_0^t ds\, R_s^2 \,\middle|\, R_t = 0\right] = E_0^{(\delta)}\left[\exp -\frac{\lambda^2}{2}\int_0^t ds\, R_s^2 \,\middle|\, R_t = a\right]$$

$$= \left(\frac{t\lambda}{\sinh(t\lambda)}\right)^{\delta/2} \exp\left(-\frac{a^2}{2t}(t\lambda\,\coth(t\lambda)-1)\right) \qquad (4.13)$$

This formula is closely related to Lévy's area formula (See [46], p. 18, and [30]) In particular, we obtain

$$E\Big[\exp -\frac{\lambda^2}{2}\int_0^1 du\, b_u^2\Big] = \Big(\frac{\lambda}{\sinh\lambda}\Big)^{1/2}$$

Using the identity $(m_u^2; u \le 1) \stackrel{law}{=} (b_u^2 + R_u^2; u \le 1)$ where R is a BES(2) independent of b (see [46], p. 44) we obtain

$$E\Big[\exp -\frac{\lambda^2}{2}\int_0^1 du\, m_u^2\Big] = \frac{1}{(\cosh\lambda)}\Big(\frac{\lambda}{\sinh\lambda}\Big)^{1/2}.$$

Moreover, from the equality

$$\int_0^1 du\, B_u^2 = g_1^2\int_0^1 du\, b_u^2 + (1-g_1)^2\int_0^1 du\, m_u^2,$$

it follows that

$$\frac{1}{(\cosh\nu)^{1/2}} = \frac{1}{\pi}\int_0^1 \frac{dt}{\sqrt{t(1-t)}}\varphi_0(\nu t)\varphi_1(\nu(1-t))$$

where $\varphi_0(\nu) = \Big(\dfrac{\nu}{\sinh\nu}\Big)^{1/2}$, $\varphi_1(\nu) = \dfrac{1}{(\cosh\nu)}\Big(\dfrac{\nu}{\sinh\nu}\Big)^{1/2}$.

It follows that there is the convolution identity

$$\frac{1}{(\cosh\nu)^{1/2}} = \frac{1}{\pi}\int_0^\nu \frac{dx}{(\sinh x)^{1/2}\,(\sinh(\nu-x))^{1/2}\,\cosh(\nu-x)}.$$

This convolution identity can also be derived from the product formula

$$E\left[\exp(-\lambda\int_0^\theta du\, B_u^2)\right] = E\left[\exp(-\lambda\int_0^{g_\theta} du\, B_u^2)\right]\, E\left[\exp(-\lambda\int_{g_\theta}^\theta du\, B_u^2)\right]$$

where θ is an exponential variable with parameter 1/2. This last formula is a consequence of the independence property discussed in subsection 2.2.

Example 4: $f_+(x) = \lambda^2/2x$

Here we have to work around the fact that f_+ is unbounded at $0+$. We are interested in the law of the principal value

$$H_t := \text{p.v.} \int_0^t \frac{ds}{B_s} := \lim_{\epsilon\to 0} \int_0^t \frac{ds}{B_s} 1_{\{|B_s|\geq\epsilon\}}$$

and the law of

$$H_t^+ := \lim_{\epsilon\to 0} \left(\int_0^t \frac{ds}{B_s} 1_{\{B_s\geq\epsilon\}} + (\ln\epsilon)\ell_t^0 \right).$$

See Biane-Yor [6] for the existence of such principal values, and Yamada [45] for a more general discussion. A decreasing solution of

$$\Upsilon''(x) = \left(\frac{\lambda^2}{x} + k^2\right) \Upsilon(x)$$

(called hereafter the Whittaker function- see Lebedev [24] p. 274 and 279) is obtained as follows:

$$\Upsilon(z) = e^{-kz}\, 2kz\, \Gamma(1+a)\, U\,(1+a; 2; 2kz) \tag{4.14}$$

where

$$\Gamma(1+a)\, U(1+a;2;z) = \int_0^\infty dt\, e^{-zt} \Big(\frac{t}{1+t}\Big)^a \text{ and } a = \frac{\lambda^2}{2k}.$$

Since $\Upsilon(0) = 1$ we obtain, for $x > 0$

$$\begin{aligned} \Phi^{f_+}(k,x) &:= E_x\Big[\exp -\Big(\frac{k^2}{2}T_0 + \frac{\lambda^2}{2}\int_0^{T_0}\frac{ds}{B_s}\Big)\Big] \\ &= 2kx\, \exp(-kx) \int_0^\infty dt \exp(-2kxt) \Big(\frac{t}{1+t}\Big)^{\lambda^2/2k}. \end{aligned}$$

As a first step, we compute the law of the pair $(H_{\tau_t}^+, \tau_t^+)$ where $\tau_t^+ := \int_0^\infty da\, \ell_{\tau_t}^a$ and (τ_t) is the inverse of $(\ell_t^0, t \geq 0)$. Here, as in the previous section, $(\ell_t^a; t\geq 0, a \in I\!R)$ is the process of Brownian local times. From the definition of H_t^+,

$$E\left[\exp\left(-\frac{\lambda^2}{2}\, H_{\tau_t}^+ - \frac{k^2}{2}\tau_t^+\right)\right] = \lim_{\epsilon\to 0}\left[I_\epsilon\, \exp\left(-\frac{t\lambda^2}{2}\ln\epsilon\right)\right]$$

where

$$I_\epsilon := E\,\left[\exp\left(-\frac{\lambda^2}{2}\int_\epsilon^\infty \frac{da}{a}\ell_{\tau_t}^a - \frac{k^2}{2}\int_\epsilon^\infty da\, \ell_{\tau_t}^a\right)\right]$$

From corollary (3.4), it follows that $I_\epsilon = \exp\left(\frac{t}{2}u'_{(\epsilon)}(k,0+)\right)$ where $u_{(\epsilon)}(k,x)$ is the decreasing solution of

$$\left\{ \begin{array}{rcll} u''(x) &=& (\dfrac{\lambda^2}{x} + k^2)u(x) & \text{on } [\epsilon,\infty[\\ u(x) &=& \alpha x + 1 & \text{on } [0,\epsilon] \end{array} \right.$$

for some α determined by the continuity properties of u and u' at $x = \epsilon$. Hence $u'_{(\epsilon)}(0+) = \alpha = \dfrac{\Upsilon'(\epsilon)}{\Upsilon(\epsilon) - \epsilon\Upsilon'(\epsilon)}$, where Υ is the Whittaker function defined in (4.14). Therefore, the problem reduces to the computation of

$$\lim_{\epsilon \to 0} \left(\frac{\Upsilon'(\epsilon)}{\Upsilon(\epsilon) - \epsilon\Upsilon'(\epsilon)} - \lambda^2 \ln \epsilon \right)$$

and this last quantity is equal to

$$\lim_{\epsilon \to 0} \left(\Upsilon'(\epsilon) - \lambda^2 (\ln \epsilon)(\Upsilon(\epsilon) - \epsilon\Upsilon'(\epsilon)) \right)$$

since $\lim_{x\to 0} x\Upsilon'(x) = 0$ and $\Upsilon(0) = 1$. The derivative of Υ is given by

$$\Upsilon'(\epsilon) = 2k \left[-\frac{1}{2}\Upsilon(\epsilon) - \frac{\lambda^2}{2k} e^{k\epsilon} \int_0^\infty du\, \frac{\exp(-u)}{u + 2k\epsilon} \left(\frac{u}{u + 2k\epsilon} \right)^{\lambda^2/2k} \right]$$

Using the asymptotic behavior, when $z \to 0$

$$\int_0^\infty du\, \frac{\exp(-u)}{u+z} \left(\frac{u}{u+z} \right)^a = -(\ln z + \frac{\Gamma'}{\Gamma}(a+1)) + O(z|\ln z|)\,,$$

it follows that the needed limit is

$$-k + \lambda^2 (\ln(2k) + \frac{\Gamma'}{\Gamma}(\frac{\lambda^2}{2k} + 1))$$

We now symmetrize the formula, paying attention to the positive part and the negative part of the excursions, and working in the complex field, and we obtain

$$E\left[\exp\left(i\lambda\, H_{\tau_t} - \frac{k^2}{2}\tau_t \right) \right] = \exp\left\{ -t \left(k + 2\lambda \mathrm{Im}\, \frac{\Gamma'}{\Gamma} \left(1 + \frac{i\lambda}{k} \right) \right) \right\}$$

and from the series development of $\dfrac{\Gamma'}{\Gamma}$, we find that

$$k + 2\lambda \mathrm{Im}\, \frac{\Gamma'}{\Gamma}(1 + \frac{i\lambda}{k}) = \pi\lambda \coth(\pi\frac{\lambda}{k})\,.$$

This formula was obtained in Biane-Yor[6] by a different argument involving excursion theory and time changes.

Example 5: generic powers $f_+(x) = \lambda^\beta x^\alpha$

The previous examples share the common property that f is a multiple of a power of x. Here, we try to obtain results for general powers. Let f be defined as $f(x) = |x|^\alpha$ and $\tilde{f}(x) = (\mathrm{sgn}\, x)|x|^\alpha$. If $\alpha > -1$, the processes A^f and $A^{\tilde{f}}$ are well defined because $\int_0^t ds\, |B_s|^\alpha < \infty$ almost surely. Moreover, it is well known [6] that

$$E(\exp -\lambda A^f(\tau_t)) = \exp\left(-tc_\alpha \lambda^{1/(\alpha+2)} \right)$$

where

$$c_\alpha := \frac{\pi 2^\nu}{\nu \sin(\pi\nu)} \left(\frac{\nu^\nu}{\Gamma(\nu)}\right)^2 ; , \ \nu := \frac{1}{\alpha+2}$$

When $-1 \geq \alpha > -\frac{3}{2}$, it is possible (see Biane-Yor [6], Yamada [45] and Yor [48]) to extend the definition of $A^{\tilde{f}}$ as follows

$$A^{\tilde{f}}(t) = \lim_{\epsilon \to 0} \int_0^t du |B_u|^\alpha \operatorname{sgn}(B_u) 1_{\{|B_u| \geq \epsilon\}}$$

The process $A^{\tilde{f}}$ has paths of unbounded variation with zero quadratic variation. It is known that the process $(A^{\tilde{f}}(\tau_t), t \geq 0)$ is a symmetric stable Lévy process [6].

We now point out some consequences of the scaling property of Brownian motion, which are reflected in the corresponding Sturm-Liouville computations when $f(x) = \lambda^\beta |x|^\alpha$ for a general α. Here, we take $\beta = 1 + \dfrac{\alpha}{2}$, for convenience in the subsequent computations. We first note that, thanks to the scaling property, the function of three variables

$$u_{(\alpha)} : (\lambda, k, b) \to u^\lambda_{(\alpha)}(k, b) := E_b \left[\exp - \left(\frac{k^2}{2} T_0 + \lambda^\beta \int_0^{T_0} ds\, B_s^\alpha\right)\right],$$

where $\lambda, k, b > 0$, may be expressed in terms of a function of two variables only.

Lemma 4.2 *The above function $u_{(\alpha)}$ satisfies*

$$u^\lambda_{(\alpha)}(k, b) = w_{\lambda/k^2}(kb)$$

where, for $\mu > 0$, w_μ is the only positive, decreasing, solution of the Sturm-Liouville equation

$$w''(x) = (1 + 2\mu^\beta x^\alpha) w(x), \ w(0) = 1$$

Except for $\alpha = 2$ and $\alpha = \pm 1$, we do not know of any reference to the function w_μ in the literature of Sturm-Liouville equations. Nevertheless, from the previous general probabilistic discussion (i.e., relations between Sturm-Liouville equations and Feynman-Kac solutions) it turns out that certain quantities associated with w_μ may be written in terms of the following Laplace transforms $\varphi_{(\alpha)}, \varphi^0_{(\alpha)}, \varphi^1_{(\alpha)}, \varphi^3_{(\alpha)}$, where

$$\varphi_{(\alpha)}(\lambda) = E\left[\exp\left(-\lambda^\beta \int_0^1 dt\, |B_t|^\alpha\right)\right] \qquad \varphi^0_{(\alpha)}(\lambda) = E\left[\exp\left(-\lambda^\beta \int_0^1 dt\, |b_t|^\alpha\right)\right]$$

$$\varphi^1_{(\alpha)}(\lambda) = E\left[\exp\left(-\lambda^\beta \int_0^1 dt\, m_t^\alpha\right)\right] \qquad \varphi^3_{(\alpha)}(\lambda) = E\left[\exp\left(-\lambda^\beta \int_0^1 dt\, r_t^\alpha\right)\right]$$

Here, $(B_t, 0 \leq t \leq 1)$ denotes a 1-dimensional Brownian motion starting from 0, $(b_t, 0 \leq t \leq 1)$ a 1-dimensional Brownian bridge (tied to 0 from both $t = 0$ and $t = 1$), $(m_t, 0 \leq t \leq 1)$ a Brownian meander, and $(r_t, 0 \leq t \leq 1)$ a 3-dimensional

Bessel bridge. The mnemonic for the suffixes 0,1,3 is that 0 indicates a $(0,0)$ Brownian bridge, 1 indicates the meander obtained by Brownian scaling on $[g_1, 1]$, and 3 indicates the 3-dimensional Bessel bridge. We also use the notation $m^{(s)}$ (resp. $b^{(s)}$) for the meander of length s (resp. the bridge of length s). Here are some relations between the functions $\varphi^0, \varphi^1, \varphi^3$ and w_μ where we have suppressed the subscript (α):

Proposition 4.3 *The following identities hold*

$$\begin{aligned}
&1) \qquad & \int_0^\infty db\, w_\mu(b) &= \frac{1}{\sqrt{\mu}} \int_0^\infty \frac{dt}{\sqrt{2\pi t}} \exp(-\frac{t}{2\mu})\, \varphi^1(t) \\
&2) \qquad & -w'_\mu(0+) &= \sqrt{\mu} \left[\int_0^\infty \frac{dt}{\sqrt{2\pi t}} \exp(-\frac{t}{2\mu})\, \varphi^0(t) \right]^{-1} \\
&3) \qquad & -w'_\mu(0+) &= \sqrt{\mu} \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \left(1 - \exp(-\frac{t}{2\mu})\, \varphi^3(t) \right)
\end{aligned}$$

Proof: Using the equality (for a symmetric function f)

$$\begin{aligned}
E\left[\exp -\left(A^f_{\theta_k} - A^f_{g_{\theta_k}}\right)\right] &= k \int_0^\infty da\, E_a \left[\exp - \left(\frac{k^2}{2} T_0 + A^f_{T_0} \right) \right] \\
&= k \int_0^\infty \frac{ds}{\sqrt{2\pi s}} \exp\left(-\frac{k^2 s}{2} \right) E\left[\exp - \int_0^s dt\, f(m_t^{(s)}) \right]
\end{aligned}$$

we obtain, in the case $f(x) = \lambda^\beta |x|^\alpha$

$$k \int_0^\infty da\, w_{\lambda/k^2}(ka) = \frac{k}{\sqrt{\lambda}} \int_0^\infty \frac{dt}{\sqrt{2\pi t}} \exp\left(-\frac{k^2 t}{2\lambda} \right) \varphi^1(t)$$

which proves the first part of the proposition.
From the formula

$$P^{g_{\theta_k}} \equiv k \int_0^\infty d\ell \Big(\exp\Big(-\frac{k^2}{2} \tau_\ell \Big) \cdot P^{\tau_\ell} \Big) = k \int_0^\infty \frac{ds}{\sqrt{2\pi s}} \exp\left(-\frac{k^2 s}{2} \right) \cdot Q^s$$

where Q^s denotes the law of the Brownian bridge $(b_t^{(s)}, t \le s)$, we deduce

$$\begin{aligned}
\frac{-1}{u'_{(\alpha)}(k, 0+)} &= \int_0^\infty \frac{ds}{\sqrt{2\pi s}} \exp\left(-\frac{k^2 s}{2} \right) E\left[\exp\left(-\lambda^\beta \int_0^s dt\, |b_t^{(s)}|^\alpha \right) \right] \\
&= \int_0^\infty \frac{ds}{\sqrt{2\pi s}} \exp\left(-\frac{k^2 s}{2} \right) E\left[\exp\left(-(\lambda s)^\beta \int_0^1 dt\, |b_t|^\alpha \right) \right] \\
&= \int_0^\infty \frac{ds}{\sqrt{2\lambda\pi s}} \exp\left(-\frac{k^2 s}{2\lambda} \right) E\left[\exp\left(-s^\beta \int_0^1 dt\, |b_t|^\alpha \right) \right]
\end{aligned}$$

Setting $\mu = \lambda/k^2$, and using the relation between the functions w_μ and u, it follows that

$$\frac{-1}{w'_\mu(0+)} = \frac{1}{\sqrt{\mu}} \int_0^\infty \frac{ds}{\sqrt{2\pi s}}\, e^{-\frac{s}{2\mu}}\, E\left[\exp\left(-s^\beta \int_0^1 dt\, |b_t|^\alpha \right) \right]$$

which is part 2 of the proposition. Part 3 follows from (3.11). □

We also note the following relations between $\varphi, \varphi^0, \varphi^1, \varphi^3$

Proposition 4.4 *The following identities hold*

$$1) \qquad \varphi(\lambda) = \int_0^1 \frac{ds}{\pi\sqrt{s(1-s)}}\, \varphi^0(\lambda s)\, \varphi^1(\lambda(1-s))$$

$$2) \qquad \left(\int_0^\infty \frac{dt}{\sqrt{2\pi t}} \exp\left(-\frac{t}{2\mu}\right) \varphi^0(t)\right)\left(\int_0^\infty \frac{ds}{\sqrt{2\pi s^3}}\left(1 - \exp\left(-\frac{s}{2\mu}\right) \varphi^3(s)\right)\right) = 1$$

Proof: Thanks to the scaling property, we obtain

$$\int_0^1 ds |B_s|^\alpha \stackrel{law}{=} g_1^\beta \left(\int_0^1 ds\, |b_s|^\alpha\right) + (1-g_1)^\beta \left(\int_0^1 ds\, m_s^\alpha\right)$$

therefore

$$\varphi_{(\alpha)}(\lambda) = \int_0^1 \frac{ds}{\pi\sqrt{s(1-s)}}\, \varphi^0_{(\alpha)}(\lambda s)\, \varphi^1_{(\alpha)}(\lambda(1-s))$$

The second equality follows from the identities 2) and 3) in proposition 4.3. □

Example 6: Exponential functions $f_\pm(x) = \dfrac{\lambda^2}{2} e^{\alpha x}$

This case has been studied in [46, 47] where other results are stated. We give here the formulae in the case $\alpha = 2$, the general case follows easily, using the scaling property. Let us define

$$A_t^\pm := \int_0^t ds \exp(2B_s)\, 1_{\{B_s \in \mathbb{R}_\pm\}}\ ,\ \tau_\ell^\pm := \int_0^{\tau_\ell} du 1_{\{B_u \in \mathbb{R}_\pm\}}$$

The associated Sturm-Liouville equation is easily solved, and one has, for $a \geq 0,\ k \geq 0,\ \lambda > 0$:

$$E_a\Big[\exp -(\frac{k^2}{2}T_0 + \frac{\lambda^2}{2}A_{T_0})\Big] = \frac{K_k(\lambda e^a)}{K_k(\lambda)} \tag{4.15}$$

$$E_{-a}\Big[\exp -(\frac{k^2}{2}T_0 + \theta^2 A_{T_0})\Big] = \frac{I_k(\lambda e^{-a})}{I_k(\lambda)}\,,$$

where I_k and K_k are modified Bessel functions and

$$\begin{aligned} E_0\Big[\exp -(\frac{k^2}{2}\tau_\ell^+ + \frac{\lambda^2}{2}A^+_{\tau_\ell})\Big] &= \exp -\frac{\ell}{2}\Big(\frac{\lambda K_{k+1}(\lambda)}{K_k(\lambda)} - k\Big) \\ E_0\Big[\exp -(\frac{k^2}{2}\tau_\ell^- + \frac{\lambda^2}{2}A^-_{\tau_\ell})\Big] &= \exp -\frac{\ell}{2}\Big(\frac{\lambda I_{k-1}(\lambda)}{I_k(\lambda)} - k\Big) \end{aligned}$$

This leads to analogous formulae for 3-dimensional Bessel bridge:

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\Big(1 - \exp(-\frac{k^2 t}{2})\ E_0^{(3)}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t ds\ \exp(-2R_s)\Big)\Big| R_t = 0\Big]\Big)$$

is equal to

$$\frac{\lambda I_{k-1}(\lambda)}{I_k(\lambda)} - k = \frac{\lambda I_{k+1}(\lambda)}{I_k(\lambda)} + k\,,$$

and

$$\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\Big(1-\exp(-\frac{k^2 t}{2})\; E_0^{(3)}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t ds\;\exp(2R_s)\Big)\Big| R_t = 0\Big]\Big)$$

is equal to

$$\frac{\lambda K_{k+1}(\lambda)}{K_k(\lambda)} - k = \frac{\lambda K_{k-1}(\lambda)}{K_k(\;\lambda)} + k\,.$$

It follows, using properties of Bessel functions (see e.g. [47, formule 17]), that $E[\exp(-\frac{\lambda^2}{2}A_{g_{\theta_k}})] = 2kI_k(\lambda)K_k(\lambda)$.

We also obtain similar results for symmetric functionals, for example

$$E\Big[\exp(-\frac{\lambda^2}{2}\int_0^{g_{\theta_k}} du\;\exp(2|B_u|)\,)\Big] = \frac{k}{(\lambda K_{k-1}(\lambda)/K_k(\lambda))+k}$$

Throughout this example, the modified Bessel functions play an essential role. This can be explained in terms of Lamperti's representation of a geometric Brownian motion as a time-changed Bessel process, that is

$$\exp(B_t+\nu t) = R\left(\int_0^t ds\exp 2(B_s+\nu s)\right) \qquad (t\geq 0)$$

where $(R(u), u\geq 0)$ is a Bessel process of dimension $d = 2(1+\nu)$. In particular, this shows that formula (4.15) yields, for $b = e^a > 1$

$$E_b^{(2)}\left[\exp -\frac{1}{2}\left(k^2\int_0^{T_1}\frac{du}{R_u^2}+\lambda^2 T_1\right)\right] = \frac{K_k(\lambda b)}{K_k(\lambda)}$$

where $E_b^{(2)}$ denotes expectation with respect to a probability governing $(R_u, u\geq 0)$ as a Bessel process of dimension 2 started at b. For $0<b<1$ there is a similar formula with I_k instead of K_k.

Example 7: $f(x) = \dfrac{\lambda^2}{2}\coth\nu x$

Following Alili's recent thesis [2], let us denote

$$H_t^{(\nu)} := \text{p.v.}\int_0^t ds\;\coth(\nu B_s) := \lim_{\epsilon\to 0}\int_0^t ds\;\coth(\nu B_s)1_{\{|B_s|>\epsilon\}}$$

and also

$$H_t^{+(\nu)} := \lim_{\epsilon\to 0}\left\{\int_0^t ds\;\coth(\nu B_s)1_{\{B_s>\epsilon\}} + \frac{\ln\epsilon}{\nu}\ell_t^0\right\}$$

Using the same notation as in example 4

$$E\left[\exp\frac{1}{2}\left(\lambda^2 H^{+(\nu)}_{\tau_t}+k^2\tau^+_t\right)\right] \tag{4.16}$$

$$=\lim_{\epsilon\to 0}E\left[\exp\left(-\frac{1}{2}\int_\epsilon^\infty da\,(\lambda^2\coth(\nu a)+k^2)\,\ell^a_{\tau_t}\right)\right]\ \exp(-\frac{\lambda^2 t}{2\nu}\ln\epsilon)$$

As in the previous section, it suffices to solve the associated Sturm-Liouville equation, which turns out to be

$$\begin{cases} u''(x) &=& (\lambda^2\coth(\nu x)+k^2)u(x) \qquad \text{on } [\epsilon,\infty[\\ u(x) &=& \alpha x+1 \quad \text{on } [0,\epsilon] \end{cases}$$

and to compute the value of α. This is done in Alili's thesis, using the hypergeometric functions of the first kind ${}_2F_1$ (see Lebedev [24] chap. 9). Hence, the quantity in (4.16) equals

$$\exp\frac{t}{2}\Big[-\sqrt{k^2+\lambda^2}+\frac{\lambda^2}{\nu}\Big(2\gamma+\frac{\Gamma'}{\Gamma}(\alpha-\beta+1)+\frac{\Gamma'}{\Gamma}(\alpha+\beta+1)+\ln(2\nu)\Big)\Big]$$

where $\alpha=\dfrac{1}{2\nu}\sqrt{k^2+\lambda^2}$, $\beta=\dfrac{1}{2\nu}\sqrt{k^2-\lambda^2}$ and γ is Euler's constant.

It can then be established that

$$E(\exp(i\lambda H^{(\nu)}_{\tau_t}-k\tau_t))=\exp\left(-\frac{\pi t|\lambda|}{\nu}\coth\left(\frac{\pi}{\nu}\sqrt{(k^2+\lambda^2)^{1/2}-k}\right)\right)$$

Using excursion theory, it follows that for $\nu\neq 0$ the law of

$$\nu^2\,\{(\int_0^1 ds\ \coth(\nu r_s))^2-1\}$$

does not depend on ν, where $(r_s, s\leq 1)$ is a standard Brownian excursion. This surprising fact is discussed in Alili[3]. Working at an independent exponential time, there is the equality

$$E(\exp(i\lambda H^{(\nu)}_{\theta_k}))=\frac{k}{\sqrt{k^2+\lambda^2}}\ \frac{1}{\cosh\left(\frac{\pi}{\nu}\sqrt{(k^2+\lambda^2)^{1/2}-k}\right)}$$

See [3] and other papers in the same monograph for further developments in this vein.

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