Hitting, occupation, and inverse local times of one-dimensional diffusions: martingale and excursion approaches *

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Technical Report No. 607

November 2, 2001

Abstract

Basic relations between the distributions of hitting, occupation, and inverse local times of a one-dimensional diffusion process X, first discussed by Itô-McKean, are reviewed from the perspectives of martingale calculus and excursion theory. These relations, and the technique of conditioning on L_T^y, the local time of X at level y before a suitable random time T, yield formulae for the joint Laplace transform of L_T^y and the times spent by X above and below level y up to time T.

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*Research supported in part by N.S.F. Grant DMS-0071448

1 Introduction

Itô-McKean [17, Sections 2.6,4.6,4.10,6.1,6.2] established some basic relations between the distributions of the following functionals of a diffusion process X with state space some subinterval I of $\mathbb{R}$: the hitting times

$$H_y := \inf\{t > 0 : X_t = y\} \qquad (y \in I);$$

the occupation times

$$A_T^{y,+} := \int_0^T 1(X_s > y)ds \text{ and } A_T^{y,-} := \int_0^T 1(X_s \leq y)ds, \tag{1}$$

for T which might be either a fixed or random time; and the inverse local times

$$\tau_\ell^y := \inf\{t : L_t^y > \ell\} \qquad (\ell \geq 0) \tag{2}$$

where $(L_t^y, t \geq 0)$ is a continuous local time process for X at level y. Itô-McKean [17, Sections 6.1 and 6.2] assumed X was a recurrent diffusion, and left to the reader the necessary modifications for transient X. Such modifications were indicated without proof by Pitman-Yor [30, (9.8)(ii)] and Borodin-Salminen [6, II.2].

This paper reviews the basic relations between hitting, occupation, and inverse local times, from the perspectives of two different approaches to one-dimensional diffusions, martingale calculus and excursion theory [37, 36], which have been developed largely since the publication of Itô-McKean [17].

We also show how these basic relations, combined with the technique of conditioning on L_T^y, yield general expressions for the Laplace transform

$$P_x[\exp(-\alpha L_T^y - \beta A_T^{y,+} - \gamma A_T^{y,-})] \tag{3}$$

for various random times T. Here P_x stands for the probability or expectation operator governing the diffusion process X started at $X_0 = x \in I$. Typically, such formulae have been derived in the literature for particular diffusions X by some combination of martingale calculus, excursion theory, and the method of Feynman-Kac [21]. But the simplicity of general expressions of these formulae, and their close connection to the basic relations described by Itô-McKean [17], does not seem to have been fully appreciated.

Following Itô-McKean [17], Rogers-Williams [37, V.44-54], Revuz-Yor [36, VII (3.12)], we assume that X is a regular diffusion whose state space is some interval $I \subseteq \mathbb{R}$, with no killing and infinitesimal generator

$$\mathcal{G} := \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} \tag{4}$$

acting on a domain of functions subject to appropriate smoothness and boundary conditions discussed in [17, 6, 37, 36]. Throughout the paper, we assume for simplicity that for $x \in \text{int}(I)$, the interior of I,

$$a(x) \text{ is strictly positive and continuous and } b(x) \text{ is locally integrable.} \tag{5}$$

Then, without regard to boundary conditions, $\mathcal{G}$ can be rewritten as

$$\mathcal{G} = \frac{1}{m(x)}\frac{d}{dx}\,\frac{1}{s'(x)}\frac{d}{dx} \tag{6}$$

where $m(x)$ is the density of the speed measure of X relative to Lebesgue measure, and $s'(x)$ is the derivative of the scale function $s(x)$. These functions are related to $a(x)$ and $b(x)$ via the formulae [6, II.9]

$$s'(x) = \exp\left(-\int^x \frac{2b(y)}{a(y)}dy\right) \text{ and } m(x) = \frac{2}{s'(x)a(x)}. \tag{7}$$

So both $m(x)$ and $s'(x)$ are continuous and strictly positive. In particular, for $a(x) \equiv 1$ and b a bounded Borel function, Zvonkin [48] showed that X with state space $I = \mathbb{R}$ can be constructed with $X_0 = x$ as the unique pathwise strong solution of the stochastic differential equation

$$X_t = x + B_t + \int_0^t b(X_s)ds \tag{8}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion with $B_0 = 0$. More generally, X can be constructed from B by the method of space transformation after a suitable time change [17], [37, V.47].

There is the following well known formula for the P_x Laplace transform of the hitting time H_y: for $x, y \in I$ and $\lambda \geq 0$

$$P_x[\exp(-\lambda H_y)] = \begin{cases} \Phi_{\lambda,-}(x)/\Phi_{\lambda,-}(y) & \text{if } x < y \\ \Phi_{\lambda,+}(x)/\Phi_{\lambda,+}(y) & \text{if } x > y. \end{cases} \tag{9}$$

for a pair of functions $\Phi_{\lambda,\pm}$, with $\Phi_{\lambda,-}$ increasing and $\Phi_{\lambda,+}$ decreasing, which are determined uniquely up to constant factors as a pair of increasing and decreasing non-negative solutions Φ of the differential equation

$$\mathcal{G}\Phi = \lambda\Phi \tag{10}$$

subject to appropriate boundary conditions [17],[6]. For many particular diffusions of interest in applications, the differential equation (10) yields explicit expressions for $\Phi_{\lambda,\pm}$ in terms of classical special functions [6, 30]. The Laplace transform (9) can often be inverted by a spectral expansion, which in some cases (e.g. the Bessel process started at $x = 0$) leads to the conclusion that the P_x distribution of H_y is that of the sum of an infinite sequence of independent exponential variables [23, 24]. While such expansions are of interest in a number of contexts [4], the corresponding representations of the density or cumulative distribution function of H_y can be difficult to work with.

Consider next the local times (occupation densities)

$$L_t^y := L_t^y(X) := a(y) \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1(|X_s - y| \leq \epsilon)ds \tag{11}$$

where the limit exists and defines a continuous increasing process $(L_t^y(X), t \geq 0)$ almost surely P_x for all x. The factor $a(y)$, which is the diffusion coefficient in the generator (4), has been put in the definition (11) of $L_t^y(X)$ to agree with the general Meyer-Tanaka definition of local times of a semimartingale, which we review in Section 4.

Assuming now that X is *recurrent*, meaning that $P_x(L^y_\infty = \infty) = 1$ for all $x, y \in \mathrm{int}(I)$, the right continuous inverse $(\tau^y_\ell, \ell \geq 0)$ of $(L^y_t, t \geq 0)$ is a *subordinator*, that is an increasing process with stationary independent increments, with $\tau^y_\ell < \infty$ almost surely. It is a well known consequence of the decompositions

$$A^{y,+}_{\tau^y_\ell} = \sum_{0<s\leq\ell} \int_{\tau^y_{s-}}^{\tau^y_s} 1(X_t > y)dt \text{ and } A^{y,-}_{\tau^y_\ell} = \sum_{0<s\leq\ell} \int_{\tau^y_{s-}}^{\tau^y_s} 1(X_t \leq y)dt$$

and Itô's excursion theory [36, XII] that the processes $(A^{y,+}_{\tau^y_\ell}, \ell \geq 0)$ and $(A^{y,-}_{\tau^y_\ell}, \ell \geq 0)$ are two independent subordinators, and for $\ell, \lambda > 0$ and $\pm = +$ or $-$ there is the formula

$$P_y[\exp(-\lambda A^{y,\pm}_{\tau^y_\ell})] = \exp\left[-\ell\, \psi^{y,\pm}(\lambda)\right] \tag{12}$$

for a pair of Laplace exponents $\psi^{y,\pm}(\lambda)$. With the choice (11) of normalization of local time, there is the following *key formula* for these Laplace exponents in terms of the functions $\Phi_{\lambda,\pm}$ and their derivatives $\Phi'_{\lambda,\pm}$:

$$-\psi^{y,\pm}(\lambda) = \pm\frac{1}{2}\frac{\Phi'_{\lambda,\pm}(y)}{\Phi_{\lambda,\pm}(y)} = \pm\frac{1}{2}\left.\frac{d}{dx}\right|_{x=y\pm} P_x[\exp(-\lambda H_y)] \tag{13}$$

where the second equality is read from (9). These formulae (9), (12) and (13) are the basic relations involving the distributions of hitting, occupation and inverse local times of a recurrent diffusion process X, and the solutions $\Phi_{\lambda,\pm}$ of $\mathcal{G}\Phi = \lambda\Phi$. As discussed in Section 2, the key formula (13) is the Laplace transform of an expression for the corresponding Lévy measures due to Itô-McKean [17, 6.2].

The rest of this paper is organized as follows. In Section 2 we present in Theorem 1 an expansion of the above discussion to include X which may be either recurrent or transient. We then give a proof of Theorem 1 using Itô's excursion theory. In Section 3, we apply the results of Theorem 1 to give general expressions of the joint Laplace transform (3) for various random times T. At the end of that section, we explain briefly how our method is related to the method of Feynman-Kac for deriving the distribution of additive functionals of diffusion processes [21], [25, 4.2.4], [19]. Finally, Section 4 presents a generalization and proof of the key formula (13), using the Meyer-Tanaka theory of local times of semi-martingales.

2 Hitting times and inverse local times

Suppose throughout this section that X is a diffusion process with state space an interval $I \subseteq \mathbb{R}$, with generator (4) subject to the regularity assumption (5), and let $y \in \operatorname{int}(I)$. Note that we assume that $P_x(X_t \in I) = 1$ for all $x \in I$ and $t \geq 0$. But, the boundary points $\inf(I)$ and $\sup(I)$, if one or both belongs to I, might be either instantaneously or slowly reflecting, or absorbing. We note that the results can also be adapted to diffusions with killing, but leave that to the reader.

2.1 Inverse local time processes

Let $(\tau_\ell^y, \ell \geq 0)$ denote the right continuous inverse of the local time process $(L_t^y, t \geq 0)$ with the Meyer-Tanaka normalization (11). By the strong Markov property of X, the process $(\tau_\ell^y, \ell \geq 0)$ is a subordinator, with some Laplace exponent $\psi^y(\lambda)$ defined by the formula

$$P_y[\exp(-\lambda \tau_\ell^y)] = \exp\left[-\ell\, \psi^y(\lambda)\right]. \tag{14}$$

Note in particular that

$$P_y(L_\infty^y > \ell) = P(\tau_\ell^y < \infty) = e^{-\ell \psi^y(0)}. \tag{15}$$

It is elementary and well known [17, 6] that either X is *recurrent*, in which case $\psi^y(0) = 0$ and $P_y(\tau_\ell^y < \infty) = 1$ for all ℓ, or X is *transient*, in which case $\psi^y(0) > 0$ and the process $(\tau_\ell^y)_{\ell \geq 0}$ jumps to ∞ at local time $\ell = L_\infty^y$ which is exponentially distributed with rate $\psi^y(0)$.

Theorem 1 [17, §6.1], [30, (9.8)(ii)] *The Laplace exponent $\psi^y(\lambda)$ in (14) is given by the formula*

$$\psi^y(\lambda) = \psi^{y,+}(\lambda) + \psi^{y,-}(\lambda) \tag{16}$$

where for $\pm = +$ or $-$

$$-\psi^{y,\pm}(\lambda) = \pm \frac{1}{2} \frac{\Phi'_{\lambda,\pm}(y)}{\Phi_{\lambda,\pm}(y)} = \pm \frac{1}{2} \left.\frac{d}{dx}\right|_{x=y\pm} P_x[\exp(-\lambda H_y)] \tag{17}$$

where $\Phi_{\lambda,-}$ and $\Phi_{\lambda,+}$ are increasing and decreasing solutions Φ of $\mathcal{G}\Phi = \lambda\Phi$. Moreover,

$$\psi^{y,\pm}(\lambda) = \int_{]0,\infty]} (1 - e^{-\lambda t}) \nu^{y,\pm}(dt) = \lambda \int_0^\infty e^{-\lambda t} \nu^{y,\pm}[t, \infty] dt \tag{18}$$

or equivalently

$$\nu^{y,\pm}[t,\infty] = \pm\frac{1}{2}\left.\frac{d}{dx}\right|_{x=y\pm} P_x(H_y \geq t) \qquad (0 < t \leq \infty) \tag{19}$$

for some Lévy measures $\nu^{y,\pm}$ *on* $(0,\infty]$, *with atoms at infinity of magnitudes*

$$\psi^{y,\pm}(0) = \nu^{y,\pm}[\infty,\infty] = -\pm\frac{1}{2}\lim_{\lambda\downarrow 0}\frac{\Phi'_{\lambda,\pm}(y)}{\Phi_{\lambda,\pm}(y)}. \tag{20}$$

Itô-McKean [17, 6.2] gave equivalents of these formulae in the recurrent case when $\psi^{y,\pm}(0) = 0$, with $d/ds(x)$ instead of $(1/2)d/dx$, and $\tilde{\tau}^y_\ell := \inf\{u : \tilde{L}^y_u > \ell\}$ instead of τ^y_ℓ, where $(\tilde{L}^x_u)$ is the jointly continuous family of local times with the Itô-McKean normalization so that

$$\int_0^t g(X_u)du = \int g(x)\tilde{L}^x_t m(x)dx \tag{21}$$

for arbitrary non-negative Borel functions g. By comparing (21) and (11), we find that

$$\tilde{L}^x_t = \frac{s'(x)}{2}L^x_t \text{ and hence } \tau^x_\ell(X) = \tilde{\tau}^x_{\ell s'(x)/2}. \tag{22}$$

This allows the formulae of [17] to be rewritten in the form (17) and (19) with $d/ds(x) = (1/s'(x))d/dx$ multiplied by $s'(x)/2$ to obtain $(1/2)d/dx$. The extension of these formulae for a transient diffusion X was indicated in [30, (9.8)(ii)] and [6, II No. 14]. Note that the equality of middle and right quantities in (17) follows immediately from the basic formula (9) for the P_x Laplace transform of H_y in terms of $\Phi_{\lambda,\pm}$.

In Section 2.2 we indicate a proof of Theorem 1 by interpeting the Laplace exponents $\psi^{y,\pm}(\lambda)$ in terms of Itô's excursion theory. As we note in Section 2.4, some care is necessary in the transient case to avoid erroneous statements. In Section 4 we offer a different approach to Theorem 1 by martingale calculus. The passage from (17) to (19) via (18) involves some smoothness in t of $\nu^{y,\pm}[t,\infty)$. In fact it is known as a consequence of Krein's theory of strings [2, 9.2.3], that $\nu^{y,\pm}[t,\infty)$ is a completely monotone function of t, so each of the Lévy measures $\nu^{y,\pm}$ has a smooth density $\nu^{y,\pm}(dt)/dt = \int_0^\infty e^{-zt}\mu^{y,\pm}(dz)$ for some measures $\mu^{y,\pm}$.

Define intervals

$$I^{y,-} :=]-\infty, y] \text{ and } I^{y,+} :=]y,\infty[$$

and as in (1) let

$$A_t^{y,\pm} := \int_0^t 1(X_s \in I^{y,\pm})ds.$$

The proof of formula (17) given in Section 2.2 involves the following generalization of the formulae (12) and (13) for a recurrent diffusion:

Corollary 2 [30, Remark (9.8) (ii)]. *For each $\ell > 0$, the processes $(A^{y,\pm}_{\tau_h^y})_{0\le h\le\ell}$ conditioned on $(\tau_\ell^y < \infty)$ are two independent subordinators parameterized by $[0,\ell]$, with Laplace exponents $\psi_0^{y,\pm}(\lambda)$ given by*

$$-\psi_0^{y,\pm}(\lambda) = \psi^{y,\pm}(0) - \psi^{y,\pm}(\lambda) = \pm\frac{1}{2}\left.\frac{d}{dx}\right|_{x=y\pm} P_x[\exp(-\lambda H_y)\,|\,H_y < \infty] \quad (23)$$

for $\psi^{y,\pm}(\lambda)$ as in (17) and (18). That is to say,

$$P_y[\exp(-\lambda A^{y,\pm}_{\tau_h^y})\,|\,\tau_\ell^y < \infty] = \exp\left[-h\psi_0^{y,\pm}(\lambda)\right] \qquad (0 \le h \le \ell). \qquad (24)$$

Moreover, in the transient case when $\psi^y(0) > 0$, provided $h < \ell$ formula (24) holds also with conditioning on $(L_\infty^y = \ell)$ instead of $(\tau_\ell^y < \infty) = (L_\infty^y > \ell)$, where L_∞^y has exponential distribution with rate $\psi^y(0)$.

2.2 Interpretation in terms of excursions

Let n_y now denote the characteristic measure of Itô's (possibly terminating) Poisson point process of excursions of X away from y, allowing additional λ-*marking* according to an independent PPP of rate λ, as considered in [15],[37, VI.49]. So $n_y(\cdots)$ is the Poisson intensity of excursions of type $\cdots$ relative to dL_t^y. According to Itô's theory of Poisson point processes of excursions [16, 28], [37, VI.49], the process of excursions of X away from y, when indexed by local time at y, is a homogeneous Poisson point process killed at an independent random local time L_∞^y which is exponentially distributed with rate $\psi^y(0)$. There is the following basic interpretation of the Laplace exponent:

$$\psi^y(\lambda) = n_y(\lambda\text{-marked excursions}). \qquad (25)$$

Note that a terminal excursion (if any) of infinite lifetime is λ-marked with probability one. So (25) for $\lambda = 0$ reduces to

$$\psi^y(0) = n_y(\text{excursions with infinite lifetime}). \qquad (26)$$

By continuity of paths of X, each excursion away from y is either a $+$ excursion which lies entirely in $I^{y,+} := [y, \infty[$, or a $-$ excursion which lies entirely in $I^{y,-} :=]-\infty, y]$. Immediately from Itô's theory of excursions, it is clear that (16) holds with

$$\psi^{y,\pm}(\lambda) = n_y(\lambda\text{-marked} \pm \text{excursions}), \tag{27}$$

$$\psi^{y,\pm}(0) = n_y(\pm \text{ excursions with infinite lifetime}). \tag{28}$$

For $y < z$ with $z \in \operatorname{int}(I)$, a $+$ excursion of infinite lifetime must first reach z and then stay above y thereafter, so

$$\psi^{y,+}(0) = n_y(H_z < \infty)\, P^z(H_y = \infty). \tag{29}$$

Combine (29) with the following lemma, and let $z \downarrow y$ to obtain the $+$ case of (17) for $\lambda = 0$, or equivalently (19) for $t = \infty$.

Lemma 3 *For* $\inf(I) < y < z < \sup(I)$,

$$n_y(H_z < \infty) = \frac{s'(y)}{2(s(z) - s(y))} \tag{30}$$

Proof. By the compensation formula of excursion theory [26, (9.7)]

$$P_y(L^y_{H_z})\, n_y(H_z < \infty) = P_y(H_z < \infty).$$

But since $s(X_t)$ is a semimartingale whose bounded variation part moves only when X is at a boundary point [27], by Tanaka's formula (75) and the change of variable formula (85) below, under P_y with $y < z$, the process

$$\int_0^t 1(X_u > y) d_u(s(X_u)) = (s(X_t) - s(y))^+ - \frac{s'(y)}{2} L^y_t$$

when stopped at H_z is a local martingale. This yields by optional sampling

$$\frac{s'(y)}{2} P_y(L^y_{H_z}) = (s(z) - s(y)) P_y(H_z < \infty)$$

and (30) follows. $\square$

The equality of the left and right quantities in formula (17) can now be derived as follows. Clearly, it is enough to deal with the $+$ case. Since we have just checked the case $\lambda = 0$, it suffices to check (23). Let $N^{y,z}(t)$ denote the number of upcrossings of $[y, z]$ by X up to time t. Given that there are at least k such upcrossings, let $H^{z,y}(k)$ denote the length of the subsequent downcrossing. Then, under P_y given $\tau_\ell^y < \infty$

$$A^{y,+}_{\tau_\ell^y} = \sum_{k=1}^{N^{y,z}(\tau_\ell^y)} H^{z,y}(k) + \epsilon^{y,z}(\tau_\ell^y) \tag{31}$$

where the last term, which counts time spent above y during upcrossings of $[y, z]$ up to time τ_ℓ^y, is less than the total time in $[y, z]$ up to time τ_ℓ^y, so can be neglected in the limit as $z \downarrow y$ on the event $\tau_\ell^y < \infty$. According to Itô, given $\tau_\ell^y < \infty$, the number $N^{y,z}(\tau_\ell^y)$ is Poisson with mean

$$P_y[N^{y,z}(\tau_\ell^y) \,|\, \tau_\ell^y < \infty] = \ell\, n_y(H_z < \infty)\, P_z(H_y < \infty) = \frac{\ell s'(y) P_z(H_y < \infty)}{2(s(z) - s(y))},$$

from (30), and given $\tau_\ell^y < \infty$ and $N^{y,z}(\tau_\ell^y) = n$ the $H^{z,y}(k)$ for $1 \le k \le n$ are independent with the P_z distribution of H_y given $H_y < \infty$. Thus, the sum in (31) (ignoring the last term) is a compound Poisson variable whose Laplace transform can be written in terms of $P_x[\exp(-\lambda H_y) \,|\, H_y < \infty]$. Formula (23) now follows easily by letting $z \downarrow y$.

The key to the formulae for Laplace transforms in Section 3 is the last sentence of Corollary 2. This follows from Itô's excursion theory by an argument of Greenwood-Pitman [15].

To conclude this section, we recall from [31, §3] that the basic differentation formulae (17) and (19) for the Lévy measures $\nu^{y,\pm}$ can be extended to corresponding formulae for the restrictions $n_{y,\pm}$ of n_y to excursions in $I^{y,\pm}$. See for instance [19, Cor. 3.4] for a typical application.

2.3 Remarks on the recurrent case

Suppose in this section that X is recurrent, that is $\psi^y(0) = 0$. Then, according to Corollary 2, the two processes $(A^{y,\pm}_{\tau_\ell^y}, \ell \ge 0)$ are simply two independent subordinators with Laplace exponents $\psi_0^{y,\pm}(\lambda) = \psi^{y,\pm}(\lambda)$. Since $t = A_t^{y,+} + A_t^{y,-}$ there is the decomposition

$$\tau_\ell^y = A^{y,+}_{\tau_\ell^y} + A^{y,-}_{\tau_\ell^y} \quad (\ell \ge 0). \tag{32}$$

So the sum of the two independent subordinators $(A^{y,\pm}_{\tau^y_\ell}, \ell \geq 0)$ is the subordinator $(\tau^y_\ell)_{\ell \geq 0}$ with Laplace exponent $\psi^y(\lambda) = \psi^{y,+}(\lambda) + \psi^{y,-}(\lambda)$.

It is well known [6, No. II.12] that the speed measure $m(x)dx$ serves as an invariant measure for a recurrent diffusion process X. As well as the *local formulae* (17) and (19) for the Laplace exponents $\psi^{y,\pm}(\lambda)$ and corresponding Lévy measures $\nu^{y,\pm}$, there are the *global formulae*

$$\psi^{y,\pm}(\lambda) = \frac{\lambda}{a(y)m(y)} \int_{I^{y,\pm}} m(x)dx P^x[\exp(-\lambda H_y)] \qquad (\lambda > 0) \qquad (33)$$

and

$$\nu^{y,\pm}[t, \infty] = \frac{1}{a(y)m(y)} \int_{I^{y,\pm}} m(x) P_x(H_y \in dt)/dt \qquad (t > 0). \qquad (34)$$

Either formula can be deduced from the other via (18). As discussed in [34, §4], apart from the normalization by $a(y)m(y)$ which is dictated by our choice of local times, formula (34) is a particular instance for one-dimensional diffusions of a general formula which expresses Itô's law of excursions from a point of a recurrent ergodic Markov process X in terms of the stationary version of X. This formula was given by Bismut [5] for a one-dimensional Brownian motion, and extended to a general recurrent Markov process X by Pitman [29]. Formula (34) appears in the present setting of a recurrent diffusion process X, with $a(x) \equiv 1$, in Truman-Williams-Yu [41, Proposition 1]. In [34, (71)] we showed how to derive (33) from (17) assuming that X is on natural scale (i.e. $s(x) = x$), by consideration of the differential equation (10) satisfied by $\Phi_{\lambda,-}$ and $\Phi_{\lambda,+}$. (Note that the measure $m(dx)$ in [34] is related to the present $m(x)$ by $2m(dx) = m(x)dx$). The general case of (33) is easily reduced to the case with natural scale by transformation from X to $s(X)$.

2.4 Remarks on the transient case

In the transient case, Corollary 2 is complicated by the fact that while (32) still holds, the two processes $(A^{y,\pm}_{\tau^y_\ell}, \ell \geq 0)$ are not independent since L^y_∞ can be recovered as a function of either one of them. Rather, the two processes are conditionally independent given L^y_∞. If the last excursion of X away from y is an upward one, which happens with probability $\psi^{y,+}(0)/\psi^y(0)$, then the process $(A^{y,+}_{\tau^y_\ell})_{\ell \geq 0}$ jumps to ∞ at local time $\ell = L^y_\infty$, while the process $(A^{y,-}_{\tau^y_\ell})_{\ell\geq 0}$ is stopped at local time L^y_∞. A similar remark applies if the last excursion of X away from y is a downward one. Thus, with the usual definition of a subordinator with state space $[0,\infty]$, allowing a jump to ∞ at an independent exponential time, for each particular choice of sign $\pm$, the process $(A^{y,\pm}_{\tau^y_\ell})_{\ell\geq 0}$ is a subordinator with Lévy measure $\nu^{y,\pm}$ only if $\psi^y(0) = \psi^{y,\pm}(0)$. Similarly, for each particular choice of sign $\pm$, formula (12) holds for some exponent $\psi^{y,\pm}(\lambda)$ if and only if $\psi^y(0) = \psi^{y,\pm}(0)$, in which case (12) holds with $\psi^{y,\pm}(\lambda)$ given by (13).

2.5 Excursions below the maximum

From Theorem 1 we immediately deduce the result of [17, 4.10 and 6.2] that under P_x the right continuous inverse of the past maximum process $(\max_{0\leq s\leq t} X_s)_{t\geq 0}$, that is the first passage process $(H_{z+})_{z\geq x}$, is an increasing process with independent increments, with $H_{z+} = H_z$ a.s. for each fixed $z > x$, and H_z has the infinitely divisible distribution on $[0,\infty]$ with Lévy-Khintchine representation

$$P_x[e^{-\lambda H_z}] = \exp\left[-\int_{[0,\infty]}(1-e^{-\lambda t})\nu_{x,z}(dt)\right] \tag{35}$$

for the Lévy measure on $[0,\infty]$

$$\nu_{x,z}(\cdot) = 2\int_x^z \nu^{y,-}(\cdot)dy$$

with $\nu^{y,-}$ as in (18) the Lévy measure associated with the lengths of downward excursions of X below level y. Consequently, the point process on $[x,\sup(I)[\times]0,\infty]$ with a point at each $(y, H_{y+} - H_y)$ such that $H_{y+} - H_y > 0$, is distributed under P_x like the point process derived from a Poisson point process with intensity $dy\,\nu^{y,-}(dt)$ by killing all points (y,t) such that $y > K_x$, where K_x is the least $y > x$ such that (y,∞) is a point of the Poisson process, with the convention that $K_x := \sup(I)$ if there is no such point. Thus the P_x distribution of $\sup_t X_t$ is that of K_x.

This Poisson description of the jumps of the inverse $(H_{z+})_{z\geq x}$ of the past maximum process of X extends straightforwardly to the point process of excursions of X below its past maximum process:

Proposition 4 (Fitzsimmons [11]) *For y with $H_y < H_{y+}$ let $\mathbf{e}_y$ denote the excursion of X below y over the interval $[H_y, H_{y+}]$, so $\mathbf{e}_y$ belongs to a suitable measurable space Ω^{exc} of excursion paths which may start and end at any level y with $y > x$. Under P_x the point process on $[x, \sup(I)[\times \Omega^{\text{exc}}$ with a point at each $(y, \mathbf{e}_y)$ such that $H_{y+} - H_y > 0$, has the same distribution as that of a Poisson point process with intensity $2dy\, n_{y,-}(d\mathbf{e})$ after killing all points $(y, \mathbf{e})$ such that $y > K_x$, where $n_{y,-}$ is the restriction of Itô's excursion law n_y to excursions below y, and K_x is the least $y > x$ such that there is a point $(y, \mathbf{e})$ of the Poisson point process for which $\mathbf{e}$ has infinite lifetime.*

Fitzsimmons [11] formulated a generalization of this result for diffusions with killing, and proved it by general techniques of Markovian excursion theory [26]. Another proof can be given by supposing that X is constructed from a Brownian motion B started at $B_0 = 0$ by a suitable space transformation and random time substitution [37, V.47]. The point process of excursions of X below its past maximum process M_X is then a push-forward of the point process of excursions of B below its past-maximum process M_B, with suitable killing. The homogeneous Poisson character of excursions of $M_B - B$ away from 0 is a well known consequence of Lévy's theorem that $M_B - B \stackrel{d}{=} |B|$. The description of excursions of X below M_X is then derived by standard results on transformation of Poisson processes.

Greenwood-Pitman [15] and Fitzsimmons [11, 10] showed how the Poisson structure of excursions below the maximum could be used to derive Williams' path decompositions [44] for Lévy processes or diffusions. See also [6, 8, 33] for further results related to the decomposition of diffusion paths at a maximum.

2.6 Relation to the Green function

Let $p(t; x, y)$ be the transition probability function of X relative to the speed measure $m(y)dy$:

$$P_x(X_t \in dy) = p(t; x, y)m(y)dy. \tag{36}$$

It is known [17, 4.11] that $p(t; x, y)$ can be chosen to be jointly continuous in $(t, x, y) \in (0, \infty) \times (\text{int}(I))^2$, and that the corresponding Green function is then

$$G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t; x, y)dt = {w_\lambda}^{-1} \Phi_{\lambda,-}(x \wedge y)\Phi_{\lambda,+}(x \vee y) \tag{37}$$

where the *Wronskian*

$$w_\lambda := \frac{\Phi'_{\lambda,-}(y)\Phi_{\lambda,+}(y) - \Phi_{\lambda,-}(y)\Phi'_{\lambda,+}(y)}{s'(y)} \tag{38}$$

depends only on λ and not on y. For an explanation of (37) in terms of excursion theory, see [37, (54.1)]. Suppose that P_x governs ε_λ as an exponential random variable with rate λ, with ε_λ independent of the diffusion process X started at $X_0 = x$. For $\tilde{L}^y_t$ the local time process at y with the Itô-McKean normalization (21) there is the formula [17, 5.4]

$$P_x(\tilde{L}^y_{\varepsilon_\lambda}) = P_x\left[\int_0^\infty \lambda e^{-\lambda t}\tilde{L}^y_t dt\right] = P_x\left[\int_0^\infty e^{-\lambda t} d\tilde{L}^y_t\right] = G_\lambda(x,y).$$

For our normalization of local time we find instead from (22) and (37) that

$$P_x(L^y_{\varepsilon_\lambda}) = g_\lambda(x,y) := \frac{2}{s'(y)}G_\lambda(x,y) = \frac{2}{s'(y)}\frac{\Phi_{\lambda,-}(x\wedge y)\Phi_{\lambda,+}(x\vee y)}{w_\lambda}. \tag{39}$$

Combine (16), (17) and (38) to see that the Laplace exponent $\psi^y(\lambda)$ of our inverse local time process (τ^y_ℓ) is

$$\psi^y(\lambda) = \frac{1}{g_\lambda(y,y)} = \frac{s'(y)}{2G_\lambda(y,y)} = \frac{s'(y)w_\lambda}{2\Phi_{\lambda,-}(y)\Phi_{\lambda,+}(y)}. \tag{40}$$

This is the equivalent via (22) of [17, 6.2.2)]. Borodin-Salminen [6, Appendix 1] tabulate the Green functions $G_\lambda(x,y)$ and Wronskians w_λ for about twenty different diffusions X. The increasing and decreasing solutions $\Phi_{\lambda,\pm}$ of $\mathcal{G}\Phi = \lambda\Phi$ can be found by inspection of these formulae using (39).

2.7 The example of Brownian motion with drift

To quickly illustrate and check the formulae in the previous section, consider the example with $I = \mathbb{R}$, $a(x) \equiv 1$ and $b(x) \equiv \mu > 0$, so X is Brownian motion on $\mathbb{R}$ with drift μ. The scale function is

$$s(x) = -e^{-2\mu x} \tag{41}$$

The increasing and decreasing solutions Φ of $\mathcal{G}\Phi = \lambda\Phi$ are

$$\Phi_{\lambda,\pm}(y) = \exp(-\pm y(\sqrt{2\lambda+\mu^2} \pm \mu)). \tag{42}$$

Hence from (17)

$$\psi^{y,\pm}(\lambda) = -\pm\frac{1}{2}\frac{\Phi'_{\lambda,\pm}(y)}{\Phi_{\lambda,\pm}(y)} = \frac{1}{2}(\sqrt{2\lambda+\mu^2}\pm\mu) \tag{43}$$

and the Laplace exponent of $(\tau^y_\ell, \ell \geq 0)$ is

$$\psi^y(\lambda) = \psi^{y,+}(\lambda) + \psi^{y,-}(\lambda) = \sqrt{2\lambda+\mu^2} \tag{44}$$

which is correct also for $\mu = 0$. In connection with (40) we find

$$w_\lambda\, s'(y) = 2\sqrt{2\lambda+\mu^2}\, e^{-2\mu y} \tag{45}$$

and

$$2\Phi_{\lambda,-}(y)\Phi_{\lambda,+}(y) = 2e^{-2\mu y} \tag{46}$$

and (45) divided by (46) is (44) in keeping with (40). Using the above expressions for $\psi^{y,\pm}(\lambda)$, our later formula (68) in this case agrees with [6, p. 205 (1.6.1)].

2.8 The example of Bessel processes

Consider now $BES(\nu)$, the Bessel process on $[0,\infty[$ with $a(x) \equiv 1$ and $b(x) = (\nu+1/2)/x$ for some $\nu > -1$, with 0 an instantaneously reflecting boundary point if $\nu \in]-1,0[$ and an entrance-non-exit boundary point if $\nu \in [0,\infty[$. It is well known [7, 22, 6] that we can take

$$\Phi_{\lambda,-}(x) = x^{-\nu} I_\nu(\sqrt{2\lambda}x); \quad \Phi_{\lambda,+}(x) = x^{-\nu} K_\nu(\sqrt{2\lambda}x) \tag{47}$$

where I_ν and K_ν are the usual modified Bessel functions. From (17) we obtain as in [30, (9.s7)] the Laplace exponents

$$\psi^{y,-}(\lambda) = \sqrt{\frac{\lambda}{2}}\frac{I'_\nu}{I_\nu} - \frac{\nu}{2y} = \sqrt{\frac{\lambda}{2}}\frac{I_{\nu+1}}{I_\nu} \tag{48}$$

$$\psi^{y,+}(\lambda) = -\sqrt{\frac{\lambda}{2}}\frac{K'_\nu}{K_\nu} + \frac{\nu}{2y} = \sqrt{\frac{\lambda}{2}}\frac{K_{\nu-1}}{K_\nu} + \frac{\nu}{y} \tag{49}$$

where all the Bessel functions and their derivatives are evaluated at $\sqrt{2\lambda}y$, and the expressions involving $I_{\nu+1}$ and $K_{\nu-1}$ follow from classical recurrences

$$I'_\nu(z) = I_{\nu+1}(z) + \nu I_\nu(z)/z = I_{\nu-1} - \nu I_\nu(z)/z \tag{50}$$

$$K'_\nu(z) = -K_{\nu+1}(z) + \nu K_\nu(z)/z = -K_{\nu-1}(z) - \nu K_\nu(z)/z. \qquad (51)$$

Formulae (48) and (49) can be checked against formulae of [6, e.g. 6.4.4.1 and 6.4.5.1 for $\nu > 0$]. From (20) and the classical asymptotics of $I_\nu(z)$ and $K_\nu(z)$ as $z \to 0$ we see that

$$\psi^{y,-}(0) = 0 \text{ and } \psi^y(0) = \psi^{y,+}(0) = \frac{\nu \vee 0}{y},$$

corresponding to the well known facts that $BES(\nu)$ is recurrent for $\nu \in]-1,0]$, and upwardly transient for $\nu > 0$, when, starting from any level $x \in [0,y]$, by (15) the distribution of L^y_∞ is exponential with rate ν/y, in agreement with [30, (9.s1)] and [6, 6.4.02]. See [30] and [20] for further applications of the formulae (48) and (49).

3 Some joint Laplace transforms

3.1 Last exit times

Consider first for a transient diffusion X and $y \in \mathrm{int}(I)$ the last exit time

$$\Lambda_y := \sup\{t : X_t = y\}$$

and the times $A^{y,\pm}_{\Lambda_y}$ spent above and below y up to time Λ_y. As a consequence of (24), by conditioning on L^y_∞, and using the fact that $\tau^y_h \uparrow \Lambda_y$ as $h \uparrow L^y_\infty$, we deduce the following generalization of results for Bessel processes found in [30, (9.s5)], which was suggested in [30, Remark (9.8)(ii)]:

Corollary 5 *For any transient diffusion X, the trivariate distribution of $(L^y_\infty, A^{y,+}_{\Lambda_y}, A^{y,-}_{\Lambda_y})$ is given by the following Laplace transform: for $\alpha, \beta, \gamma > 0$*

$$P_y[\exp(-\alpha L^y_\infty - \beta A^{y,+}_{\Lambda_y} - \gamma A^{y,-}_{\Lambda_y})] = \frac{\psi^y(0)}{\alpha + \psi^{y,+}(\beta) + \psi^{y,-}(\gamma)}. \qquad (52)$$

for $\psi^{y,\pm}$ and $\psi^y = \psi^{y,+} + \psi^{y,-}$ as in Theorem 1.

In particular, for $\alpha = 0$ and $\beta = \gamma = \lambda \geq 0$, we find using (16) and (40) that

$$P_y[\exp(-\lambda\Lambda_y)] = \frac{\psi^y(0)}{\psi^y(\lambda)} = \frac{G_\lambda(y,y)}{G_0(y,y)}. \qquad (53)$$

By application of the strong Markov property of X and the first passage Laplace transforms (9), as in [37, V (50.7)], formula (53) extends easily to

$$P_x[\exp(-\lambda\Lambda_y)] = \frac{G_\lambda(x,y)}{G_0(y,y)} \tag{54}$$

and hence there is the following formula of [30, Theorem (6.1), (6.e)],[38]:

$$\frac{P_x[\Lambda_y \in dt]}{dt} = \frac{p(t;x,y)}{G_0(y,y)} \qquad (t>0) \tag{55}$$

where $p(t;x,y)$ is the transition density function of X relative to $m(y)dy$, as in (36). For more about last exit times of one-dimensional diffusions, see [38]. See also [13], [12], [26], [37, VI.50] regarding last exit times of more general Markov processes.

We also deduce the following from Corollary 5:

Corollary 6 *Suppose that X is a transient diffusion with $\lim_{t\to\infty} X_t = \sup I$ almost surely, so that $\psi^y(0) = \psi^{y,+}(0) > 0$ for all $y \in \operatorname{int}(I)$. Then the total time $A_\infty^{y,-}$ that X spends below y has Laplace transform*

$$P_y[\exp(-\gamma A_\infty^{y,-})] = \frac{\psi^y(0)}{\psi^y(0) + \psi^{y,-}(\gamma)} \tag{56}$$

for $\psi^{y,\pm}$ and $\psi^y = \psi^{y,+} + \psi^{y,-}$ as in Theorem 1.

As indicated in [30], formula (56) combined with (9), (47) and (48) yields after simplification with (50) the famous result of Ciesielski-Taylor [7, 14] that for $y > 0$ and $\nu > 0$ the distribution of $A_\infty^{y,-}$ for $BES_0(\nu)$ is identical to the distribution of H_y for $BES_0(\nu - 1)$. See also [3, 45, 46] for alternative approaches to this identity, and various extensions.

To give another application of formula (56), let us compute for a Brownian motion with drift $\mu > 0$ started at 0 the Laplace transform of the total time spent in an interval $[\alpha, \beta]$ for some $0 \le \alpha < \beta < \infty$. By obvious reductions, this distribution depends only on the length of the interval, say $c := \beta - \alpha$, and is the distribution of the total time spent below c by a Brownian motion on $[0, \infty)$ with drift $\mu > 0$ and a reflecting boundary at 0. For this diffusion we identify $\Phi_{\lambda,\pm}$ and hence $\psi^{y,+}$ and $\psi^{y,-}$ from the formulae of [6, Appendix 1.14], and deduce that the required Laplace transform in λ, with $\gamma := \sqrt{2\lambda + \mu^2}$ is

$$\frac{e^{\mu c}}{\cosh(\gamma c) + \frac{\gamma^2+\mu^2}{2\mu\gamma}\sinh(\gamma c)} = \frac{4\gamma\mu}{(\gamma+\mu)^2 e^{(\gamma-\mu)c} - (\gamma-\mu)^2 e^{-(\gamma+\mu)c}} \tag{57}$$

which agrees with a formula of Evans [9] for $\mu = c = 1$, and with the formula of [47, p. 47] after correction of a typographical error whereby $(\alpha - \beta)$ should be replaced twice by $(\beta - \alpha)$.

3.2 Hitting times

The following consequence of Corollary 5 extends results of [35, (12),(13)] which were formulated in the recurrent case.

Corollary 7 *For X which may be either transient or recurrent, and $y \neq z$, let*

$$\Lambda_{y,z} := \sup\{t < H_z : X_t = y\}$$

be the time of the last exit from y before the first hit of z. Let $\alpha, \beta, \gamma > 0$. Then the trivariate distribution of $(L^y_{H_z}, A^{y,+}_{\Lambda_{y,z}}, A^{y,-}_{\Lambda_{y,z}})$ under P_y is determined by the Laplace transform

$$P_y[\exp(-\alpha L^y_{H_z} - \beta A^{y,+}_{\Lambda_{y,z}} - \gamma A^{y,-}_{\Lambda_{y,z}})] = \frac{\psi^{y,z}(0)}{\alpha + \psi^{y,z,+}(\beta) + \psi^{y,z,-}(\gamma)} \tag{58}$$

where $\psi^{y,z} = \psi^{y,z,-} + \psi^{y,z,+}$ for exponents $\psi^{y,z,\pm}$ determined by the formula

$$\frac{1}{\psi^{y,z}(\lambda)} = g_\lambda(y,y) - (P_y e^{-\lambda H_z}) g_\lambda(z,y) \qquad (\lambda > 0) \tag{59}$$

with $\psi^{y,z,-} = \psi^{y,-}$ if $y < z$ and $\psi^{y,z,+} = \psi^{y,+}$ if $y > z$. Moreover, the random variables $\Lambda_{y,z}$ and $H_z - \Lambda_{y,z}$ are independent, with Laplace transforms

$$P_y[\exp(-\lambda \Lambda_{y,z})] = \frac{\psi^{y,z}(0)}{\psi^{y,z}(\lambda)} \tag{60}$$

and

$$P_y[\exp(-\lambda(H_z - \Lambda_{y,z}))] = P_y(e^{-\lambda H_z})\frac{\psi^{y,z}(\lambda)}{\psi^{y,z}(0)} \tag{61}$$

and the P_y joint law of $(L^y_{H_z}, A^{y,+}_{H_z}, A^{y,-}_{H_z})$ is determined by

$$P_y[\exp(-\alpha L^y_{H_z} - \beta A^{y,+}_{H_z} - \gamma A^{y,-}_{H_z})] = \frac{P_y(e^{-\beta H_z})\psi^{y,z}(\beta)}{\alpha + \psi^{y,z,+}(\beta) + \psi^{y,z,-}(\gamma)}. \tag{62}$$

Proof. Suppose first that $y < z$. Apply Corollary 7 to the diffusion process on $I \cap (-\infty, z]$ with z as an absorbing boundary point, obtained by stopping X at time H_z. This gives (58) with the Laplace exponents $\psi^{y,z,\pm}$ associated with the diffusion stopped at H_z, and obviously $\psi^{y,z,-} = \psi^{y,-}$ if $y < z$. Since the right side of (59) is $g^z_\lambda(y,y)$ for g^z_λ the Green function of the stopped diffusion, with normalization as in (39), formula (59) is read from (40). The argument for $y > z$ is similar. The remaining formulae follow immediately from (59) and the last exit decomposition at time $\Lambda_{y,z}$. □

To be more explicit, for $y < z$ we find from (9), (39) and (59) that for $\lambda > 0$ and $y < z$

$$\frac{1}{\psi^{y,z}(\lambda)} = \frac{2}{s'(y) w_\lambda} \frac{\Phi_{\lambda,-}(y)}{\Phi_{\lambda,-}(z)} [\Phi_{\lambda,-}(z)\Phi_{\lambda,+}(y) - \Phi_{\lambda,-}(y)\Phi_{\lambda,+}(z)]. \tag{63}$$

As a check, we can calculate $\psi^{y,z,+}$ in another way by application of (17) to the diffusion X stopped at z. This gives

$$\psi^{y,z,+}(\lambda) = -\frac{1}{2} \frac{{\Phi^z_{\lambda,+}}'(y)}{\Phi^z_{\lambda,+}(y)}$$

where $\Phi^z_{\lambda,+}$ is the analog of $\Phi_{\lambda,+}$ for X stopped on hitting z, which is a decreasing solution of $\mathcal{G}\Phi = \lambda\Phi$ with $\Phi(z) = 0$, for instance

$$\Phi^z_{\lambda,+}(x) = \frac{\Phi_{\lambda,+}(x)}{\Phi_{\lambda,+}(z)} - \frac{\Phi_{\lambda,-}(x)}{\Phi_{\lambda,-}(z)}.$$

The identity (63) for $\psi^{y,z} = \psi^{y,z,+} + \psi^{y,z,-}$ can now be verified by elementary algebra, using (17) for $\psi^{y,z,-} = \psi^{y,-}$, and (39).

For a transient diffusion X, the exponent $\psi^{y,z}(0)$ in (58) can be evaluated by simply using (59) for $\lambda = 0$. For a recurrent X, the right side of (59) for $\lambda = 0$ reads $\infty - \infty$. But in either case $\psi^{y,z}(0)$ can be evaluated as $\psi^{y,z}(0+)$. For $y < z$, by (28) applied to X stopped on hitting z, we can also express $\psi^{y,z}(0) = \psi^{y,-}(0) + \psi^{y,z,+}(0)$ in terms of the scale function $s(x)$ using (30), which makes

$$\psi^{y,z,+}(0) = n_y(H_z < \infty) = \frac{s'(y)}{2(s(z) - s(y))}. \tag{64}$$

Since $\psi^{y,z,+}(\lambda) - \psi^{y,z,+}(0)$ is the rate of λ-marked + excursions from y which fail to reach z, this quantity decreases to 0 as $z \downarrow y$. Since $\psi^{y,-}(\lambda) \equiv$

$\psi^{y,z}(\lambda) - \psi^{y,z,+}(\lambda)$ for $y < z$ it follows that

$$\psi^{y,-}(\lambda) = \lim_{z \downarrow y} [\psi^{y,z}(\lambda) - \psi^{y,z,+}(0)] \tag{65}$$

where $\psi^{y,z}(\lambda)$ is determined by (59) and $\psi^{y,z,+}(0)$ by (64). We apologize for presenting this awkward evaluation of $\psi^{y,-}(\lambda)$ in [35, (9)] without mentioning the much simpler evaluation of $\psi^{y,-}(\lambda)$ by (17).

3.3 Independent exponential times

For X a Brownian motion started at y, Kac [21] derived Lévy's arcsine law for the distribution of $A_t^{y,+}/t$ after determining the double Laplace transform

$$\int_0^\infty e^{-\lambda t} P_y \left(e^{-\beta A_t^{y,+}} \right) dt = \frac{1}{\sqrt{\lambda(\lambda + \beta)}}$$

by what is now known as the method of Feynman-Kac. A number of generalizations of Kac's formula for this double Laplace transform have been obtained using either Kac's approach or excursion theory: see for instance Barlow-Pitman-Yor [1, (4.2)], Truman-Williams [39], [40], Truman-Williams-Yu [41], Watanabe [42, Corollary 2], Weber [43], Jeanblanc-Pitman-Yor [19]. Usually, the processes involved have been assumed to be recurrent. The following proposition presents a rather general result in this vein for a one-dimensional diffusion X, without any recurrence assumption.

Corollary 8 *For X which may be either transient or recurrent, and $\lambda > 0$, let ε_λ denote an exponential variable with rate λ which is independent of X. Let $\Lambda(\varepsilon_\lambda) := \sup\{t < \varepsilon_\lambda : X_t = y\}$ be the time of the last exit from y before time ε_λ, and set $\Delta_\lambda^{y,\pm} := A_{\varepsilon_\lambda}^{y,\pm} - A_{\Lambda(\varepsilon_\lambda)}^{y,\pm}$. Let $\alpha, \beta, \gamma > 0$. Under P_y the random vectors $(L_{\varepsilon_\lambda}^y, A_{\Lambda(\varepsilon_\lambda)}^{y,+}, A_{\Lambda(\varepsilon_\lambda)}^{y,-})$ and $(\Delta_\lambda^{y,+}, \Delta_\lambda^{y,+})$ are independent, with Laplace transforms*

$$P_y[\exp(-\alpha L_{\varepsilon_\lambda}^y - \beta A_{\Lambda(\varepsilon_\lambda)}^{y,+} - \gamma A_{\Lambda(\varepsilon_\lambda)}^{y,-})] = \frac{\psi^y(\lambda)}{\alpha + \psi^{y,+}(\lambda + \beta) + \psi^{y,-}(\lambda + \gamma)} \tag{66}$$

where $\psi^y = \psi^{y,+} + \psi^{y,-}$ for Laplace exponents $\psi^{y,\pm}$ determined by formula (17), and

$$P_y[\exp(-\beta(\Delta_\lambda^{y,+}) - \gamma(\Delta_\lambda^{y,+}))] = \frac{\lambda}{\psi^y(\lambda)} \left(\frac{\psi^{y,+}(\lambda + \beta)}{\lambda + \beta} + \frac{\psi^{y,-}(\lambda + \gamma)}{\lambda + \gamma} \right) \tag{67}$$

whence

$$P_y\left[\int_0^\infty \exp(-\lambda t - \alpha L_t^y - \beta A_t^{y,+} - \gamma A_t^{y,-})dt\right] = \frac{\frac{\psi^{y,+}(\lambda+\beta)}{\lambda+\beta} + \frac{\psi^{y,-}(\lambda+\gamma)}{\lambda+\gamma}}{\alpha + \psi^{y,+}(\lambda+\beta) + \psi^{y,-}(\lambda+\gamma)} \tag{68}$$

Proof. Much as in [19, 2.2], formula (66) follows from the Poisson character of the excursions by the Poisson thinning argument of [15, p. 901], which yields also the independence of the two random vectors, and

$$P(\varepsilon_\lambda - \Lambda(\varepsilon_\lambda) \in ds, X_{\varepsilon_\lambda} \in I^{y,\pm}) = \psi^y(\lambda)^{-1}\lambda e^{-\lambda s}\nu^{y,\pm}[s,\infty]ds$$

which is easily seen to be equivalent to (67), using (18). Since the left side of (68) is just $\lambda^{-1}P_y[\exp(-\alpha L_{\varepsilon_\lambda}^y - \beta A_{\varepsilon_\lambda}^{y,+} - \gamma A_{\varepsilon_\lambda}^{y,-})]$, formula (68) follows by multiplying formulae (66) and (67) and cancelling the factor of λ. □

3.4 Relation to the method of Feynman-Kac

It is instructive to compare the derivation of (68) for $\alpha = 0$ with the more traditional approach of Feynman-Kac. That would be to show that the function

$$F(x) := P_x(S_\infty) \text{ where } S_t := \int_0^t f(X_s)\Pi_s ds \text{ with } \Pi_s := \exp\left[\int_0^s c(X_u)du\right],$$

for suitable functions f and c, was the unique solution of a differential equation subject to appropriate boundary conditions. In the case at hand, we would have $f(x) \equiv 1$ and

$$c(x) = (\lambda+\beta)1(x>y) + (\lambda+\gamma)1(x \le y). \tag{69}$$

A well known martingale approach is to consider the martingale

$$P_x[S_\infty \mid \mathcal{F}_t] = S_t + \Pi_t F(X_t) \tag{70}$$

which leads via Itô's formula to the equation

$$f(x) + \mathcal{G}F(x) = c(x)F(x),$$

assuming that F belongs to the domain of the infinitesimal generator $\mathcal{G}$ of X. Assuming now that X is recurrent, we find it easier to recover (68) by optional sampling of (70) and related martingales at the times τ_ℓ^y. Indeed, if we fix (y,λ,β,γ), work from now on under P_y, and consider for $c(x)$ as in (69) and $f(x)\equiv 1$, then we find from (70) using $X_{\tau_\ell^y}=y$ a.s. that

$$(S_{\tau_\ell^y}-F(y)\Pi_{\tau_\ell^y})_{\ell\geq 0} \text{ is an } (\mathcal{F}_{\tau_\ell^y})_{\ell\geq 0}\text{-martingale.} \tag{71}$$

On the other hand, it follows from the definition of $\psi^{y,\pm}$ as Laplace exponents that $P_y[\Pi_{\tau_\ell^y}]=\exp(-k\ell)$ where

$$k:=k(y,\lambda,\beta,\gamma):=\psi^{y,+}(\lambda+\beta)+\psi^{y,-}(\lambda+\gamma)$$

is the denominator on the right hand side of (68) with $\alpha=0$. Hence

$$(e^{\ell k}\Pi_{\tau_\ell^y})_{\ell\geq 0} \text{ is an } (\mathcal{F}_{\tau_\ell^y})_{\ell\geq 0}\text{-martingale.} \tag{72}$$

which is equivalent by integration by parts to

$$\left(\Pi_{\tau_\ell^y}+k\int_0^\ell \Pi_{\tau_s^y}ds\right)_{\ell\geq 0} \quad \text{is an } (\mathcal{F}_{\tau_\ell^y})_{\ell\geq 0}\text{-martingale.} \tag{73}$$

But the decomposition $S_{\tau_\ell^y}=\sum_{0<s\leq\ell}(S_{\tau_s^y}-S_{\tau_s^y-})$ and the compensation formula of excursion theory [26, (9.7)] show that

$$\left(S_{\tau_\ell^y}-K\int_0^\ell \Pi_{\tau_s^y}ds\right)_{\ell\geq 0} \quad \text{is an } (\mathcal{F}_{\tau_\ell^y})_{\ell\geq 0}\text{-martingale,} \tag{74}$$

where $K:=K(y,\lambda,\beta,\gamma)$ is the numerator on the right side of (68). When we compare (71), (72) and (74) we see that necessarily $F(y)=K/k$ as in (68).

4 Martingale expression of the key formula

We begin by recalling the Meyer-Tanaka definition of local times of a continuous semi-martingale Z, following [37, IV.43],[36, VI]. Let $(L_t^z(Z),t\geq 0)$ denote the continuous increasing Meyer-Tanaka local time process of Z at level z, characterized by the property that for all $t\geq 0$

$$(Z_t-z)^+=(Z_0-z)^++\int_0^t 1(Z_s>z)dZ_s+\frac{1}{2}L_t^z(Z). \tag{75}$$

Then for each $t \geq 0$ there is the *Itô-Tanaka formula* [36, VI (1.5)]

$$g(Z_t) = g(Z_0) + \int_0^t g'_-(Z_s)dZ_s + \frac{1}{2}\int_{\mathbb{R}} L_t^z(Z)g''(dz) \tag{76}$$

for every convex function g, and the *occupation times formula* [36, VI (1.5)]

$$\int_0^t h(Z_u)d < Z, Z >_u = \int L_t^z(Z)h(z)dz \tag{77}$$

for every non-negative Borel function h.

Lemma 9 *Let $(Z_t)_{t\geq 0}$ be a continuous semimartingale with respect to some filtration $(\mathcal{F}_t)$, and let (A_t) be an $(\mathcal{F}_t)$-adapted continuous process with bounded variation, such that*

$$M_t := Z_t \exp(-A_t) \text{ is an } (\mathcal{F}_t) \text{ local martingale} \tag{78}$$

or equivalently

$$Z_t := N_t + \int_0^t Z_s dA_s \text{ for some } (\mathcal{F}_t) \text{ local martingale } (N_t). \tag{79}$$

Then

$$dM_t = \exp(-A_t)dN_t \tag{80}$$

and for each $z \neq 0$, then each of the following two processes $(M_t^{z,-})$ and $(M_t^{z,+})$ is a continuous local martingale:

$$M_t^{z,-} := (Z_t \wedge z) \exp\left[\frac{1}{2z}L_t^z(Z) - \int_0^t 1(Z_s \leq z)dA_s\right] \tag{81}$$

$$M_t^{z,+} := (Z_t \vee z) \exp\left[-\frac{1}{2z}L_t^z(Z) - \int_0^t 1(Z_s > z)dA_s\right]. \tag{82}$$

Proof. Itô's formula implies the equivalence of (78) and (79), and the consequence (80) of this relation. Fix $z \neq 0$ and set $Z'_t := Z_t \wedge z$.

In view of the equivalence of (78) and (79), to check that $(M_t^{z,-})$ is a local martingale it suffices to show that

$$Z'_t = N'_t + \int_0^t Z'_s dA'_s$$

for an $(\mathcal{F}_t)$ local martingale (N'_t) and

$$dA'_s = 1(Z_s \leq z)dA_s - \frac{1}{2z}dL^z_s(Z).$$

That is to say, the process

$$(Z_t \wedge z) - \int_0^t (Z_s \wedge z)\left(1(Z_s \leq z)dA_s - \frac{1}{2z}dL^z_s(Z)\right)$$

is an $(\mathcal{F}_t)$ local martingale. But this is readily obtained from the Itô-Tanaka formula (76) applied with $g(y) = y \wedge z$. The proof for $(M^{z,+}_t)$ is similar. □

Remarks.

a) If the indicator $1(Z_s \leq z)$ in (81) is replaced by $1(Z_s < z)$, then, $L^z_t(Z)$ should be replaced by $L^{z-}_t(Z)$ for $L^z_t(Z)$ chosen to be càdlàg in z, as in [36, VI (1.7)], with

$$L^z_t(Z) - L^{z-}_t(Z) = 2z \int_0^t 1(Z_s = z)dA_s.$$

It follows that each of the processes $(M^{z,-}_t)$ and $(M^{z,+}_t)$ admits a version which is jointly continuous in (t, z).

b) If $(Z_t, t \geq 0)$ is a positive supermartingale, then so is $(Z_t \wedge z, t \geq 0)$. Formula (81) then makes explicit the multiplicative representation of $(Z_t \wedge z)$ as the product of a local martingale and a decreasing process, due to Itô-Watanabe [18, (2.6)]. Whereas if $(Z_t, t \geq 0)$ is a positive submartingale, then so is $(Z_t \vee z, t \geq 0)$, and (82) gives the Itô-Watanabe representation of $(Z_t \vee z)$ as the product of a local martingale and an increasing process.

Theorem 10 *Let $(Z_t)_{t\geq 0}$ be a non-negative continuous semimartingale with respect to some filtration $(\mathcal{F}_t)$, and let (A_t) be an $(\mathcal{F}_t)$-adapted continuous increasing process, such that $Z_t \exp(-A_t)$ is an $(\mathcal{F}_t)$ local martingale. Suppose further that $Z_0 = z_0$ for some fixed $z_0 > 0$, and that $P(L^z_\infty(Z) = \infty) = 1$ for some $z > 0$. Let $\tau^z_\ell(Z) := \inf\{t > 0 : L^z_t(Z) > \ell\}$. Then*

$$E\left[\exp\left(-\int_0^{\tau^z_\ell(Z)} 1(Z_s \leq z)dA_s\right)\right] = \left(\left(\frac{z_0}{z}\right) \wedge 1\right) \exp\left(-\frac{\ell}{2z}\right). \qquad (83)$$

Proof. This follows from Lemma 9 by optional sampling. □

Corollary 11 *Let $(X_t)_{t\geq 0}$ be a continuous semimartingale with respect to some filtration $(\mathcal{F}_t)$, and let Φ be a strictly increasing non-negative function with continuous derivative Φ', and (A_t) be an $(\mathcal{F}_t)$-adapted continuous increasing process, such that $\Phi(X_t)\exp(-A_t)$ is an $(\mathcal{F}_t)$ local martingale. Suppose further that $X_0 = x_0$ for some fixed x_0 with $\Phi(x_0) > 0$, and that $P(L^x_\infty(X) = \infty) = 1$ for some x. Let $\tau^x_\ell(X) := \inf\{t > 0 : L^x_t(X) > \ell\}$. Then*

$$E\left[\exp\left(-\int_0^{\tau^x_\ell(X)} 1(X_s \leq x)dA_s\right)\right] = \left(\left(\frac{\Phi(x_0)}{\Phi(x)}\right)\wedge 1\right)\exp\left(-\frac{\ell}{2}\frac{\Phi'(x)}{\Phi(x)}\right). \tag{84}$$

Proof. Let $Z_t := \Phi(X_t)$. The occupation times formula (77) yields the formula [32, (A.8)], [36, VI (1.23)]

$$L^{\Phi(x)}_t(Z) = \Phi'(x)L^x_t(X) \tag{85}$$

and hence

$$\tau^z_\ell(X) = \tau^x_{\ell\Phi'(x)}(Z) \tag{86}$$

so (84) can be read from (83). □

4.1 Application to diffusions

Suppose first that X is recurrent, and that neither of the boundary points of I can be reached by X in finite time. Then the functions $\Phi_{\lambda,\pm}$ are completely specified up to constant factors as the increasing and decreasing solutions of the differential equation $\mathcal{G}\Phi = \lambda\Phi$ on $\mathrm{int}(I)$, and we deduce from Itô's formula, as in [37, V.50], that for each $x \in \mathrm{int}(I)$

$$(\Phi_{\lambda,\pm}(X_t)\exp(-\lambda t), t \geq 0) \text{ is a } P_x \text{ local martingale.} \tag{87}$$

The first equality in (17) for $\pm = +$ is now seen to be the particular case of (84), with $y = x = x_0$, $A_t = \lambda t$, and $\Phi = \Phi_{\lambda,-}$. The case $\pm = -$ follows similarly.

The proof of (17) for a recurrent X which can reach the boundary in finite time is complicated by the fact that (87) is no longer true in this case. Rather $\exp(-\lambda t)$ must be replaced by $\exp(-A_{\lambda,\pm}(t))$ where

$$A_{\lambda,\pm}(t) = \lambda t + c_{\lambda,\pm}L^{\partial(\pm)}_t$$

where $\partial(+) = \inf(I)$, $\partial(-) = \sup(I)$, the coefficient $c_{\lambda,\pm}$ does not depend on t, and $L_t^{\partial(\pm)}$ is the local time of the semi-martingale $s(X)$ at $s(\partial(\pm))$ up to time t. See for instance [27]. The argument can now be completed as before, using the fact that for instance

$$d_t A_{\lambda,-}(t) 1(X_t \leq y) = \lambda \, dt \, 1(X_t \leq y)$$

because the upper boundary correction term contributes nothing when X is below y.

The same argument can be adapted to the transient case, when $\lim_{t\to\infty} X_t$ must exist and be an endpoint of I almost surely, by appealing to some basic facts of excursion theory. To illustrate, we show how to handle the case when $\lim_{t\to\infty} X_t = \sup I$ almost surely, that is when $\psi^y(0) = \psi^{y,+}(0) > 0$ for all $y \in I$. In the setting of Theorem 10, with $z_0 = z$, if we relax the assumption that $P(L_\infty^z(Z) = \infty) = 1$, and suppose instead that $Z_t \geq z$ for all sufficiently large t a.s., then by first applying the optional sampling theorem at $\tau_\ell^z \wedge t$, then letting $t \to \infty$, we obtain by dominated convergence that

$$1 = \exp(\theta\ell) E\left[\exp\left(-\int_0^{\tau_\ell^z} 1(Z_s \leq z) dA_s\right) 1(\tau_\ell^z < \infty)\right]$$

$$+ E\left[\exp\left(\theta L_\infty^z - \int_0^\infty 1(Z_s \leq z) dA_s\right) 1(\tau_\ell^z = \infty)\right]$$

where $\tau_\ell^z := \tau_\ell^z(Z)$ and $\theta := 1/(2z)$. Applied to the diffusion X, with $z = \Phi_{\lambda,-}(y)$, $Z_t = \Phi_{\lambda,-}(X_t)$, $A_t = \lambda t$, and using the fact that $L_\infty^y(X)$ has exponential distribution with rate $\nu := \nu^{y,+}(\infty) \in (0,\infty)$, and the notation $\alpha := \Phi'_{\lambda,-}(y)/(2\Phi_{\lambda,-}(y)) - \psi^{y,-}(\lambda)$, the previous identity implies

$$1 = e^{\alpha m} e^{-\nu m} + \frac{\nu}{\nu - \alpha}\left(1 - e^{-(\nu-\alpha)m}\right)$$

for all $m > 0$, hence $\alpha = 0$ as required. The case when $\lim_{t\to\infty} X_t$ might be either end of the range can be handled similarly, but details are left to the reader.

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