# The distribution of local times of a Brownian bridge

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## 1 Introduction

Let $(L_t^x, t \geq 0, x \in \mathbb{R})$ denote the jointly continuous process of local times of a standard one-dimensional Brownian motion $(B_t, t \geq 0)$ started at $B_0 = 0$, as determined by the occupation density formula [19]

$$\int_0^t f(B_s)\, ds = \int_{-\infty}^{\infty} f(x) L_t^x \, dx$$

for all non-negative Borel functions $f$. Borodin [7, p. 6] used the method of Feynman-Kac to obtain the following description of the joint distribution of $L_1^x$ and $B_1$ for arbitrary fixed $x \in \mathbb{R}$: for $y > 0$ and $b \in \mathbb{R}$

$$P(L_1^x \in dy, B_1 \in db) = \frac{1}{\sqrt{2\pi}}(|x| + |b - x| + y)\, e^{-\frac{1}{2}(|x|+|b-x|+y)^2}\, dy\, db. \tag{1}$$

This formula, and features of the local time process of a Brownian bridge described in the rest of this introduction, are also implicit in Ray's description of the joint law of $(L_T^x, x \in \mathbb{R})$ and $B_T$ for $T$ an exponentially distributed random time independent of the Brownian motion [18, 22, 6]. See [14, 17] for various characterizations of the local time processes of Brownian bridge and Brownian excursion, and further references. These local time processes arise naturally both in combinatorial limit theorems involving the height profiles of trees and forests [1, 17] and in the study of level crossings of empirical processes [20, 4].

Section 2 of this note presents an elementary derivation of formula (1), based on Lévy's identity in distribution [15],[19, Ch. VI, Theorem (2.3)]

$$(L_t^0, |B_t|) \stackrel{d}{=} (M_t, M_t - B_t) \quad \text{where} \quad M_t := \sup_{0 \leq s \leq t} B_s, \tag{2}$$

and the well known description of this joint law implied by reflection principle [19, Ch. III, Ex. (3.14)], which yields (1) for $x = 0$. Formula (1) determines the one-dimensional distributions of the process of local times at time 1 derived from a Brownian bridge of length 1 from 0 to $b$, as follows: for $y \geq 0$

$$P(L_1^x > y \,|\, B_1 = b) = e^{-\frac{1}{2}((|x|+|b-x|+y)^2 - b^2)}. \tag{3}$$

Section 3 presents a number of identities in distribution as consequences of this formula. If $x$ is between 0 and $b$ then $|x|+|b-x| = |b|$, so the distribution of $L_1^x$ for the bridge from 0 to $b$ is the same for all $x$ between 0 and $b$. Furthermore, assuming for simplicity that $b > 0$, the process $(L_1^x, 0 \leq x \leq b \,|\, B_1 = b)$ is both reversible and stationary. Reversibility follows immediately from the fact that if $(B_s^{0\to b}, 0 \leq s \leq 1)$ denotes the Brownian bridge of length 1 from 0 to $b$ then

$$(b - B_{1-s}^{0\to b}, 0 \leq s \leq 1) \stackrel{d}{=} (B_s^{0\to b}, 0 \leq s \leq 1). \tag{4}$$

To spell out the stationarity property, for each $x > 0$ and $\theta > 0$ with $0 < x + \theta \leq b$, there is the invariance in distribution

$$(L_1^{x+a}, 0 \leq a \leq \theta \,|\, B_1 = b) \stackrel{d}{=} (L_1^a, 0 \leq a \leq \theta \,|\, B_1 = b). \tag{5}$$

Equivalently, for every non-negative measurable function $f$ which vanishes off the interval $[0, \theta]$

$$\int_0^1 f(B_s^{0\to b} - x)ds \stackrel{d}{=} \int_0^1 f(B_s^{0\to b})ds. \tag{6}$$

Howard and Zumbrun [11] proved (6) for $f$ the indicator of a Borel set, and (6) for general $f$ can be established by their method. Alternatively, (6) can be deduced from the following invariance in distribution on the path space $C[0,1]$: for each $0 \leq x \leq b$

$$({}^xB_s^{0\to b}, 0 \leq s \leq 1) \stackrel{d}{=} (B_s^{0\to b}, 0 \leq s \leq 1) \tag{7}$$

where $({}^xB_s^{0\to b}, 0 \leq s \leq 1)$ is derived from the bridge $(B_s^{0\to b}, 0 \leq s \leq 1)$ by the following path transformation:

$${}^xB_s^{0\to b} := \begin{cases} B_{\sigma_x+s}^{0\to b} - x & \text{if } 0 \leq s \leq 1 - \sigma_x \\ \\ b - B_{1-s}^{0\to b} & \text{if } 1 - \sigma_x < s \leq 1 \end{cases}$$

with $\sigma_x$ the first hitting time of $x$ by $(B_s^{0\to b}, 0 \leq s \leq 1)$. The invariance (7) can be checked by a standard technique [5, 20]: the corresponding transformation on lattice paths is a bijection, which gives simple random walk analogs of (7), (6) and (5); the results for the Brownian bridge then follow by weak convergence. Formula (7) implies also that the process $(\sigma_x, 0 \le x \le b)$ derived from $(B_s^{0\to b}, 0 \le s \le 1)$ has stationary increments. It is easily shown by similar arguments that the increments of this process are in fact exchangeable. According to the above discussion, for each $b > 0$ the process of bridge occupation times

$$\left(\int_0^1 1(0 \le B_s^{0\to b} \le x)\, ds, 0 \le x \le b\right)$$

has increments which are stationary and reversible, but not exchangeable (due to continuity of the local time process).

See [9, 16] for other examples of stationary local time processes. In particular, it is known [16, Prop. 2] that if $({}^{\delta}L_t^x, t \ge 0, x \in \mathbb{R})$ is the process of local times of a Brownian motion with drift $\delta > 0$, say ${}^{\delta}B_t := B_t + \delta t, t \ge 0$, then the process $({}^{\delta}L_\infty^x, x \ge 0)$ is a stationary diffusion, with ${}^{\delta}L_\infty^x$ exponentially distributed with rate $\delta$. The stationarity of this process is obvious by application of the strong Markov property of ${}^{\delta}B$ at its first hitting time of $x > 0$. It is also known [22, Theorem 4.5] that if $T_\delta$ is exponential with rate $\frac{1}{2}\delta^2$, independent of $B$, then for each $b > 0$

$$(B_t, 0 \le t \le T_\delta \,|\, B_{T_\delta} = b) \stackrel{d}{=} ({}^{\delta}B_t, 0 \le t \le {}^{\delta}\lambda_b) \tag{8}$$

where ${}^{\delta}\lambda_b$ is the time of the last hit of $b$ by ${}^{\delta}B$, and hence

$$(L_{T_\delta}^x, 0 \le x \le b \,|\, B_{T_\delta} = b) \stackrel{d}{=} ({}^{\delta}L_\infty^x, 0 \le x \le b). \tag{9}$$

The stationarity of $({}^{\delta}L_\infty^x, x \ge 0)$ then implies stationarity of $(L_{T_\delta}^x, 0 \le x \le b \,|\, B_{T_\delta} = b)$, which is implicit in Ray's description of this process [18, 22, 6]. This yields another proof of the stationarity of $(L_t^x, 0 \le x \le b \,|\, B_t = b)$ for arbitrary $t > 0$, by uniqueness of Laplace transforms.

## 2 Proof of formula (1).

As observed in the introduction, formula (1) for $x = 0$ is equivalent via (2) to Lévy's well known description of the joint distribution of $M_1$ and $M_1 - B_1$. The case of (1) with $x \ne 0$ will now be deduced from the case with $x = 0$. It clearly suffices to deal with $x > 0$, as will now be supposed. Let $\sigma_x := \inf\{t : B_t = x\}$, and set

$${}^{x}B_t := B_{\sigma_x + t} - x \quad (t \ge 0).$$

According to the strong Markov property of $B$, the process ${}^xB := ({}^xB_t, t \geq 0)$ is a standard Brownian motion independent of $\sigma_x$. Let ${}^xM_t$ and ${}^xL_t^0$ be the functionals of ${}^xB$ corresponding to the functionals $M_t$ and $L_t^0$ of $B$. Then for $y > 0$, $b \in \mathbb{R}$, and $a := |b - x|$, we can compute as follows:

$$
\begin{aligned}
P(L_1^x > y, B_1 \in db)/db &= P(\sigma_x < 1)\, \tfrac{1}{2}\, P({}^xL^0_{1-\sigma_x} > y, |{}^xB_{1-\sigma_x}| \in da)/da \\
&= P(\sigma_x < 1)\, \tfrac{1}{2}\, P({}^xM_{1-\sigma_x} > y, {}^xM_{1-\sigma_x} - {}^xB_{1-\sigma_x} \in da)/da \\
&= \tfrac{1}{2}\, P(M_1 > x + y, M_1 - B_1 \in da)/da \\
&= P(L_1^0 > x + y, B_1 \in da)/da.
\end{aligned}
$$

The first and third equalities are justified by the strong Markov property, the second appeals to Lévy's identity (2) applied to ${}^xB$, and the fourth uses (2) applied to $B$. The formula (1) for $x > 0$ can now be read from (1) with $x = 0$.

# 3 Some identities in distribution.

Let $R$ denote a random variable with the Rayleigh distribution

$$P(R > r) = e^{-\frac{1}{2}r^2} \qquad (r > 0). \tag{10}$$

According to formula (3),

$$(L_1^x \,|\, B_1 = 0) \stackrel{d}{=} (R - 2|x|)^+ \tag{11}$$

where the left side denotes the distribution of $L_1^x$ for a standard Brownian bridge. The corresponding result for the unconditioned Brownian motion, obtained by integrating out $b$ in (1), is

$$L_1^x \stackrel{d}{=} (|B_1| - |x|)^+. \tag{12}$$

Lévy gave both these identities for $x = 0$. For the bridge from $0$ to $b \in \mathbb{R}$ and $x > 0$ the events $(M_1 > x)$ and $(L_1^x > 0)$ are a.s. identical. So (3) for $y = 0$ reduces to Lévy's result that

$$P(M_1 > x \,|\, B_1 = b) = e^{-2x(x-b)} \qquad (0 \vee b < x).$$

Let

$$\varphi(z) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}; \quad \bar{\Phi}(x) := \int_x^\infty \varphi(z)dz = P(B_1 > x).$$

Then the mean occupation density at $x \in \mathbb{R}$ of the Brownian bridge from 0 to $b \in \mathbb{R}$ is

$$E(L_1^x \,|\, B_1 = b) = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\, \varphi\left(\frac{x - bs}{\sqrt{s(1-s)}}\right) ds = \frac{\bar{\Phi}(|x| + |b - x|)}{\varphi(b)}. \tag{13}$$

The first equality is read from the occupation density formula and the fact that $B_s^{0\to b}$ has normal distribution with mean $bs$ and variance $s(1-s)$. The second equality, which is not obvious directly, is obtained using the first equality by integration of (3). As a consequence of (13), for each $b > 0$ and each Borel subset $A$ of $[0, b]$, the expected time spent in $A$ by the Brownian bridge from 0 to $b$ is

$$E\left(\int_0^1 1(B_s^{0\to b} \in A) ds \,\middle|\, B_1 = b\right) = |A|\, \frac{\bar{\Phi}(b)}{\varphi(b)} \tag{14}$$

where $|A|$ is the Lebesgue measure of $A$. Take $A = [0, b]$ to recover the standard estimate $\bar{\Phi}(b) < \varphi(b)/b$. For each $b \in \mathbb{R}$, the function of $x$ appearing in (13) is the probability density function of $B_U^{0\to b}$ for $U$ a uniform$[0,1]$ variable independent of the bridge. In particular, for $b = 0$, formula (13) yields

$$|B_U^{0\to 0}| \stackrel{d}{=} \tfrac{1}{2} U R \tag{15}$$

where the Rayleigh variable $R$ is independent of $U$. This and related identities were found in [1], where the reflecting bridge $(|B_s^{0\to 0}|, 0 \le s \le 1)$ was used to describe the asymptotic distribution of a path derived from a random mapping.

Recall that $\sigma_x$ is the first hitting time of $x$ by the Brownian motion $B$. Let $\gamma_1^x$ denote last time $B$ is at $x$ before time 1, with the convention $\gamma_1^x = 0$ if no such time, and set $\delta_1^x = (\gamma_1^x - \sigma_x)^+$. By well known first entrance and last exit decompositions [10], given $B_1$ and $\delta_1^x$ with $\delta_1^x > 0$, the segment of $B$ between times $\sigma_x$ and $\gamma_1^x$ is a Brownian bridge of length $\delta_1^x$ from $x$ to $x$. Therefore,

$$(L_1^x \,|\, B_1 = b) \stackrel{d}{=} (\sqrt{\delta_1^x}\, R \,|\, B_1 = b) \tag{16}$$

where the Rayleigh variable $R$ is independent of $\delta_1^x$, and $R$ given $\delta_1^x > 0$ may be interpreted as the local time at 0 of the standard bridge derived by Brownian scaling of the segment of $B$ between times $\sigma_x$ and $\gamma_1^x$. By consideration of moments, formula (16) shows that the law of $(L_1^x \,|\, B_1 = b)$ displayed in formula (3) determines the law of $(\delta_1^x \,|\, B_1 = b)$, and vice versa. As indicated by Imhof [12], the distribution of $\delta_1^x$ given $B_1 = b$ can be derived by integration from the joint distribution of $\sigma_x$ and $\gamma_1^x$ given $B_1 = b$, which is easily written down. By comparison with the formula for the joint density of $\gamma_1^0$ and $|B_1|$, due to Chung [8, (2.5)], it turns out that

$$(\delta_1^x \,|\, B_1 = 0, \delta_1^x > 0) \stackrel{d}{=} (\gamma_1^0 \,|\, B_1 = 2x) \stackrel{d}{=} \frac{B_1^2}{B_1^2 + 4x^2} \tag{17}$$

where the second equality is obtained from Chung's formula by an elementary change of variable, as in [2, (6)-(8)], where the same family of distributions on $[0,1]$ appears in another context. Set $a = 2x$ and combine (11), (16) and (17) to deduce the identity

$$\sqrt{\frac{B_1^2}{B_1^2 + a^2}}\, R \stackrel{d}{=} (R - a \,|\, R > a) \qquad (a \geq 0) \tag{18}$$

where $R$ and $B_1$ are assumed independent. By consideration of moments, this identity amounts to the equality of two different integral representations for the Hermite function [13, (10.5.2) and Problem 10.8.1].

If $\varepsilon$ is independent of $B$ and exponentially distributed with rate 1, a variation of (16) gives

$$L_{2\varepsilon}^0 \stackrel{d}{=} \sqrt{\gamma_{2\varepsilon}^0}\, R \tag{19}$$

where $\gamma_{2\varepsilon}^0$ and $R$ are independent, hence

$$\varepsilon \stackrel{d}{=} |B_1|\, R \tag{20}$$

where $B_1$ and $R$ are independent. By consideration of moments, this classical identity is equivalent to the duplication formula for the gamma function [21]. Another Brownian representation of (20) is

$$|B_{2\varepsilon}^0| \stackrel{d}{=} \sqrt{2\varepsilon - \gamma_{2\varepsilon}^0}\, M_1 \tag{21}$$

where $M_1$ is the final value of a standard Brownian meander. Compare with [19, Ch. XII, Ex's (3.8) and (3.9)], and [5]. See also [3, p. 681] for an appearance of (20) in the study of random trees.

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