Applications of the continuous-time ballot theorem to Brownian motion and related processes.

by Jason Schweinsberg

Technical Report No. 583

Department of Statistics,
University of California
367 Evans Hall # 3860
Berkeley, CA 94720-3860

revised January 29, 2001

Abstract

Motivated by questions related to a fragmentation process which has been studied by Aldous, Pitman, and Bertoin, we use the continuous-time ballot theorem to establish some results regarding the lengths of the excursions of Brownian motion and related processes. We show that the distribution of the lengths of the excursions below the maximum for Brownian motion conditioned to first hit $\lambda > 0$ at time t is not affected by conditioning the Brownian motion to stay below a line segment from $(0, c)$ to (t, λ). We extend a result of Bertoin by showing that the length of the first excursion below the maximum for a negative Brownian excursion plus drift is a size-biased pick from all of the excursion lengths, and we describe the law of a negative Brownian excursion plus drift after this first excursion. We then use the same methods to prove similar results for the excursions of more general Markov processes.

1 Introduction

We use a continuous-time analog of the classical ballot theorem to prove some results pertaining to the lengths of the excursions of Brownian motion. We also extend these results to other Markov processes. This work was motivated by questions raised by an alternative construction described by Bertoin in [6] of a fragmentation process introduced by Aldous and Pitman in [4].

Before reviewing the descriptions of this process, we recall the definition of a fragmentation process, as given in [6]. For $l \geq 0$, define

$$\Delta_l = \Big\{ (x_i)_{i=1}^{\infty} : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} x_i = l \Big\}.$$

Let $\Delta = \bigcup_{l \geq 0} \Delta_l$. Suppose $\kappa_t(l)$ is a probability measure on Δ_l for all $l \geq 0$ and all $t \geq 0$. For each $L = (l_1, l_2, \ldots) \in \Delta$, let $\kappa_t(L)$ denote the distribution of the decreasing rearrangement of the terms of independent sequences $L_1, L_2, \ldots$, where L_i has distribution $\kappa_t(l_i)$ for all $i \in \mathbb{N}$. Then, for each $t \geq 0$, denote by κ_t the family of distributions $(\kappa_t(L), L \in \Delta)$, which we call the *fragmentation kernel* generated by $(\kappa_t(l), l \geq 0)$. If the fragmentation kernels $(\kappa_t, t \geq 0)$ form a semigroup, then any Δ-valued Markov process with $(\kappa_t, t \geq 0)$ as its transition semigroup is called a *fragmentation process.*

In [7], Bertoin characterizes all fragmentation processes that satisfy a kind of invariance under scaling. In [8], Bertoin extends this characterization to a class of fragmentation processes having a weaker self-similarity property. The self-similar fragmentation that has been studied the most thoroughly is the fragmentation process introduced by Aldous and Pitman in [4] and constructed another way by Bertoin in [6].

If $X = (X_t)_{t \geq 0}$ is a stochastic process such that $Z = \{t : X_t = 0\}$ is almost surely a closed set of zero Lebesgue measure, then $(0, l) \setminus Z$ almost surely consists of a finite or countable collection of disjoint open intervals whose lengths sum to l. The sequence consisting of the lengths of these intervals in decreasing order is almost surely in Δ_l, and we denote this sequence by $V_l(X)$. The distribution of $V_l(X)$ when X is Brownian motion or a Bessel process of dimension $\delta \in (0, 2)$ is studied in [21], [23], and [25]. In this case, it was shown in [19] that Z is the closure of the range of a stable subordinator of index α, where $\alpha = 1 - \delta/2$.

We now describe Bertoin's construction in [6] of a fragmentation process derived from Brownian motion with drift. Let $B = (B_t)_{t \geq 0}$ denote one-dimensional Brownian motion started at zero. Let $B^\lambda = (B_t^\lambda)_{t \geq 0} = (B_t + \lambda t)_{t \geq 0}$ denote Brownian motion with drift λ, and define $M_t^\lambda = \sup_{0 \leq s \leq t} B_s^\lambda$. Let $T_a = \inf\{t : B_t > a\}$, and let

$$F(\lambda) = V_{T_1}(M^\lambda - B^\lambda). \tag{1}$$

Thus $F(\lambda)$ consists of the lengths of the excursions below the maximum, up to time T_1, for Brownian motion with drift. Bertoin shows in [6] that $(F(\lambda))_{\lambda \geq 0}$ is a fragmentation process. To describe the fragmentation kernels, let $e = (e_t)_{0 \leq t \leq l}$ be a Brownian excursion of duration l, and define $\Psi_\lambda e = (\Psi_\lambda e_t)_{0 \leq t \leq l}$ by

$$\Psi_\lambda e_t = \sup_{0 \leq s \leq t} (\lambda s - e_s) - (\lambda t - e_t). \tag{2}$$

Let $\varphi_\lambda(l)$ be the distribution of $V_l(\Psi_\lambda e)$, and let φ_λ be the fragmentation kernel generated by $(\varphi_\lambda(l), l \geq 0)$. Bertoin shows that $(\varphi_\lambda, \lambda \geq 0)$ is the transition semigroup of $(F(\lambda))_{\lambda \geq 0}$.

In [4], Aldous and Pitman study a fragmentation process $(G(\lambda))_{\lambda \geq 0}$, where $G(\lambda)$ consists of the ranked masses of the components of the Brownian continuum random tree (see [1], [2], and [3]) when the skeleton of the tree has been subjected to a Poisson process of cuts for time λ. Aldous and Pitman show in [4] that the same fragmentation process arises by time-reversing the standard additive coalescent. To describe the distribution of $G(\lambda)$, let $B = (B_t)_{t \geq 0}$ be Brownian motion, and define $T = (T_a)_{a \geq 0}$ by $T_a = \inf\{t : B_t > a\}$. Let $(J_i)_{i=1}^{\infty}$ be the sequence of jump sizes of $(T_a)_{0 \leq a \leq \lambda}$ ranked in decreasing order. Since it is well-known that T is a stable subordinator of index $1/2$, it follows from Theorem 4 of [4] and scaling properties of stable subordinators that

$$G(\lambda) =_d (J_1, J_2, \ldots | T_\lambda = 1). \tag{3}$$

Note that $(J_i)_{i=1}^{\infty}$ is the ranked sequence of the lengths of the excursions of B below its maximum that are completed before time T_λ. That is, if $M = (M_t)_{t \geq 0}$ is defined by $M_t = \sup_{0 \leq s \leq t} B_s$ for all t, then $(J_i)_{i=1}^{\infty} = V_{T_\lambda}(M - B)$. Thus, (3) can be written as

$$G(\lambda) =_d (V_1(M - B) | T_\lambda = 1). \tag{4}$$

Bertoin shows in [6] that $(\varphi_\lambda, \lambda \geq 0)$ is also the transition semigroup for the process $(G(\lambda))_{\lambda \geq 0}$. We have $G(0) = (1, 0, 0, \ldots)$ almost surely, whereas $F(0) = V_{T_1}(M - B)$. Therefore, $(F(\lambda))_{\lambda \geq 0}$ and $(G(\lambda))_{\lambda \geq 0}$ are fragmentation processes with the same semigroup but different initial distributions. The fact that $(\varphi_\lambda, \lambda \geq 0)$ is the semigroup of $(G(\lambda))_{\lambda \geq 0}$ implies that if $e = (e_t)_{0 \leq t \leq 1}$ is a Brownian excursion of duration 1, then

$$(V_1(\Psi_\lambda e))_{\lambda \geq 0} =_d (G(\lambda))_{\lambda \geq 0}. \tag{5}$$

In [12], Chassaing and Louchard give an alternative proof of (5) based on a discrete approximation using parking functions. Note that (4) and (5) imply that

$$V_1(\Psi_\lambda e) =_d (V_1(M - B) | T_\lambda = 1) \tag{6}$$

for any fixed $\lambda > 0$. Conversely, once it is established that $(F(\lambda))_{\lambda \geq 0}$ and $(G(\lambda))_{\lambda \geq 0}$ are fragmentation processes, the equality in (6) combined with scaling arguments is sufficient to establish that the fragmentation kernels for these two processes must be the same. See section 7 of [12] for the necessary scaling arguments. In section 4, we will show how (6) follows from a path transformation result in [11] (see Lemma 16 and Remark 17).

The work in [6] raises further questions pertaining to the processes $(F(\lambda))_{\lambda \geq 0}$ and $(\Psi_\lambda e_t)_{0 \leq t \leq 1}$. The main purpose of this paper is to answer three such questions using the continuous-time ballot theorem. We introduce the three questions in subsections 1.1, 1.2, and 1.3 of the introduction. In section 2, we review the continuous-time ballot theorem. In section 3, we establish two theorems for the inverses of nondecreasing pure-jump processes with interchangeable increments. In section 4, we apply these results to prove the propositions stated in subsections 1.1, 1.2, and 1.3. In section 5, we show how, via path transformations, one of these results for Brownian motion yields information about the Brownian bridge, the Brownian excursion, the Brownian meander, and the three-dimensional Bessel process. Finally, in section 6, we apply the theorems in section 3 to obtain results about the excursion lengths of more general Markov processes.

1.1 Brownian motion conditioned to stay below a line

The equality of the transition semigroups of the processes $(F(\lambda))_{\lambda\geq 0}$ and $(G(\lambda))_{\lambda\geq 0}$ suggests the following result regarding the lengths of the excursions below the maximum for Brownian motion conditioned to first hit $\lambda+1$ at time 1.

Proposition 1 *Fix $\lambda \geq 0$. Let $W = (W_t)_{t\geq 0}$ be a process with the same law as a Brownian motion B conditioned on $T_{\lambda+1} = 1$, where $T_{\lambda+1} = \inf\{t : B_t > \lambda + 1\}$. Define $M = (M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} W_s$. Then $V_1(M - W)$ is independent of the event $\{W_t \leq 1 + \lambda t$ for all $t \in [0, 1]\}$.*

Note that $V_1(M - W)$ is the sequence of lengths of the excursions of W below its maximum up to time 1, and $\{W_t \leq 1 + \lambda t$ for all $t \in [0, 1]\}$ is the event that W does not cross the line from $(0, 1)$ to $(1, \lambda + 1)$.

We defer to section 4 a rigorous proof of Proposition 1. Now, we give an informal argument, without justifying the conditioning involved, for why we should expect Proposition 1 to follow from the equality of the transition semigroups of $(F(\lambda))_{\lambda\geq 0}$ and $(G(\lambda))_{\lambda\geq 0}$.

It follows from (4) and the definition of $F(\lambda)$ given in (1) that $G(1)$ has the same distribution as $F(0)$ conditioned on $T_1 = 1$. Since the processes $(F(\lambda))_{\lambda\geq 0}$ and $(G(\lambda))_{\lambda\geq 0}$ have the same transition semigroup, it follows that for any fixed $\lambda > 0$, the distribution of $G(\lambda + 1)$ is the same as the conditional distribution of $F(\lambda)$ given $T_1 = 1$. Thus, letting $B_t^\lambda = B_t + \lambda t$ and $M_t^\lambda = \sup_{0\leq s\leq t} B_s^\lambda$ and using (4) and (1), we obtain

$$(V_1(M - B)|T_{\lambda+1} = 1) =_d (V_1(M^\lambda - B^\lambda)|T_1 = 1). \tag{7}$$

Define $T_a^\lambda = \inf\{t : B_t^\lambda > a\}$. If $T_1 = 1$ then $T_{\lambda+1}^\lambda = 1$. Conversely, if $B_1^\lambda = \lambda + 1$, then $T_1 = 1$ if and only if $B_t^\lambda \leq 1 + \lambda t$ for all $t \in [0, 1]$. Therefore, $T_1 = 1$ if and only if $T_{\lambda+1}^\lambda = 1$ and $B_t^\lambda \leq 1 + \lambda t$ for all $t \in [0, 1]$, which means (7) can be written as

$$(V_1(M - B)|T_{\lambda+1} = 1) =_d (V_1(M^\lambda - B^\lambda)|T_{\lambda+1}^\lambda = 1 \text{ and } B_t^\lambda \leq 1 + \lambda t \text{ for all } t \in [0, 1]). \tag{8}$$

It is a consequence of Girsanov's Theorem that for all $a \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, the process $(B_t)_{0\leq t\leq 1}$ conditioned on $B_1 = a$ has the same law as $(B_t^\lambda)_{0\leq t\leq 1}$ conditioned on $B_1^\lambda = a$. Therefore, $(B_t^\lambda)_{0\leq t\leq 1}$ conditioned on $T_a^\lambda = 1$ has the same law as $(B_t)_{0\leq t\leq 1}$ conditioned on $T_a = 1$. This fact, combined with (8), implies

$$(V_1(M - B)|T_{\lambda+1} = 1) =_d (V_1(M - B)|T_{\lambda+1} = 1 \text{ and } B_t \leq 1 + \lambda t \text{ for all } t \in [0, 1]),$$

which is equivalent to the statement of the proposition.

Since Proposition 1 is just a fact about Brownian motion, the above discussion raises the question of whether one can find a proof that does not require introducing a fragmentation process. In section 4, we prove Proposition 1 using the continuous-time ballot theorem.

1.2 The length of the first excursion of $\Psi_\lambda e$

In [6], Bertoin also studies the length of the first excursion of the process $\Psi_\lambda e$ defined in (2), where e is a Brownian excursion of duration 1. We first give the following definition.

Definition 2 *Given a random sequence $V = (V_i)_{i=1}^\infty$ in Δ, a* size-biased pick from V *is a random variable V_N such that*

$$P(N = n|V) = \frac{V_n}{\sum_{i=1}^\infty V_i}. \tag{9}$$

Note that since (9) involves conditioning on the sequence V, a random variable can have the same distribution as a size-biased pick from V without being a size-biased pick from V.

Size-biased picks from random sequences in Δ_1 that are given by the interval lengths of $(0,1) \setminus Z$, where Z is a random closed set, have been studied, for example, in [20], [23], [24], and [25]. In [23], Pitman and Yor show that if Z is the zero set of Brownian motion or a Bessel process of dimension $\delta \in (0,2)$, then the length of the last interval of $(0,1) \setminus Z$, which has length $1 - \sup\{t \in (0,1) \cap Z\}$, is a size-biased pick from all of the interval lengths. Pitman and Yor show in [24] that if Z is any random self-similar closed subset of $(0,\infty)$ with zero Lebesgue measure, then $1 - \sup\{t \in (0,1) \cap Z\}$ has the same distribution as a size-biased pick from the lengths of the intervals of $(0,1) \setminus Z$ but is not necessarily a size-biased pick from these lengths.

A consequence of Proposition 10 of [6] is that the length of the first excursion interval of $\Psi_\lambda e$ has the same distribution as a size-biased pick from the interval lengths in the sequence $V_1(\Psi_\lambda e)$. Using the continuous-time ballot theorem combined with a path transformation identity proved in [11], we show that the length of the first excursion interval of $\Psi_\lambda e$ is indeed a size-biased pick from $V_1(\Psi_\lambda e)$. We state this result as Proposition 3 below.

Proposition 3 *Let $e = (e_t)_{0 \le t \le 1}$ be a Brownian excursion of length 1, and define $\Psi_\lambda e$ as in (2). Let $H = \inf\{t : \lambda t - e_t > 0\}$. Then, H is a size-biased pick from the sequence $V_1(\Psi_\lambda e)$.*

1.3 The process $(\lambda t - e_t)_{H \le t \le 1}$

We know from Proposition 3 that $H = \inf\{t : \lambda t - e_t > 0\}$ is a size-biased pick from the sequence $V_1(\Psi_\lambda e)$. It follows from results in [11] (see Theorem 2.6 and the discussion in subsection 6.3) that conditional on $H = h$, the process $(\lambda t - e_t)_{0 \le t \le H}$ has the same law as a Brownian excursion of length h. The following proposition describes the process $(\lambda t - e_t)_{H \le t \le 1}$.

Proposition 4 *Let $e = (e_t)_{0 \le t \le 1}$ be a Brownian excursion of length 1. Fix $\lambda > 0$, and let $H = \inf\{t : \lambda t - e_t > 0\}$. For all $r \ge 0$, let $W^{\lambda,r} = (W^{\lambda,r})_{t \ge 0}$ be a process with the same law as a Brownian motion B conditioned on $T_\lambda = r$, where $T_\lambda = \inf\{t : B_t > \lambda\}$. Then, the law of the process $(\lambda(t+H) - e_{t+H})_{0 \le t \le 1-H}$ conditioned on $H = h$ is the same as the conditional law of $(W_t^{\lambda,1-h})_{0 \le t \le 1-h}$ given the event $\{W_t^{\lambda,1-h} \le \lambda(t+h) \text{ for all } t \in [0,1-h]\}$.*

We show in section 4 that Proposition 4 follows from the continuous-time ballot theorem and a result in [20] pertaining to size-biased sampling from Poisson point processes.

2 The continuous-time ballot theorem

We first recall the classical ballot theorem. Suppose in an election, candidate A receives a votes and candidate B receives b votes, where $a > b$. The classical ballot theorem states that if the votes are counted in random order, then the probability that, for all $n \geq 1$, candidate A leads candidate B after n votes have been counted is $(a-b)/(a+b)$. A short proof using the reflection principle is given in section 3.3 of [13]. Another proof is given in [27].

To reformulate this result, let $\xi_i = 0$ if candidate A receives the ith vote, and let $\xi_i = 2$ if candidate B receives the ith vote. Let $X_n = \sum_{i=1}^n \xi_i$, and let $N = a + b$. Then, the classical ballot theorem states that for all even integers k less than N, we have

$$P(X_n < n \text{ for all } 1 \leq n \leq N | X_N = k) = \frac{a-b}{a+b} = 1 - \frac{2b}{N} = 1 - \frac{k}{N}. \tag{10}$$

As shown in [27] and [17], equation (10) holds whenever the vector $(\xi_1, \ldots, \xi_N)$ has nonnegative integer-valued components and its distribution is invariant under the N cyclic permutations of its components.

The ballot theorem has a natural generalization to continuous-time processes with cyclically interchangeable increments. Namely, if $T > 0$ is fixed and $(X_t)_{0 \leq t \leq T}$ is a nondecreasing process with cyclically interchangeable increments such that the derivative of $t \mapsto X_t$ is almost surely zero Lebesgue almost everywhere, then

$$P(X_t \leq t \text{ for all } 0 \leq t \leq T | X_T) = \max\left\{0, 1 - \frac{X_T}{T}\right\}. \tag{11}$$

In [27], Takács studies this generalization extensively and discusses applications to queuing processes and storage processes. See [17] for another proof of (11). See also [16] for a recent extension of the result to include processes with stationary, but not necessarily cyclically interchangeable, increments.

From (11), we easily obtain the following corollary, which we apply in section 3.

Corollary 5 *Fix $T > 0$ and $s > 0$. Let $X = (X_t)_{0 \leq t \leq T}$ be a nondecreasing stochastic process with cyclically interchangeable increments such that almost surely $X_0 = 0$ and $X_T = s$. Suppose the derivative of $t \mapsto X_t$ is almost surely zero Lebesgue almost everywhere. Fix $c \in [0, T]$. Then*

$$P(X_t \geq (t-c)\left(\frac{s}{T-c}\right) \textit{ for all } 0 \leq t \leq T) = \frac{c}{T}.$$

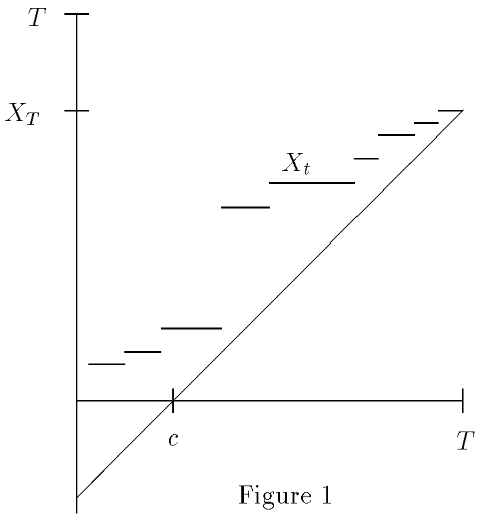

Figure 1

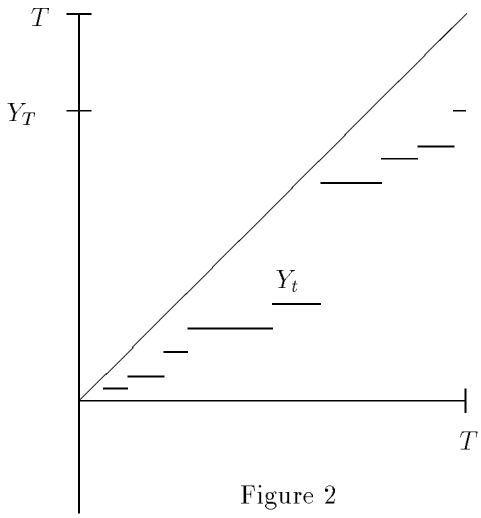

Figure 2

Proof. Let $K = (T - c)/s$. Define the process $Y = (Y_t)_{0 \le t \le T}$ by $Y_t = K(X_T - X_{T-t})$. (See Figures 1 and 2, and note that Figure 2 can be obtained from Figure 1 by a 180 degree rotation.) Note that Y is a nondecreasing process with cyclically interchangeable increments. It is also easily checked that the derivative of the function $t \mapsto Y_t$ is almost surely zero Lebesgue almost everywhere. Since $Y_T = Ks$ almost surely, (11) gives

$$P(Y_t \le t \text{ for all } 0 \le t \le T) = \max\left\{0, 1 - \frac{Ks}{T}\right\} = \max\left\{0, \frac{c}{T}\right\} = \frac{c}{T}. \tag{12}$$

Note that $Y_t \le t$ if and only if $X_{T-t} \ge s - t/K = s - st/(T-c) = (T-t-c)s/(T-c)$. Therefore, $Y_t \le t$ for all $0 \le t \le T$ if and only if $X_t \ge (t-c)s/(T-c)$ for all $0 \le t \le T$, so (12) implies the corollary. ■

3 Results for processes with interchangeable increments

In this section, we establish two theorems which apply to the inverses of nondecreasing pure-jump processes with interchangeable increments. Throughout the section, we assume that $T > 0$ is fixed and that $X = (X_a)_{0 \le a \le T}$ is a nondecreasing right-continuous process with interchangeable increments for which the closure of the range has zero Lebesgue measure. The condition that the closure of the range of X has zero Lebesgue measure is equivalent to the condition that X is a pure-jump process. We also define the inverse process $Y = (Y_t)_{0 \le t \le X_T}$ such that $Y_t = \inf\{a : X_a > t\}$ for $0 \le t < X_T$ and $Y_{X_T} = T$. It follows from Kallenberg's characterization of processes with interchangeable increments (see Theorem 2.1 of [15]) that if $J = (J_i)_{i=1}^{\infty}$ consists of the sizes of the jumps of X in decreasing order, then there exists a sequence $(U_i)_{i=1}^{\infty}$ of independent random variables with a uniform distribution on $[0, T]$ such that $(U_i)_{i=1}^{\infty}$ is independent of $(J_i)_{i=1}^{\infty}$ and

$$X_a = \sum_{i=1}^{\infty} J_i 1_{\{U_i \leq a\}} \tag{13}$$

for all $0 \leq a \leq T$.

For all $j = (j_i)_{i=1}^{\infty} \in \Delta$, we also define a process $X^j = (X_a^j)_{0 \leq a \leq T}$ by

$$X_a^j = \sum_{i=1}^{\infty} j_i 1_{\{U_i \leq a\}}. \tag{14}$$

Let $s = \sum_{i=1}^{\infty} j_i$, and define $Y^j = (Y_t^j)_{0 \leq t \leq s}$ by

$$Y_t^j = \inf\{a : X_a^j > t\} \text{ for } 0 \leq t < s, \tag{15}$$

$$Y_s^j = T. \tag{16}$$

Note that the law of X^j is the same as the conditional law of X given $J = j$, and therefore the law of Y^j is the same as the conditional law of Y given $J = j$.

The first theorem of this section states that for $c \in [0, T]$, the probability that Y does not cross a line from $(0, c)$ to (X_T, T) is c/T, and that this "crossing event" is independent of the jump sizes of X (see Figure 3).

Theorem 6 *Fix $T > 0$. Let $X = (X_a)_{0 \leq a \leq T}$ be a nondecreasing right-continuous process with interchangeable increments such that $X_0 = 0$ a.s. Let $S = X_T$, and assume $S > 0$ a.s. Let Z be the closure of $\{t : X_a = t$ for some $0 \leq a \leq T\}$, and assume Z has Lebesgue measure zero. Let $J = (J_i)_{i=1}^{\infty}$ be the sequence consisting of the lengths, in decreasing order, of the disjoint open intervals whose union is $(0, S) \setminus Z$. Let $Y = (Y_t)_{0 \leq t \leq S}$ be the right-continuous inverse of X, defined by $Y_t = \inf\{a : X_a > t\}$ for $0 \leq t < S$ and $Y_S = T$. Let $c \in [0, T]$. Then*

$$P\big(Y_t \leq c + \left(\frac{T-c}{S}\right) t \textit{ for all } 0 \leq t \leq S\big) = \frac{c}{T}. \tag{17}$$

Moreover, J is independent of the event $\big\{Y_t \leq c + \left(\frac{T-c}{S}\right) t$ for all $0 \leq t \leq S\big\}$.

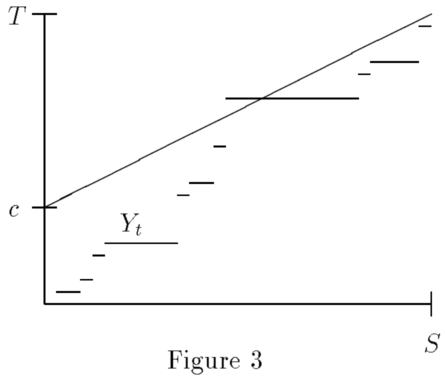

Figure 3

Proof. It suffices to show that

$$P\big(Y_t \leq c + \left(\frac{T-c}{S}\right)t \text{ for all } 0 \leq t \leq S \big| J\big) = \frac{c}{T}.$$

Since the law of the process Y^j defined by (15) and (16) is the same as the conditional law of Y given $J = j$, it suffices to prove that

$$P\big(Y_t^j \leq c + \left(\frac{T-c}{s}\right)t \text{ for all } 0 \leq t \leq s\big) = \frac{c}{T} \tag{18}$$

for all $j = (j_i)_{i=1}^{\infty} \in \Delta$, where $s = \sum_{i=1}^{\infty} j_i$.

Fix $j = (j_i)_{i=1}^{\infty} \in \Delta$, and let $s = \sum_{i=1}^{\infty} j_i$. Since $Y_t^j \leq T$ for all $0 \leq t \leq s$, clearly (18) holds when $c = T$. Assume $c < T$, and let $K = s/(T-c)$. We claim that $Y_t^j \leq c + K^{-1}t$ for all $0 \leq t \leq s$ if and only if $X_a^j \geq K(a-c)$ for all $0 \leq a \leq T$. If $Y_t^j \leq c + K^{-1}t$, then $X^j_{c+K^{-1}t} \geq t$ by the right continuity of X^j. If $Y_t^j > c + K^{-1}t$ for some $0 \leq t < s$, then there exists $\epsilon > 0$ such that $Y_t^j > c + K^{-1}(t+\epsilon)$ and thus $X^j_{c+K^{-1}(t+\epsilon)} \leq t < t + \epsilon$. Therefore, $Y_t^j \leq c + K^{-1}t$ for all $0 \leq t \leq s$ if and only if $X^j_{c+K^{-1}t} \geq t$ for all $0 \leq t \leq s$. By making the substitution $a = c + K^{-1}t$, we see that $X^j_{c+K^{-1}t} \geq t$ for all $0 \leq t \leq s$ if and only if $X_a^j \geq K(a-c)$ for all $0 \leq a \leq T$, which proves the claim.

It follows from (14) that $X_T^j = s$ and the derivative of the function $a \mapsto X_a^j$ is zero Lebesgue almost everywhere (see the Corollary on p.529 of [14]). Therefore, using Corollary 5 for the second equality, we have

$$P(Y_t^j \leq c + K^{-1}t \text{ for all } 0 \leq t \leq s) = P(X_a^j \geq K(a-c) \text{ for all } 0 \leq a \leq T) = \frac{c}{T},$$

which proves (18). ■

Remark 7 In Theorem 6, it is possible to replace the assumption that X has interchangeable increments with the weaker assumption that X has cyclically interchangeable increments. However, we will not need this generalization for the results that follow.

Before stating the second theorem of this section, we give a definition.

Definition 8 *Fix $T > 0$. Let $f : [0,T] \to \mathbb{R}$ be a function, and fix $w \in [0,T]$. Let $r_w f : [0,T] \to \mathbb{R}$ be the function defined by*

$$\begin{aligned} r_w f(t) &= f(w+t) - f(w) && \text{if } 0 \le t < T - w \\ r_w f(t) &= f(T) - f(w) + f(w+t-T) && \text{if } T - w \le t \le T. \end{aligned}$$

That is, $r_w f$ is the function obtained by cutting the function f at the point w and interchanging the segment of f from 0 to w with the segment of f from w to T. If $Y = (Y_t)_{0 \le t \le T}$ is a stochastic process and $w \in [0,T]$, then we can define a stochastic process $r_w Y = (r_w Y_t)_{0 \le t \le T}$ by replacing f with Y in the definition above.

Theorem 9 *Fix $T > 0$. Let $X = (X_a)_{0 \le a \le T}$ be a nondecreasing right-continuous process with interchangeable increments such that $X_0 = 0$ a.s. Let $S = X_T$, and assume $S > 0$ a.s. Let $\lambda = T/S$. Let Z be the closure of $\{t : X_a = t$ for some $0 \le a \le T\}$, and assume Z has Lebesgue measure zero. Let $J = (J_i)_{i=1}^\infty$ be the sequence consisting of the lengths, in decreasing order, of the disjoint open intervals whose union is $(0,S) \setminus Z$. Let $U = (U_i)_{i=1}^\infty$ be a sequence of independent random variables with a uniform distribution on $[0,T]$ such that U is independent of J and (13) holds. Let $Y = (Y_t)_{0 \le t \le S}$ be the right-continuous inverse of X, defined by $Y_t = \inf\{a : X_a > t\}$ for $0 \le t < S$ and $Y_S = T$. Let $c \in [0,T]$. Then*

(a) The process $(Y_t - \lambda t)_{0 \le t < S}$ almost surely attains its maximum at a unique time, which we denote by K. Almost surely $K = g_i$ for some i, where $g_i = X_{U_i -}$.

(b) Let $H = \inf\{t : r_K Y_t > 0\}$. Then H is a size-biased pick from $(J_i)_{i=1}^\infty$.

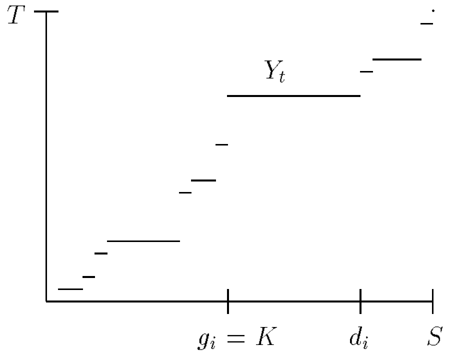

Figure 4

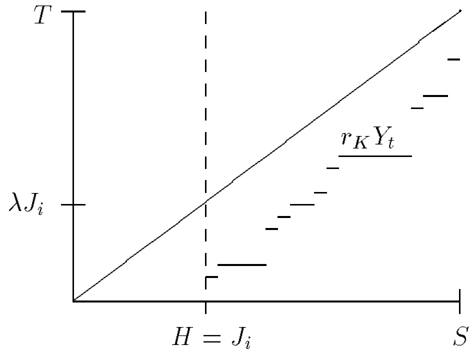

Figure 5

In Figure 4, we have labeled the time K at which $(Y_t - \lambda t)_{0 \le t < S}$ attains its maximum. Part (a) of Theorem 9 states that such a time must exist and be unique. Since the jump of X having size J_i is associated with a flat interval of Y having length J_i, part (b) of Theorem 9 implies that the length of the flat interval of Y starting at K is a size-biased pick from the lengths of all of the flat intervals of Y. Equivalently, part (b) implies that the length of the first flat interval of $r_K Y$ (see Figure 5) is a size-biased pick from the lengths of all flat intervals of $r_K Y$.

We now outline our strategy for proving Theorem 9. We first show in Lemma 10 that if $w \in [0, S]$, then $(Y_t - \lambda t)_{0 \leq t < S}$ attains a unique maximum at w if and only if $r_w Y_t < \lambda t$ for all $0 < t < S$. Then we show in Lemma 13 that if g_i is the left endpoint of the flat interval of length J_i, then

$$P(r_{g_i} Y_t < \lambda t \text{ for all } 0 < t < S | J) = J_i / S. \tag{19}$$

Since $\sum_{i=1}^{\infty} J_i = S$, equation (19) implies part (a) of Theorem 9, and part (b) follows from the fact that $J_i = \inf\{t : r_{g_i} Y_t > 0\}$.

To see how (19) follows from the continuous-time ballot theorem, consider Figure 5. Note that $r_{g_i} Y_t = 0$ for $0 \leq t < J_i$, so $r_{g_i} Y_t < \lambda t$ for all $0 < t < S$ if and only if $(r_{g_i} Y_t)_{J_i \leq t \leq 1}$ stays below the line from $(J_i, \lambda J_i)$ to (S, T). Since the portion of Figure 5 to the right of the dashed line looks like Figure 3, the probability of this event, conditional on J_i, is $\lambda J_i / T = J_i / S$.

We now begin the formal proof. We first establish two deterministic lemmas.

Lemma 10 *Fix $s > 0$, $T > 0$, and $w \in [0, s]$. Let $\lambda = T/s$. Suppose $f : [0, s] \to [0, T]$ is a function such that $f(0) = 0$ and $f(s) = T$. Then $r_w f(t) - \lambda t \leq 0$ for all $0 \leq t \leq s$ if and only if*

$$f(w) - \lambda w = \max_{0 \leq t \leq s} (f(t) - \lambda t).$$

Also, $r_w f(t) - \lambda t < 0$ for all $0 < t < s$ if and only if $f(w) - \lambda w > f(t) - \lambda t$ for all t such that $0 < |t - w| < s$.

Proof. If $0 \leq t < s - w$, then

$$r_w f(t) - \lambda t = f(w + t) - f(w) - \lambda t = \big(f(w + t) - \lambda(w + t)\big) - (f(w) - \lambda w). \tag{20}$$

If $s - w \leq t \leq s$, then

$$\begin{aligned} r_w f(t) - \lambda t &= f(s) - f(w) + f(w + t - s) - \lambda t \\ &= (f(s) - \lambda s) - (f(w) - \lambda w) + \big(f(w + t - s) - \lambda(w + t - s)\big) \\ &= \big(f(w + t - s) - \lambda(w + t - s)\big) - (f(w) - \lambda w). \end{aligned} \tag{21}$$

Equations (20) and (21) imply both statements of the lemma. ■

Lemma 11 *Fix $T > 0$. Choose a sequence $j = (j_i)_{i=1}^{\infty}$ in Δ and a sequence $u = (u_i)_{i=1}^{\infty}$ in $[0, T]^{\infty}$. Let $s = \sum_{i=1}^{\infty} j_i$. Define a function $f : [0, T] \to [0, s]$ by*

$$f(t) = \sum_{i=1}^{\infty} j_i 1_{\{u_i \leq t\}}.$$

Define a function $m : [0,T] \times [0,T] \to (0,T]$ such that $m(u,w) = u - w$ if $u > w$ and $m(u,w) = u - w + T$ if $u \le w$. Then,

$$r_w f(t) = \sum_{i=1}^{\infty} j_i 1_{\{m(u_i,w) \le t\}} \tag{22}$$

for all $w \in [0,T]$.

Proof. If $0 \le t < T - w$, then

$$r_w f(t) = f(w+t) - f(w) = \sum_{i=1}^{\infty} j_i 1_{\{w < u_i \le w+t\}} = \sum_{i=1}^{\infty} j_i 1_{\{m(u_i,w) \le t\}}. \tag{23}$$

If $T - w \le t \le T$, then

$$\begin{aligned} r_w f(t) = f(T) - f(w) + f(w+t-T) &= s - \sum_{i=1}^{\infty} j_i 1_{\{w+t-T < u_i \le w\}} \\ = s - \sum_{i=1}^{\infty} j_i 1_{\{m(u_i,w) > t\}} &= \sum_{i=1}^{\infty} j_i 1_{\{m(u_i,w) \le t\}}. \end{aligned} \tag{24}$$

Equations (23) and (24) establish (22). ∎

Lemma 12 *Fix $j = (j_i)_{i=1}^{\infty} \in \Delta$, and let $s = \sum_{i=1}^{\infty} j_i$. Define X^j as in (14) and Y^j as in (15) and (16). Fix $i \in \mathbb{N}$. Define $g_i = X^j_{U_i-}$ and $d_i = X^j_{U_i}$. Then, we have $r_{g_i} Y^j_t = 0$ for $0 \le t < j_i$ a.s. and $r_{g_i} Y^j_{t+j_i} = \inf\{a : r_{U_i} X^j_a > t\}$ for $0 \le t < s - j_i$ a.s.*

Proof. It follows from (14) that $d_i = X^j_{U_i} = j_i + g_i$ and $Y^j_t = U_i$ for $g_i \le t < d_i$. Also, $X^j_0 = 0$ a.s. because $U_j = 0$ a.s. for all $j \in \mathbb{N}$. We now prove the lemma by considering three cases.

Case 1: Suppose $0 \le t < j_i$. Then $g_i + t < j_i + g_i = d_i \le s$, so $r_{g_i} Y^j_t = Y^j_{t+g_i} - Y^j_{g_i} = U_i - U_i = 0$, as claimed.

Case 2: Suppose $0 \le t < s - d_i$. Then $X^j_T - X^j_{U_i} = s - d_i > t$, so $\inf\{a : X^j_{U_i+a} - X^j_{U_i} > t\} \le T - U_i$. It follows from this inequality and the fact that $X^j_0 = 0$ a.s. that $\inf\{a : r_{U_i} X^j_a > t\} = \inf\{a : X^j_{U_i+a} - X^j_{U_i} > t\} = \inf\{b - U_i : X^j_b > t + d_i\} = Y^j_{t+d_i} - U_i = Y^j_{g_i+t+j_i} - Y^j_{g_i} = r_{g_i} Y^j_{t+j_i}$, as claimed.

Case 3: Suppose $s - d_i \le t < s - j_i$. If $a < T - U_i$, then $X^j_{U_i+a} - X^j_{U_i} \le s - d_i \le t$. Therefore, $\inf\{a : r_{U_i} X^j_a > t\} = \inf\{a : s - X^j_{U_i} + X^j_{U_i+a-T} > t\} = \inf\{b + T - U_i : X^j_b > t + d_i - s\} = Y^j_{t+d_i-s} + T - U_i = T - Y^j_{g_i} + Y^j_{g_i+t+j_i-s} = r_{g_i} Y^j_{t+j_i}$, as claimed. ∎

Lemma 13 *Fix $j = (j_i)_{i=1}^{\infty} \in \Delta$, and let $s = \sum_{i=1}^{\infty} j_i$. Define X^j as in (14) and Y^j as in (15) and (16). For all $i \in \mathbb{N}$, define $g_i = X^j_{U_i-}$ and $d_i = X^j_{U_i}$. Let $\lambda = T/s$. Then, for all $i \in \mathbb{N}$ such that $j_i > 0$, we have $P(r_{g_i} Y^j_t < \lambda t$ for all $0 < t < s) = j_i/s$.*

Proof. It is easy to verify the lemma if $j_i = 0$ for all $i \geq 2$, so we will assume $j_2 > 0$. Fix $i \in \mathbb{N}$ such that $j_i > 0$. Define a process $R^i = (R^i_t)_{0 \leq t \leq s-j_i}$ by $R^i_t = r_{g_i} Y^j_{t+j_i}$. By Lemma 12, we have $r_{g_i} Y^j_t = 0 < \lambda t$ for $0 < t < j_i$. Therefore $r_{g_i} Y^j_t < \lambda t$ for all $0 < t < s$ if and only if $r_{g_i} Y^j_t < \lambda t$ for all $j_i \leq t < s$, or, equivalently, if and only if $R^i_t < \lambda(t + j_i)$ for all $0 \leq t < s - j_i$.

Define $X^{j,i} = (X^{j,i}_a)_{0 \leq a \leq T}$ such that $X^{j,i}_a = r_{U_i} X^j_a$ if $0 \leq a < T$ and $X^{j,i}_T = s - j_i$. By Lemma 11, we have

$$X^{j,i}_a = \sum_{k \neq i} j_k 1_{\{m(U_k, U_i) \leq a\}} \tag{25}$$

for all $0 \leq a \leq T$, where m is the function defined in Lemma 11. Since the random variables $m(U_k, U_i)$ for $k \neq i$ are independent and have a uniform distribution on $[0, T]$, equation (25) implies that $X^{j,i}$ is a nondecreasing, right-continuous process with interchangeable increments such that $X^{j,i}_0 = 0$ a.s. and $X^{j,i}_T = s - j_i > 0$. Also, the closure of $\{t : X^{j,i}_a = t \text{ for some } 0 \leq a \leq T\}$ has Lebesgue measure zero. By Lemma 12, $R^i_t = r_{g_i} Y^j_{t+j_i} = \inf\{a : r_{U_i} X^j_a > t\} = \inf\{a : X^{j,i}_a > t\}$ for all $0 \leq t < s - j_i$, and $R^i_{s-j_i} = r_{g_i} Y^j_s = T$. Let $0 \leq \epsilon < 1$, and let $c = (1-\epsilon) j_i T / s$. By Theorem 6,

$$P\left(R^i_t \leq c + \left(\frac{T - c}{s - j_i}\right) t \text{ for all } 0 \leq t \leq s - j_i\right) = \frac{c}{T} = \frac{(1-\epsilon) j_i}{s}. \tag{26}$$

Note that

$$c + \left(\frac{T-c}{s-j_i}\right) t = \frac{(1-\epsilon) j_i T}{s} + \left(\frac{T - (1-\epsilon) j_i T / s}{s - j_i}\right) t = \lambda\left((1-\epsilon) j_i + \left(\frac{s - (1-\epsilon) j_i}{s - j_i}\right) t\right),$$

which equals $\lambda(t + j_i)$ if $\epsilon = 0$ and is less than $\lambda(t + j_i)$ if $\epsilon > 0$ and $0 \leq t < s - j_i$. Therefore, equation (26) when $\epsilon = 0$ becomes

$$P(R^i_t \leq \lambda(t + j_i) \text{ for all } 0 \leq t \leq s - j_i) = \frac{j_i}{s}.$$

Since we always have $R^i_t \leq \lambda(t + j_i)$ when $t = s - j_i$, it follows that

$$P(R^i_t < \lambda(t + j_i) \text{ for all } 0 \leq t < s - j_i) \leq \frac{j_i}{s}. \tag{27}$$

Using (26) when $\epsilon > 0$, we obtain

$$\begin{aligned} P(R^i_t < \lambda(t + j_i) \text{ for all } 0 \leq t < s - j_i) &\geq P\left(R^i_t \leq c + \left(\frac{T-c}{s-j_i}\right) t \text{ for all } 0 \leq t \leq s - j_i\right) \\ &= \frac{(1-\epsilon) j_i}{s}. \end{aligned}$$

Letting $\epsilon \downarrow 0$ gives $P(R^i_t < \lambda(t + j_i) \text{ for all } 0 \leq t < s - j_i) \geq j_i / s$, which, combined with (27), implies the lemma. ■

Proof of Theorem 9. If $J_i > 0$, then $g_i < S$. Therefore, by Lemma 10, $(Y_t - \lambda t)_{0 \le t < S}$ attains its maximum only at time g_i if and only if $r_{g_i} Y_t < \lambda t$ for all $0 < t < S$. Since, for all $j \in \Delta$, the distribution of Y^j is the same as the conditional distribution of Y given $J = j$, Lemma 13 implies that

$$P(r_{g_i} Y_t < \lambda t \text{ for all } 0 < t < S | J) = J_i / S$$

on the event $\{J_i > 0\}$. Since $\sum_{i=1}^{\infty} J_i = S$, it follows that almost surely $(Y_t - \lambda t)_{0 \le t < S}$ attains its maximum at a unique time K, and

$$P(K = g_i | J) = J_i / S \tag{28}$$

for all $i \in \mathbb{N}$, which proves part (a) of Theorem 9. Part (b) will also follow from (28) if we can show that $H = J_i$ a.s. on the event $\{K = g_i\}$ whenever $J_i > 0$. Note that $r_{g_i} Y_t = Y_{t+g_i} - Y_t = U_i - U_i = 0$ a.s. if $0 \le t < J_i$. If $d_i = S$, then $r_{g_i} Y_{J_i} = Y_S - Y_{g_i} + Y_0 \ge T - U_i > 0$ a.s. because almost surely $U_i < T$ for all i, so in this case $\inf\{t : r_{g_i} Y_t > 0\} = J_i$. If instead $d_i < S$, then for $0 < t < S - d_i$, we have $r_{g_i} Y_{J_i + t} = Y_{d_i + t} - Y_{g_i} > 0$, since $Y_{d_i + t} > U_i$ by the right continuity of X. Hence, $\inf\{t : r_{g_i} Y_t > 0\} = J_i$, so $H = J_i$ a.s. on $\{K = g_i\}$, as claimed. ∎

4 Proofs of Propositions 1, 3, and 4

In this section, we prove Propositions 1, 3, and 4 in the introduction by applying Theorems 6 and 9. We first introduce some notation. Let $B^{|br|,r} = (B_t^{|br|,r})_{0 \le t \le r}$ be a reflecting Brownian bridge from $(0,0)$ to $(r,0)$, and let $(L_t^{|br|,r})_{0 \le t \le r}$ be its local time at zero, meaning that $L_t^{|br|,r}$ is the local time of $B^{|br|,r}$ at zero up to time t. By Lemma 12 of [22], for each $r > 0$ there exists, on the path space of continuous functions defined on $[0,r]$, a unique family of conditional laws $(P^{\lambda,r}, \lambda \ge 0)$ for $B^{|br|,r}$ given $L_r^{|br|,r} = \lambda$ that is weakly continuous in λ. Let $A^{\lambda,r} = (A_t^{\lambda,r})_{0 \le t \le r}$ be a process with law $P^{\lambda,r}$, and let $L^{\lambda,r} = (L_t^{\lambda,r})_{0 \le t \le r}$ denote its local time at zero. Define another process $W^{\lambda,r} = (W_t^{\lambda,r})_{0 \le t \le r}$ by $W_t^{\lambda,r} = L_t^{\lambda,r} - A_t^{\lambda,r}$. We claim that for $\lambda > 0$, the process $W^{\lambda,r}$ has the same law as Brownian motion conditioned to first hit λ at time r.

To prove this claim, let $(B_t)_{t \ge 0}$ be Brownian motion, and let $(L_t)_{t \ge 0}$ be its local time at zero. Let $M_t = \sup_{0 \le s \le t} M_s$ for all $t \ge 0$. For all $\lambda \ge 0$, let $T_\lambda = \inf\{t : B_t > \lambda\}$ and let $\tau_\lambda = \inf\{t : L_t > \lambda\}$. By equation (5.a) of [23], the conditional law of $(B_t^{|br|,r})_{0 \le t \le r}$ given $L_r^{|br|,r} = \lambda$ is the same as the conditional law of $(|B_t|)_{0 \le t \le r}$ given $\tau_\lambda = r$. By Lévy's Theorem, $(M_t - B_t, M_t)_{t \ge 0} =_d (|B_t|, L_t)_{t \ge 0}$. Therefore, for all $\lambda > 0$, the conditional law of the process $(M_t - B_t, M_t)_{0 \le t \le r}$ given $T_\lambda = r$ is the same as the conditional law of $(|B_t|, L_t)_{0 \le t \le r}$ given $\tau_\lambda = r$. Thus, the process $(A_t^{\lambda,r}, L_t^{\lambda,r})_{0 \le t \le r}$ has the same law as the conditional law of $(M_t - B_t, M_t)_{0 \le t \le r}$ given $T_\lambda = r$. It follows that $(W_t^{\lambda,r}, L_t^{\lambda,r})_{0 \le t \le r}$ has the same law as the conditional law of $(M_t - (M_t - B_t), M_t)_{0 \le t \le r} = (B_t, M_t)_{0 \le t \le r}$ given $T_\lambda = r$, which implies the claim.

A construction of $A^{\lambda,r}$ for $\lambda \geq 0$ and $r > 0$ is sketched in the proof of Lemma 12 of [22]. This construction in the case when $r = 1$ is described in section 6 of [23] and subsection 6.3 of [11]. We record the construction below.

Construction 14 Fix $\lambda \geq 0$ and $r > 0$. Let $J = (J_i)_{i=1}^\infty$ be a random sequence having the same distribution as $V_r(A^{\lambda,r})$. For a description of this distribution when $r = 1$, see subsection 6.3 of [11]. Independently of J, let $U = (U_i)_{i=1}^\infty$ be a sequence of i.i.d. random variables with a uniform distribution on $[0, \lambda]$. Define a process $X = (X_a)_{0 \leq a \leq \lambda}$ by

$$X_a = \sum_{i=1}^{\infty} J_i 1_{\{U_i \leq a\}}. \tag{29}$$

For all $i \in \mathbb{N}$, let $d_i = X_{U_i}$ and $g_i = X_{U_i -}$. Independently of (J, U), let $(e^i)_{i=1}^\infty$ be a sequence of independent Brownian excursions of length 1. Define $A^{\lambda,r} = (A_t^{\lambda,r})_{0 \leq t \leq r}$ by

$$A_t^{\lambda,r} = \sqrt{d_i - g_i}\, e^i_{(t-g_i)/(d_i-g_i)}$$

for $t \in (g_i, d_i)$ and $A_t^{\lambda,r} = 0$ for $t \notin (0, r) \setminus \bigcup_{i=1}^\infty (g_i, d_i)$. Then, $A^{\lambda,r}$ has law $P^{\lambda,r}$. Moreover, it is stated in [11] and [23] that if $L^{\lambda,r}$ is the local time of $A^{\lambda,r}$ at zero, then $L_t^{\lambda,r} = U_i$ for $t \in (g_i, d_i)$.

Using the notation of Construction 14, note that if $t \in (g_i, d_i)$ for some $i \in \mathbb{N}$, then we have $\inf\{a : X_a > t\} = U_i = L_t^{\lambda,r}$. Since $t \mapsto L_t^{\lambda,r}$ is a.s. nondecreasing and continuous, it follows that

$$L_t^{\lambda,r} = \inf\{a : X_a > t\} \quad \text{for all } 0 \leq t < r \quad \text{a.s.} \tag{30}$$

Likewise, if we define $W^{\lambda,r} = L^{\lambda,r} - A^{\lambda,r}$ and $M_t^{\lambda,r} = \sup_{0 \leq s \leq t} W_s^{\lambda,r}$, then $M_t^{\lambda,r} = U_i$ for all $g_i < t < d_i$ and $t \mapsto M_t^{\lambda,t}$ is a.s. nondecreasing and continuous. Therefore,

$$L_t^{\lambda,r} = M_t^{\lambda,r} \quad \text{for all } 0 \leq t \leq r \quad \text{a.s.,} \tag{31}$$

and so

$$(J_i)_{i=1}^\infty = V_r(A^{\lambda,r}) = V_r(L^{\lambda,r} - W^{\lambda,r}) = V_r(M^{\lambda,r} - W^{\lambda,r}) \quad \text{a.s.} \tag{32}$$

It follows from (32) that the terms of the sequence $(J_i)_{i=1}^\infty$ are both the lengths of excursions of $A^{\lambda,r}$ away from zero and the lengths of excursions of $W^{\lambda,r}$ below its current maximum.

It is clear from (29) that $(X_a)_{0 \leq a \leq \lambda}$ is a nondecreasing right-continuous process with interchangeable increments such that $X_0 = 0$ a.s. and $X_\lambda = r > 0$ a.s. Also, (29) implies that if Z is the closure of $\{t : X_a = t \text{ for some } 0 \leq a \leq \lambda\}$, then Z has Lebesgue measure zero a.s. Therefore, X satisfies the hypotheses of Theorems 6 and 9. From (30) and the fact that $L_r^{\lambda,r} = \lambda$, we see that the process $L^{\lambda,r}$ plays the role of Y in those theorems. Also, J is the sequence of ranked lengths of the open intervals whose union is $(0, r) \setminus Z$. Therefore, by applying Theorem 6, we obtain Proposition 15 below. From Proposition 15, we can deduce Proposition 1 by putting $\lambda + 1$ in place of λ and setting $r = c = 1$.

Proposition 15 *Fix $\lambda > 0$ and $r > 0$. Let $(W_t^{\lambda,r})_{0 \leq t \leq r}$ be a process with the same law as a Brownian motion B conditioned on $T_\lambda = r$, where $T_\lambda = \inf\{t : B_t > \lambda\}$. Fix $c \in [0, \lambda]$, and define $M_t^{\lambda,r} = \sup_{0 \leq s \leq t} W_s^{\lambda,r}$. Then*

$$P\big(W_t^{\lambda,r} \leq c + \left(\frac{\lambda - c}{r}\right) t \text{ for all } t \in [0, r]\big) = \frac{c}{\lambda}. \tag{33}$$

Moreover, $V_r(M^{\lambda,r} - W^{\lambda,r})$ is independent of the event $\big\{W_t^{\lambda,r} \leq c + \left(\frac{\lambda - c}{r}\right) t$ for all $t \in [0, r]\big\}$.

Proof. From Theorem 6 and equations (30) and (32), we obtain the conclusions of the proposition with $L_t^{\lambda,r}$ in place of $W_t^{\lambda,r}$ in (33) and in the definition of the event at the end of the statement of the proposition. The conclusions of the proposition then follow from (31) and the fact that the events $\{M_t^{\lambda,r} \leq c + ((\lambda - c)/r)t$ for all $t \in [0, r]\}$ and $\{W_t^{\lambda,r} \leq c + ((\lambda - c)/r)t$ for all $t \in [0, r]\}$ are the same. ∎

Our next goal is to prove Propositions 3 and 4. For the rest of this section, we will fix $\lambda > 0$ and we will use the notation of Construction 14 and the discussion preceding Proposition 15. Also, we will define $A = A^{\lambda,1}$, $L = L^{\lambda,1}$, $W = W^{\lambda,1}$, and $M = M^{\lambda,1}$. By part (a) of Theorem 9, there is almost surely a unique time K at which $(L_t - \lambda t)_{0 \leq t < 1}$ attains its maximum, and almost surely $K = g_i$ for some i. The fact that $(L_t - \lambda t)_{0 \leq t < 1}$ attains its maximum at a unique time also follows from Theorem 2.6 of [11]. Let

$$H = \inf\{t : r_K L_t > 0\}.$$

We have $H = J_i$ a.s. on the event $\{K = g_i\}$, as shown in the proof of Theorem 9. It follows from the description of the distribution of $(J_i)_{i=1}^\infty$ in section 6.3 of [11] that $J_1 > J_2 > \ldots > 0$ a.s. Therefore

$$\{H = J_i\} = \{K = g_i\} \tag{34}$$

up to a null set. Since $A_{g_i} = 0$ for all $i \in \mathbb{N}$, we have $L_K = W_K$ a.s. Since $W_t \leq L_t$ for all t, it follows that K is also the unique time at which $(W_t - \lambda t)_{0 \leq t < 1}$ attains its maximum.

Lemma 16 *Let $(\tilde{e}_t)_{0 \leq t \leq 1}$ be a Brownian excursion. Then $(r_K W_t)_{0 \leq t \leq 1} =_d (\lambda t - \tilde{e}_t)_{0 \leq t \leq 1}$.*

Proof. Corollary 2.9 of [11] states that the process $(e_t)_{0 \leq t \leq 1}$ defined by

$$e_t = A_{K+t} + \lambda t - L_{K+t} + L_K \qquad \text{if } K + t < 1 \tag{35}$$

$$e_t = A_{K+t-1} + \lambda t - L_{K+t-1} + L_K - \lambda \qquad \text{if } K + t \geq 1 \tag{36}$$

is a Brownian excursion. If $K + t < 1$, then $\lambda t - e_t = L_{K+t} - L_K - A_{K+t} = W_{K+t} - W_K = r_K W_t$. If $K + t \geq 1$, then $\lambda t - e_t = \lambda + L_{K+t-1} - L_K - A_{K+t-1} = \lambda + W_{K+t-1} - W_K = W_1 + W_{K+t-1} - W_K = r_K W_t$. Thus, $r_K W_t = \lambda t - e_t$ for all $0 \leq t \leq 1$, which proves the lemma. ∎

Remark 17 Note that equation (6) in the introduction can be deduced from Lemma 16 because $V_1(\Psi_\lambda e)$ is the sequence consisting of the ranked lengths of the excursions of $(\lambda t - e_t)_{t\geq 0}$ below its maximum, and $(V_1(M-B)|T_\lambda = 1)$ consists of the ranked excursion lengths of $(W_t)_{t\geq 0}$ below its maximum, or, equivalently, the ranked excursion lengths of $(r_K W_t)_{t\geq 0}$ below its maximum. Therefore, Lemma 16, combined with scaling arguments, can be used to establish the equality of the transition semigroups for the two fragmentation processes discussed in the introduction.

Proof of Proposition 3. Define a Brownian excursion $e = (e_t)_{0\leq t\leq 1}$ as in (35) and (36), so $\lambda t - e_t = r_K W_t$ for $0 \leq t \leq 1$. Then, we have

$$H = \inf\{t : r_K L_t > 0\} = \inf\{t : r_K W_t > 0\} = \inf\{t : \lambda t - e_t > 0\}.$$

By part (b) of Theorem 9, H is a size-biased pick from $(J_i)_{i=1}^{\infty}$. Therefore, to prove Proposition 3, it suffices to prove that $(J_i)_{i=1}^{\infty} = V_1(\Psi_\lambda e)$ a.s., where $\Psi_\lambda e$ is as defined in (2). By (32), it suffices to show that $V_1(M-W) = V_1(\Psi_\lambda e)$ a.s. Note that $M_t - W_t = 0$ if and only if $W_t = \sup_{0\leq s\leq t} W_s$, and $\Psi_\lambda e_t = 0$ if and only if $\lambda t - e_t = \sup_{0\leq s\leq t}(\lambda s - e_s)$ or, equivalently, if and only if $r_K W_t = \sup_{0\leq s\leq t} r_K W_s$. Since $M_0 = W_0 = 0$, $M_K = W_K$, and $M_1 = W_1 = \lambda$ a.s., Definition 8 implies that the following hold up to a null set:

$$\{t \leq 1-K : r_K W_t = \sup_{0\leq s\leq t} r_K W_s\} = \{t - K : t \geq K,\, W_t = \sup_{0\leq s\leq t} W_s\} \tag{37}$$

$$\{t \geq 1-K : r_K W_t = \sup_{0\leq s\leq t} r_K W_s\} = \{t + 1 - K : t \leq K,\, W_t = \sup_{0\leq s\leq t} W_s\}. \tag{38}$$

Equations (37) and (38) imply that $V_1(M-W) = V_1(\Psi_\lambda e)$, which proves the proposition. ■

To prove Proposition 4, we will need the following lemma, which can be deduced from equation (4.i) of [20].

Lemma 18 *Let $(Z_i)_{i=1}^{\infty}$ be the points of a Poisson point process N on $(0,\infty)$ with mean measure μ. Assume that μ is σ-finite and $\mu((0,\infty)) = \infty$. Also, assume $T = \sum_{i=1}^{\infty} Z_i$ is a.s. finite. Let N' be a point process obtained by deleting a point Z from N, where Z is a size-biased pick from $(Z_i)_{i=1}^{\infty}$. Let $T' = T - Z$. Then, the conditional distribution of N' given $T = t$ and $T' = t'$ is the same as the conditional distribution of N given $T = t'$.*

Remark 19 Recall that the jump sizes of a subordinator run for time t have the same distribution as the points of a Poisson point process on $(0,\infty)$ whose mean measure is t times the Lévy measure of the subordinator. The Lévy measure ν of a stable subordinator of index $1/2$ is given by $\nu(dx) = Cx^{-3/2}\,dx$, where C is a constant. Therefore, $t\nu$ is σ-finite and $t\nu((0,\infty)) = \infty$.

Proof of Proposition 4. For all $l \in \mathbb{N}$, let $J^{(-l)} = (J_i^{(-l)})_{i=1}^{\infty}$ be a sequence of random variables such that $J_i^{(-l)} = J_i$ for $i < l$ and $J_i^{(-l)} = J_{i+1}$ for $i \geq l$. Define $(U_i^{(-l)})_{i=1}^{\infty}$ by $U_i^{(-l)} = m(U_i, U_l)$

for $i < l$ and $U_i^{(-l)} = m(U_{i+1}, U_l)$ for $i \geq l$, where m is as defined in Lemma 11 with λ in place of T. It follows from Lemma 11 and the fact that $m(U_l^{(-l)}, U_l^{(-l)}) = \lambda$ for all $l \in \mathbb{N}$ that

$$r_{U_l} X_a = \sum_{i=1}^{\infty} J_i^{(-l)} 1_{\{U_i^{(-l)} \leq a\}} \tag{39}$$

for all $l \in \mathbb{N}$ and $0 \leq a < \lambda$. Now define $J' = (J_i')_{i=1}^{\infty}$ and $U' = (U_i')_{i=1}^{\infty}$ such that for all $i \in \mathbb{N}$ and $l \in \mathbb{N}$, we have $J_i' = J_i^{(-l)}$ and $U_i' = U_i^{(-l)}$ on $\{K = g_l\}$. Define $X' = (X_a')_{0 \leq a \leq \lambda}$ by

$$X_a' = \sum_{i=1}^{\infty} J_i' 1_{\{U_i' \leq a\}}. \tag{40}$$

By Lemma 12, we have $r_{g_l} L_{t+J_l} = \inf\{a : r_{U_l} X_a > t\}$ for all $l \in \mathbb{N}$ and $0 \leq t < 1 - J_l$. Since $H = J_l$ and $X' = r_{U_l} X$ a.s. on $\{K = g_l\}$, it follows that

$$r_K W_{t+H} = r_K L_{t+H} - r_K A_{t+H} = \inf\{a : X_a' > t\} - r_K A_{t+H} \tag{41}$$

for all $0 \leq t < 1 - H$.

Define $W^{\lambda,1-h} = L^{\lambda,1-h} - A^{\lambda,1-h}$, where $A^{\lambda,1-h}$ and $L^{\lambda,1-h}$ are obtained from $\tilde{J} = (\tilde{J}_i)_{i=1}^{\infty}$, $\tilde{U} = (\tilde{U}_i)_{i=1}^{\infty}$, and a sequence of Brownian excursions $(\tilde{e}^i)_{i=1}^{\infty}$ as in Construction 14. Define

$$\tilde{X}_a = \sum_{i=1}^{\infty} \tilde{J}_i 1_{\{\tilde{U}_i \leq a\}} \tag{42}$$

for all $0 \leq a \leq \lambda$. Then, using (30), we obtain

$$W_t^{\lambda,1-h} = L_t^{\lambda,1-h} - A_t^{\lambda,1-h} = \inf\{a : \tilde{X}_a > t\} - A_t^{\lambda,1-h} \tag{43}$$

for $0 \leq t < 1 - h$. We now claim that the conditional distribution of $(\tilde{J}, \tilde{U})$ given the event $\{W_t^{\lambda,1-h} \leq \lambda(t+h) \text{ for all } t \in [0, 1-h]\}$, which we denote hereafter by E_h, is the same as the conditional distribution of (J', U') given $H = h$. Recall that $(r_K A_{t+H})_{0 \leq t \leq 1-H}$ and $(A_t^{\lambda,1-h})_{0 \leq t \leq 1-h}$ were constructed from independent Brownian excursions over the flat intervals of $(r_K L_{t+H})_{0 \leq t \leq 1-H}$ and $(L_t^{\lambda,1-h})_{0 \leq t \leq 1-h}$ respectively. Therefore, by equations (40), (41), (42), and (43), the claim implies that the conditional law of $W^{\lambda,1-h}$ given E_h is the same as the conditional law of $(r_K W_{t+H})_{0 \leq t \leq 1-H}$ given $H = h$. Thus, by Lemma 16, the claim proves Proposition 4. To prove the claim, it suffices to prove the following two statements:

(a) The conditional distribution of $\tilde{J}$ given E_h is the same as the conditional distribution of J' given $H = h$.

(b) The conditional distribution of $\tilde{U}$ given E_h and given $\tilde{J} = j$ is the same as the conditional distribution of U' given $H = h$ and $J' = j$.

We first prove (a). Let B be a Brownian motion, and let $(L_t)_{t\geq 0}$ be the local time of B at zero. Define $\tau_\lambda = \inf\{t : L_t > \lambda\}$. As shown in the second paragraph of this section, the law of $M - W$ is the same as the conditional law of $|B|$ given $\tau_\lambda = 1$. Therefore, using (32), we see that J has the same distribution as the conditional distribution of $V_1(B)$ given $\tau_\lambda = 1$, which is the same as the conditional distribution of $V_{\tau_\lambda}(B)$ given $\tau_\lambda = 1$. By (34) and part (b) of Theorem 9, J' is obtained from J by deleting a point H, where H is a size-biased pick from J. Therefore, if $V'_{\tau_\lambda}(B)$ is a sequence obtained by removing a size-biased pick Z from the sequence $V_{\tau_\lambda}(B)$, then the conditional distribution of J' given $H = h$ is the same as the conditional distribution of $V'_{\tau_\lambda}(B)$ given $\tau_\lambda = 1$ and $Z = h$, which is the same as the conditional distribution of $V'_{\tau_\lambda}(B)$ given $\tau_\lambda = 1$ and $\tau_\lambda - Z = 1 - h$. Recall that $V_{\tau_\lambda}(B)$ consists of the jump sizes of a stable subordinator of index $1/2$ run for time λ, and τ_λ is the sum of these jump sizes. Therefore, by Lemma 18 and Remark 19, the conditional distribution of $V'_{\tau_\lambda}(B)$ given $\tau_\lambda = 1$ and $\tau_\lambda - Z = 1 - h$ is the same as the conditional distribution of $V_{\tau_\lambda}(B)$ given $\tau_\lambda = 1 - h$, which by (32) is the same as the distribution of $\tilde{J}$. By Proposition 15, $\tilde{J}$ is independent of E_h. Therefore, the distribution of $\tilde{J}$ is the same as the conditional distribution of $\tilde{J}$ given E_h, which establishes (a).

We now prove (b). For all $h \in (0,1)$ and all $j \in \Delta_{1-h}$, there exists a subset $D_{j,h}$ of $[0,\lambda]^\infty$ such that if $\tilde{J} = j$ then E_h occurs if and only if $\tilde{U} \in D_{j,h}$. Let μ_λ denote Lebesgue measure on $[0,\lambda]$, normalized by $1/\lambda$, and let A be a Borel subset of $[0,\lambda]^\infty$. Proposition 15 implies that $P(E_h) = h$ and E_h is independent of $\tilde{J}$. Fix $j \in \Delta_{1-h}$. Since $\tilde{U}$ has distribution μ_λ^∞ and is independent of $\tilde{J}$, we have

$$P(\tilde{U} \in A | E_h, \tilde{J} = j) = \frac{P(\{\tilde{U} \in A\} \cap E_h | \tilde{J} = j)}{P(E_h | \tilde{J} = j)} = \frac{P(\tilde{U} \in A \cap D_{j,h} | \tilde{J} = j)}{h} = \frac{\mu_\lambda^\infty(A \cap D_{j,h})}{h}. \tag{44}$$

Fix $l \in \mathbb{N}$ such that $l - 1$ is the number of terms in the sequence j greater than h, and let $j^{(+h)}$ be the sequence in Δ_1 whose terms include h and all of the terms of j. By Lemma 10, we have $K = g_l$ if and only if $r_{g_l} L_t < \lambda t$ for all $0 < t < 1$. Since $(L_t - \lambda t)_{0 \leq t < 1}$ attains its maximum at a unique time, Lemma 10 also implies that up to a null set, the condition that $r_{g_l} L_t < \lambda t$ for all $0 < t < 1$ is equivalent to the condition that $r_{g_l} L_t \leq \lambda t$ for all $0 \leq t \leq 1$, which by Lemma 12 is equivalent to the condition that $r_{g_l} L_{t+J_l} \leq \lambda(t + J_l)$ for all $0 \leq t \leq 1 - J_l$. By (39) and Lemma 12, we have $r_{g_l} L_{t+J_l} \leq \lambda(t + J_l)$ for all $0 \leq t \leq 1 - J_l$ if and only if $U^{(-l)} \in D_{J^{(-l)}, J_l}$. Since $\{U' = U^{(-l)}\} = \{K = g_l\}$ by the definition of U', it follows from (34) that

$$\{U' \in A\} \cap \{H = J_l\} = \{U' \in A\} \cap \{U^{(-l)} \in D_{J^{(-l)}, J_l}\} = \{U^{(-l)} \in A \cap D_{J^{(-l)}, J_l}\}$$

up to a null set. Since $U^{(-l)}$ has distribution μ_λ^∞ and is independent of J, and since H is a size-biased pick from J, we have

$$P(U' \in A | H=h, J'=j) = P(U' \in A | H=h, J=j^{(+h)}) = \frac{P(\{U' \in A\} \cap \{H=h\} | J=j^{(+h)})}{P(H=h|J=j^{(+h)})}$$
$$= \frac{P(U^{(-l)} \in A \cap D_{j,h} | J=j^{(+h)})}{h} = \frac{\mu_\lambda^\infty(A \cap D_{j,h})}{h}. \tag{45}$$

Equations (44) and (45) imply (b). ■

5 Results obtainable by path transformations

In this section, we present some corollaries of Proposition 15 that relate to the Brownian bridge, the Brownian excursion, the Brownian meander, and the three-dimensional Bessel process. We prove these results by applying well-known path transformations that enable us to construct one of these processes from another. See [9] for a discussion of a large collection of such transformations. Lemma 20 below contains the path transformation results that we will use.

Lemma 20 *Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion started at zero, and let $(R_t)_{t\geq 0}$ be a three-dimensional Bessel process started at zero. Then, the following hold:*

(a) The processes $\big((1-t)B_{t/(1-t)}\big)_{0\leq t\leq 1}$ and $\big(tB_{(1-t)/t}\big)_{0\leq t\leq 1}$ are Brownian bridges.

(b) The processes $\big((1-t)R_{t/(1-t)}\big)_{0\leq t\leq 1}$ and $\big(tR_{(1-t)/t}\big)_{0\leq t\leq 1}$ are Brownian excursions.

(c) For all $\lambda > 0$, define $T_\lambda = \inf\{t : B_t = \lambda\}$ and $L_\lambda = \sup\{t : R_t = \lambda\}$. Then, the processes $(R_t)_{0\leq t\leq L_\lambda}$ and $(\lambda - B_{T_\lambda - t})_{0\leq t\leq T_\lambda}$ have the same law.

Part (a) of Lemma 20 is part of exercise 3.10 in chapter I of [26]. The fact that $\big(tR_{(1-t)/t}\big)_{0\leq t\leq 1}$ is a Brownian excursion is stated in the proof of Proposition 10 in [6]. It then follows from the invariance of Brownian excursions under time reversal (see Corollary 4.3 in chapter XII of [26]) that $\big((1-t)R_{t/(1-t)}\big)_{0\leq t\leq 1}$ is a Brownian excursion. Part (c) is a time-reversal theorem proved by Williams in [28] and is also Corollary 4.4 in chapter XII of [26]. LeGall gives an alternative approach to this result in [18].

We begin with the following corollary pertaining to the Brownian bridge.

Corollary 21 *Let $B^{br} = (B_t^{br})_{0\leq t\leq 1}$ be a Brownian bridge. Fix $\lambda > 0$ and $c \in [0,\lambda]$. Let $\tau = \inf\{t : B_t^{br} = (1-t)\lambda\}$. Then*

$$P\Big(B_t^{br} \leq c + \left(\frac{\lambda(1-\tau)-c}{\tau}\right)t \text{ for all } 0\leq t\leq \tau \Big| \tau\Big) = \frac{c}{\lambda}. \tag{46}$$

Likewise, let $\gamma = \sup\{t : B_t^{br} = \lambda t\}$. Then

$$P\Big(B_t^{br} \leq \frac{\gamma(\lambda-c)+(c-\lambda\gamma)t}{1-\gamma} \text{ for all } \gamma\leq t\leq 1 \Big| \gamma\Big) = \frac{c}{\lambda}. \tag{47}$$

Equation (46) states that if $(\tau, \lambda - \lambda\tau)$ is the point at which B^{br} first crosses the line segment from $(0, \lambda)$ to $(1, 0)$, then B^{br} crosses the line segment from $(0, c)$ to $(\tau, (1-\tau)\lambda)$ with probability c/λ (see Figure 6). Equation (47) states that if γ is the last time that B^{br} crosses the line segment from $(0, 0)$ to $(1, \lambda)$, then B^{br} crosses the line segment from $(\gamma, \lambda\gamma)$ to $(1, c)$ with probability c/λ (see Figure 7).

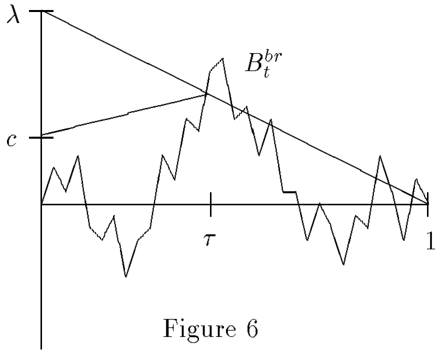

Figure 6

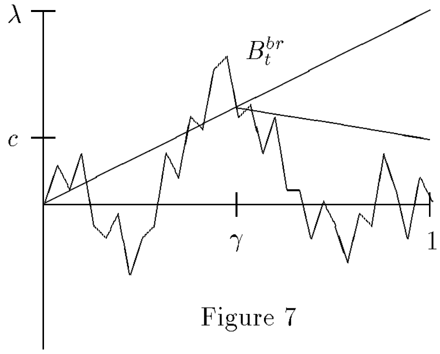

Figure 7

Proof. Let $B = (B_t)_{t\geq 0}$ be Brownian motion, and let $T_\lambda = \inf\{t : B_t = \lambda\}$. By part (a) of Lemma 20, we may assume that $B_t^{br} = (1-t)B_{t/(1-t)}$ for all $0 \leq t \leq 1$. Then, we have $B_t^{br} = (1-t)\lambda$ if and only if $B_{t/(1-t)} = \lambda$, so $T_\lambda = \tau/(1-\tau)$. Furthermore, since

$$(1-t)\left(c + \left(\frac{\lambda - c}{T_\lambda}\right)\left(\frac{t}{1-t}\right)\right) = c + \left(\frac{\lambda(1-\tau) - c}{\tau}\right)t,$$

it follows from Proposition 15 that

$$\begin{aligned} P\big(B_t^{br} &\leq c + \left(\frac{\lambda(1-\tau)-c}{\tau}\right)t \text{ for all } 0 \leq t \leq \tau \big| \tau\big) \\ &= P\big(B_t \leq c + \left(\frac{\lambda - c}{T_\lambda}\right)t \text{ for all } 0 \leq t \leq T_\lambda \big| T_\lambda\big) = \frac{c}{\lambda}, \end{aligned}$$

which proves (46). We can deduce (47) from (46) by time reversal, replacing t by $1-t$. Alternatively, we can establish (47) by assuming $B_t^{br} = tB_{(1-t)/t}$ and giving another argument similar to that above. ■

We now use Williams' time-reversal theorem to obtain a result about the three-dimensional Bessel process. See Figure 8 for the associated picture.

Corollary 22 *Let $(R_t)_{t\geq 0}$ be a three-dimensional Bessel process started at zero. Fix $\lambda > 0$ and $r > 0$, and fix $c \in [0, \lambda]$. Then, $P(R_t \geq ct/r$ for all $0 \leq t \leq r | R_r = \lambda) = (\lambda - c)/\lambda$.*

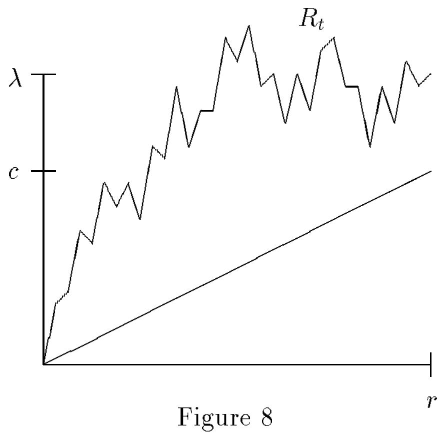

Figure 8

Proof. Let $(B_t)_{t\geq 0}$ be Brownian motion. Define $T_\lambda = \inf\{t : B_t = \lambda\}$ and $L_\lambda = \sup\{t : R_t = \lambda\}$. Let $a = \lambda - c$. Then, $B_t \leq a + (\lambda - a)t/T_\lambda$ for all $0 \leq t \leq T_\lambda$ if and only if $\lambda - B_{T_\lambda - t} \geq (\lambda - a)t/T_\lambda = ct/T_\lambda$ for all $0 \leq t \leq T_\lambda$. It follows from Proposition 15 and part (c) of Lemma 20 that

$$\begin{aligned} & P(R_t \geq ct/r \text{ for all } 0 \leq t \leq r | L_\lambda = r) \\ & \quad = P(\lambda - B_{T_\lambda - t} \geq ct/r \text{ for all } 0 \leq t \leq r | T_\lambda = r) \\ & \quad = P\big(B_t \leq a + \left(\frac{\lambda - a}{r}\right) t \text{ for all } 0 \leq t \leq r \big| T_\lambda = r\big) = \frac{a}{\lambda} = \frac{\lambda - c}{\lambda}. \end{aligned} \tag{48}$$

It is stated in the proof of Theorem 3 of [10] that $(R_t)_{0\leq t\leq r}$ conditioned on $L_\lambda = r$ has the same law as $(R_t)_{0\leq t\leq r}$ conditioned on $R_r = \lambda$. This result, combined with (48), establishes the corollary. ■

Remark 23 Let $(m_t)_{0\leq t\leq 1}$ be a normalized Brownian meander. It is proved in [10] that if f is a nonnegative measurable function whose domain is the set of all continuous $[0, \infty)$-valued functions defined in $[0, 1]$, then

$$E\big[f\big((m_t)_{0\leq t\leq 1}\big)\big] = E\left[f\big((R_t)_{0\leq t\leq 1}\big)\sqrt{\frac{\pi}{2}\frac{1}{R_1}}\right].$$

Therefore, the process $(m_t)_{0\leq t\leq 1}$ conditioned on $m_1 = \lambda$ has the same law as $(R_t)_{0\leq t\leq 1}$ conditioned on $R_1 = \lambda$. Thus, Corollary 22 gives $P(m_t \geq ct \text{ for all } 0 \leq t \leq 1 | m_1 = \lambda) = (\lambda - c)/\lambda$ for all $c \in [0, \lambda]$.

We now show how Corollary 22 gives rise to a result for Brownian excursions. See figures 9 and 10 for the pictures associated with equations (49) and (50) respectively.

Corollary 24 *Let $e = (e_t)_{0 \le t \le 1}$ be a normalized Brownian excursion. Fix $\lambda > 0$ and $u \in (0,1)$. Fix $c \in [0, \lambda]$. Then*

$$P\big(e_t \ge \left(\frac{c}{u}\right) t \text{ for all } 0 \le t \le u \big| e_u = \lambda\big) = \frac{\lambda - c}{\lambda} \tag{49}$$

and

$$P\big(e_t \ge \left(\frac{c}{1-u}\right)(1-t) \text{ for all } u \le t \le 1 \big| e_u = \lambda\big) = \frac{\lambda - c}{\lambda}. \tag{50}$$

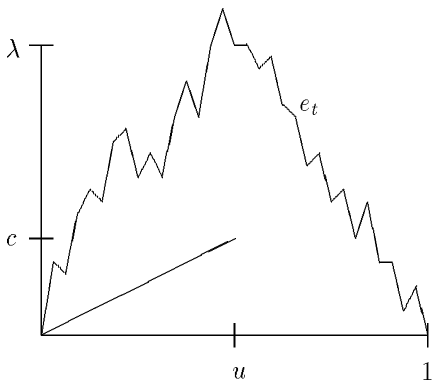

Figure 9

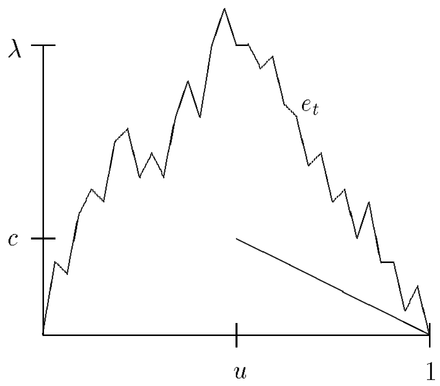

Figure 10

Proof. Let $(R_t)_{t \ge 0}$ be a three-dimensional Bessel process. By part (b) of Lemma 20, we may assume that $e_t = (1-t)R_{t/(1-t)}$ for all $0 \le t \le 1$. Then $e_u = \lambda$ if and only if $R_{u/(1-u)} = \lambda/(1-u)$. Therefore, using Corollary 22 for the next-to-last equality, we have

$$\begin{aligned}
&P(e_t \ge ct/u \text{ for all } 0 \le t \le u | e_u = \lambda) \\
&\quad = P\big((1-t)R_{t/(1-t)} \ge ct/u \text{ for all } 0 \le t \le u | R_{u/(1-u)} = \lambda/(1-u)\big) \\
&\quad = P\big(R_{t/(1-t)} \ge ct/u(1-t) \text{ for all } 0 \le t \le u | R_{u/(1-u)} = \lambda/(1-u)\big) \\
&\quad = P\big(R_s \ge cs/u \text{ for all } 0 \le s \le u/(1-u) | R_{u/(1-u)} = \lambda/(1-u)\big) \\
&\quad = P\left(R_s \ge \left(\frac{c}{1-u}\right)\left(\frac{u}{1-u}\right)^{-1} s \text{ for all } 0 \le s \le \frac{u}{1-u} \middle| R_{u/(1-u)} = \frac{\lambda}{1-u}\right) \\
&\quad = \left(\frac{\lambda}{1-u} - \frac{c}{1-u}\right) \Big/ \left(\frac{\lambda}{1-u}\right) = \frac{\lambda - c}{\lambda},
\end{aligned}$$

which proves (49). Equation (50) follows easily from the symmetry of Brownian excursion under time reversal. Alternatively, (50) can be proved by assuming $e_t = tR_{(1-t)/t}$ and following steps similar to those above. ■

6 Excursions of Markov processes

In this section, we show how Theorems 6 and 9 lead to results pertaining to the excursions of more general Markov processes. We consider a Markov process $\xi = (\xi_t)_{t \geq 0}$ which is "nice" in the sense defined at the beginning of chapter IV of [5]. That is, we assume ξ is an $\mathbb{R}^d$-valued stochastic process with right-continuous sample paths that is adapted to a complete right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ and satisfies a Markov property. The Markov property is defined in [5] as the property that there exists a family of probability measures $(P^x, x \in \mathbb{R}^d)$ such that for every stopping time $T < \infty$, the shifted process $(\xi_{T+t})_{t \geq 0}$ conditional on $\xi_T = x$ is independent of $\mathcal{F}_T$ and has law P^x. As noted in [5], Feller processes satisfy these conditions.

We assume that $\xi_0 = 0$ a.s. We also assume that 0 is a *regular point*, which means that $\inf\{t > 0 : \xi_t = 0\} = 0$ a.s., and an *instantaneous point*, meaning that $\inf\{t > 0 : \xi_t \neq 0\} = 0$ a.s. Thus, ξ does not hold in its initial state, but it returns to that state at arbitrarily small positive times. We also assume that 0 is *recurrent*, meaning that $\sup\{t : \xi_t = 0\} = \infty$ a.s. Let Z denote the closure of $\{t : \xi_t = 0\}$. Then, $(0, \infty) \setminus Z$ consists of a collection of disjoint open intervals, which we call the *excursion intervals* of ξ away from 0.

In section 2 of chapter IV of [5], Bertoin constructs a process $(L_t)_{t \geq 0}$ called the *local time* of ξ, which is determined up to an arbitrary positive constant. By Theorem 4 in chapter IV of [6], the process $(L_t)_{t \geq 0}$ is continuous and nondecreasing and satisfies $L_0 = 0$. The same theorem states that Z is the support of the Stieltjes measure dL, so L is constant on the excursion intervals of ξ. Still following [5], we define the inverse local time process $\tau = (\tau_a)_{a \geq 0}$ by $\tau_a = \inf\{t : L_t > a\}$. Then, by Proposition 7 in chapter IV of [5], the following two equations hold for all $t > 0$:

$$\tau_{L_t} = \inf\{s > t : \xi_s = 0\}, \tag{51}$$

$$\tau_{L_{t-}} = \sup\{s < t : \xi_s = 0\}. \tag{52}$$

By Theorem 8 in chapter IV of [5], the process $(\tau_a)_{a \geq 0}$ is a subordinator. For this result, we need the assumption that 0 is recurrent, which ensures that $\lim_{t \to \infty} L_t = \infty$ almost surely.

Lemma 25 *The set Z is the closure of $\{t : \tau_a = t$ for some $a\}$.*

Proof. Since $\xi_0 = 0$, clearly $0 \in Z$. If $L_t = 0$ for some $t > 0$, then the Stieltjes measure dL is supported on $[t, \infty)$, which contradicts that Z is the is the support of dL. Thus, $\tau_0 = 0$. Now suppose $t > 0$ and $\xi_t = 0$. By (51), if $0 < \epsilon < t$, then $\tau_{L_{t-\epsilon}} = \inf\{s > t - \epsilon : \xi_s = 0\}$, which is in the interval $[t - \epsilon, t]$. Therefore, t is in the closure of $\{t : \tau_a = t$ for some $a\}$. It follows that Z is contained in the closure of $\{t : \tau_a = t$ for some $a\}$.

Next, suppose $t > 0$ and $\tau_a = t$ for some a. Since $(L_s)_{s \geq 0}$ is continuous and $\lim_{s \to \infty} L_s = \infty$ a.s., there exists $u > 0$ such that $L_u = a$. By (51), we have $\tau_a = \tau_{L_u} = \inf\{s > u : \xi_s = 0\} \in Z$. It follows that $\{t : \tau_a = t$ for some $a\} \subset Z$, so the closure of $\{t : \tau_a = t$ for some $a\}$ is contained in Z. ■

Corollary 26 *Fix $T > 0$. Let $\xi = (\xi_t)_{t \geq 0}$ be a Markov process which is "nice" in the sense defined at the beginning of this section. Assume $\xi_0 = 0$ a.s. and that 0 is regular, instantaneous, and recurrent. Let Z be the closure of $\{t : \xi_t = 0\}$, and assume Z has Lebesgue measure zero a.s. Let $(L_t)_{t \geq 0}$ be the local time of ξ at zero. Let $S = \inf\{t : L_t > T\}$, and let $\lambda = T/S$. Let $(J_i)_{i=1}^{\infty}$ be the sequence consisting of the lengths, in decreasing order, of the disjoint open intervals whose union is $(0, S) \setminus Z$. Then for all $c \in [0, T]$,*

$$P\big(L_t \leq c + \left(\frac{T-c}{S}\right)t \text{ for all } 0 \leq t \leq S\big) = \frac{c}{T},$$

and $(J_i)_{i=1}^{\infty}$ is independent of the event $\{L_t \leq c + ((T-c)/S)t \text{ for all } 0 \leq t \leq S\}$. Moreover, $(L_t - \lambda t)_{0 \leq t < S}$ almost surely attains its maximum at a unique time, which we denote by K. If $H = \inf\{t : r_K L_t > 0\}$, then H is a size-biased pick from $(J_i)_{i=1}^{\infty}$.

Proof. Define $\tau = (\tau_a)_{a \geq 0}$ by $\tau_a = \inf\{t : L_t > a\}$. Since τ is a subordinator, τ has interchangeable increments. Recall that $\tau_0 = 0$ as shown in the proof of Lemma 25, and $S = \tau_T > 0$ a.s. because $(L_t)_{t \geq 0}$ is continuous. By Lemma 25, the closure of $\{t : \tau_a = t \text{ for some } a\}$ equals Z, which has Lebesgue measure zero by assumption. By (52), $\tau_{L_t -} \leq t$ for all $t > 0$, so $L_t \leq \inf\{a : \tau_a > t\}$ for all $t > 0$. By the continuity of $(L_t)_{t \geq 0}$, we have $\tau_{L_t + \epsilon} > t$ for all $t > 0$ and all $\epsilon > 0$, so $L_t + \epsilon \geq \inf\{a : \tau_a > t\}$ for all $t > 0$ and all $\epsilon > 0$. Also, $\tau_a > 0$ for all $a > 0$, so $L_0 = 0 = \inf\{a : \tau_a > 0\}$. Hence, $L_t = \inf\{a : \tau_a > t\}$ for all $0 \leq t < S$, and $L_S = T$ by the continuity of $(L_t)_{t \geq 0}$. Thus, Corollary 26 follows from Theorems 6 and 9. ■

Note that $(J_i)_{i=1}^{\infty}$ consists of the lengths of the excursions of ξ away from 0 that are completed before local time T. Corollary 26 thus states that the event that the local time process stays below the line from $(0, c)$ to (S, T) occurs with probability c/T and is independent of the excursion lengths. Also, note that H is the length of the excursion of ξ that begins at K, so the corollary shows that the length of the excursion that begins at the unique time when $(L_t - \lambda t)_{0 \leq t < S}$ attains its maximum is a size-biased pick from all of the excursion lengths.

Acknowledgments

The author thanks Jim Pitman for helpful discussions and for many comments on earlier drafts of this work. He also thanks Marc Yor for some suggestions.

References

[1] D. Aldous. The continuum random tree I. *Ann. Probab.*, 19:1–28, 1991.

[2] D. Aldous. The continuum random tree II: an overview. In M. Barlow and N. Bingham, editors, *Stochastic Analysis*, pages 23–70. Cambridge University Press, 1991.

[3] D. Aldous. The continuum random tree III. *Ann. Probab.*, 21:248–289, 1993.

[4] D. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.*, 26:1703–1726, 1998.

[5] J. Bertoin. *Lévy Processes.* Cambridge University Press, 1996.

[6] J. Bertoin. A fragmentation process connected to Brownian motion. *Probab. Theory Related Fields*, 117:289–301, 2000.

[7] J. Bertoin. Homogeneous fragmentation processes. Preprint, 2000.

[8] J. Bertoin. Self-similar fragmentations. Preprint, 2000.

[9] J. Bertoin and J. Pitman. Path transformations connecting Brownian bridge, excursion, and meander. *Bull. Sci. Math.*, 118:147–166, 1994.

[10] P. Biane, J.-F. LeGall, and M. Yor. Un processus qui ressemble au pont Brownien. In *Séminaire des Probabilités, XXI, Lecture Notes in Mathematics, Vol. 1247*, pages 270–275. Springer, Berlin, 1987.

[11] P. Chassaing and S. Janson. A Vervaat-like path transformation for the reflected Brownian bridge conditioned on its local time at 0. Available via http://altair.iecn.u-nancy.fr/~chassain, 1999.

[12] P. Chassaing and G. Louchard. Phase transition for parking blocks, Brownian excursion, and coalescence. Available via http://altair.iecn.u-nancy.fr/~chassain, 2000.

[13] R. Durrett. *Probability: Theory and Examples. 2nd ed.* Duxbury Press, Belmont, CA, 1996.

[14] F. Jones. *Lebesgue Integration on Euclidean Space.* Jones and Bartlett, Boston, 1993.

[15] O. Kallenberg. Canonical representations and convergence criteria for processes with interchangeable increments. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, 27:23–36, 1973.

[16] O. Kallenberg. Ballot theorems and sojourn laws for stationary processes. *Ann. Probab.*, 27:2011–2019, 1999.

[17] T. Konstantopoulos. Ballot theorems revisited. *Statist. Probab. Lett.*, 24:331–338, 1995.

[18] J.-F. LeGall. Une approche élémentaire des théorèms de décomposition de Williams. In *Séminaire des Probabilités, XX, Lecture Notes in Mathematics, Vol. 1204*, pages 447–464. Springer, Berlin, 1986.

[19] S.A. Molchanov and E. Ostrovski. Symmetric stable processes as traces of degenerate diffusion processes. *Theory Probab. Appl.*, 14:128–131, 1969.

[20] M. Perman, J. Pitman, and M. Yor. Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields*, 92:21–39, 1992.

[21] J. Pitman. Partition structures derived from Brownian motion and stable subordinators. *Bernoulli*, 3:79–96, 1997.

[22] J. Pitman. The SDE solved by local times of a Brownian excursion or bridge derived from the height profile of a random tree or forest. *Ann. Probab.*, 27:261–283, 1999.

[23] J. Pitman and M. Yor. Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc.*, 65:326–356, 1992.

[24] J. Pitman and M. Yor. Random discrete distributions derived from self-similar random sets. *Electron. J. Probab.*, 1:1–28, 1996.

[25] J. Pitman and M. Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *Ann. Probab.*, 25:855–900, 1997.

[26] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion. 3rd ed.* Springer-Verlag, Berlin, 1999.

[27] L. Takács. *Combinatorial Methods in the Theory of Stochastic Processes.* Wiley, New York, 1967.

[28] D. Williams. Path decomposition and continuity of local time for one-dimensional diffusions, I. *Proc. London Math. Soc.*, 28:738–768, 1974.