Stopped Markov chains with stationary occupation times

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Abstract: Let E be a finite set equipped with a group G of bijective transformations and suppose that X is an irreducible Markov chain on E that is equivariant under the action of G. In particular, if $E = G$ with the corresponding transformations being left or right multiplication, then X is a random walk on G. We show that when X is started at a fixed point there is a stopping time U such that the distribution of the random vector of pre-U occupation times is invariant under the action of G. When G acts transitively (that is, E is a homogeneous space), any non-zero, finite expectation stopping time with this property can occur no earlier than the time S of the first return to the starting point after all states have been visited. We obtain an expression for the joint Laplace transform of the pre-S occupation times for an arbitrary finite chain and show that even for random walk on the group of integers mod r the pre-S occupation times do not generally have a group invariant distribution. This appears to contrast with the Brownian analog, as there is considerable support for the conjecture that the field of local times for Brownian motion on the circle prior to the counterpart of S is stationary under circular shifts.

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1. Introduction.

In Pitman (1995a) it was shown that if a circular Brownian motion with constant drift is started at a fixed initial point, say 0, and stopped when it first returns to 0 after a complete loop around the circle, then the resulting local time field is stationary with respect to shifts around the circle. The same is true for stopping at the first return after a clockwise loop or an anti-clockwise loop.

There is an immediate consequence of this fact for random walks. For a positive integer r let $\mathbb{Z}_r = \{0, 1, 2, \ldots, r-1\}$ denote the group of integers with addition modulo r, and let $(X_t, \mathbb{P}^x)$ be a continuous time simple (that is, nearest neighbour) random walk on $\mathbb{Z}_r$, not necessarily symmetric. Think of the circle as the interval $[0, 1[$ equipped with addition mod 1 and identify the elements of $\mathbb{Z}_r$ with the points $\{0, 1/r, \ldots (r-1)/r\}$. Define the *occupation time process* of X to be the vector-valued process $((\Lambda_t(x))_{x \in \mathbb{Z}_r})_{t \geq 0}$ given by

$$\Lambda_t(x) = (|\{0 \leq s < t : X_s = x\}|)_{x \in \mathbb{Z}_r}$$

where $|\cdot|$ denotes Lebesgue measure. Let $V_\pm$ (respectively, V_+, V_-) denote the first time the random walk returns to its starting point after a loop (respectively, anti-clockwise loop, clockwise loop) around the circle. Recall that for Brownian motion with drift started at x, the local time at x prior to the first visit to $\{x-a, x+a\}$ has an exponential distribution for any x and any $a > 0$, and both the mean of this local time and the probability that the first visit is to $x + a$ can varied arbitrarily by suitably choosing the diffusion and drift parameters of the Brownian motion. Combining these observations, we obtain the following:

Proposition (1.1). *The $\mathbb{P}^0$ distribution of $(\Lambda_{V_\pm}(x+y))_{x \in \mathbb{Z}_r}$ is the same for all $y \in \mathbb{Z}_r$. The same is true for V_+ or V_- instead of $V_\pm$.*

There are other stopping times with the same property. Put

$$T_0 = 0, \; T_k = \min\{t \geq T_{k-1} : X_t = k\} \text{ for } k = 1, \ldots, r-1,$$

and

$$U_+ = T_r = \min\{t \geq T_{r-1} : X_t = 0\}.$$

It is a consequence of Theorem 3.1 below that the $\mathbb{P}^0$ distribution of $(\Lambda_{U_+}(x+y))_{x\in\mathbb{Z}_r}$ is the same for all $y \in \mathbb{Z}_r$. We remark that the two stopping times $V_\pm$ and U_+ are not comparable, in the sense that neither inequality $V_\pm \leq U_+$ nor $V_\pm \geq U_+$ holds almost surely.

In view of these examples, it is natural to ask the following question. Given an irreducible random walk on a finite group, when is there a stopping time T such that the distribution of the pre-T occupation times is stationary under the action of the group? We will show in §3 that such a stopping time always exists, as a consequence of the following more general proposition. If an irreducible Markov chain on a finite state space is equivariant under the action of a group of transformations on the state space (see §2 for the relevant definition), then there exists a non-randomised stopping time T such that the distribution of the pre-T occupation times is invariant under the action of the group. Moreover, $\exp(aT)$ has finite expectation for sufficiently small $a > 0$.

Note that for the walk on $\mathbb{Z}_r$ the stopping times $V_\pm$ and U_+ occur after the first time S that all states have been visited (the *cover time* of the walk). Also, the stopping times $V_\pm$ and U_+ occur at the time of a return to the starting point. According to Proposition (2.1), for an equivariant chain, both of these features are essentially necessary conditions on a stopping time T for the distribution of the pre-T occupation times to be invariant. This raises the question of stationarity of occupation times prior to the time of the first return to the starting point after the cover time. In §4 we obtain an explicit formula for the joint Laplace transform of these occupation times for an arbitrary finite chain. For the walk on $\mathbb{Z}_r$ we evaluate this formula to show that in this case the occupation times are not stationary for $r = 3$ or $r = 4$. Finally, in §5 we make some comments about asymptotics for the random walk following from results for Brownian motion on the circle.

While the results of this paper are presented for Markov chains with continuous time parameter, these results also have discrete time analogs with positive integer valued occupation times instead of positive real occupation times. Results for an embedded discrete time jumping chain can be deduced by decomposition of the continuous occupation times as sums of numbers of i.i.d. exponential variables determined by occupation counts for the jumping chain. In particular, joint probability generating functions for discrete occupation counts can be read from joint Laplace transforms for continuous occupation times by conditioning on the discrete counts. Further, due to the law of large numbers, asymptotics for occupation counts of a symmetric discrete time walk on $\mathbb{Z}_r$, with counts normalized by r, are the same as asymptotics for occupation times of the corresponding continuous time walk on $\mathbb{Z}_r$, with holding times normalized to have mean $1/r$.

2. Some general observations.

Let E be finite set and $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x;\ t \geq 0,\ x \in E)$ be an irreducible, continuous time, Markov chain on E. Assume that X has right-continuous paths of jump-hold type.

Assume also that Ω is equipped in the usual way with shift operators $\theta_t : \Omega \to \Omega$ for $t \geq 0$, so that $X_s \circ \theta_t = X_{s+t}$, $s \geq 0$.

The *occupation times process* for X is the vector-valued process $((\Lambda_t(x))_{x \in E})_{t \geq 0}$ given by $\Lambda_t(x) = |\{0 \leq s < t : X_s = x\}|$.

Let G be a finite group of bijective transformations on E. The group G acts *transitively* if the orbit of any $x \in E$ under G is all of E, and in this case (E, G) is said to be a *homogeneous space*. The chain X is *equivariant* under the action of G if, for all $x \in E$, $g \in G$, and $A \in \mathcal{F}$,

$$\mathbb{P}^x\{g \circ X \in A\} = \mathbb{P}^{gx}\{X \in A\}.$$

A random vector $(Z_x)_{x \in E}$ has a distribution that is *invariant* under the action of G if the distribution of $(Z_{gx})_{x \in E}$ is the same for all $g \in E$.

The *cover time*, W, for X is the first time that all states have been visited. Let S denote the first time after W that X returns to its starting point.

Proposition (2.1). *Suppose that (E, G) is a homogeneous space and X is equivariant under the action of G. Fix $e \in E$.*

i) If T is a stopping time such that $\mathbb{P}^e[T] < \infty$ and $\mathbb{P}^e[\Lambda_T(x)]$ is the same for all $x \in E$, then $X_T = e$, $\mathbb{P}^e$-a.s.
ii) It T is a stopping time such that $T > 0$, $\mathbb{P}^e$-a.s. and the $\mathbb{P}^e$ distribution of $\Lambda_T(x)$ is the same for all $x \in E$, then $T \geq W$, $\mathbb{P}^e$-a.s.

Proof. (i) Let A denote the infinitesimal generator of X. By assumption on X, the uniform distribution on E is the stationary measure for X, and so $\sum_{x \in E} Af(x) = 0$ for any function f on E. Thus,

$$\begin{aligned}\mathbb{P}^e[f(X_T)] &= f(e) + \mathbb{P}^e\left[\int_0^T Af(X_s)\, ds\right] \\ &= f(e) + \sum_{x \in E} Af(x)\mathbb{P}^e[\Lambda_T(x)] = f(e),\end{aligned}$$

and so $X_T = e$, $\mathbb{P}^e$-a.s.

(ii) If $\mathbb{P}^e\{T > 0\} = 1$, then $\mathbb{P}^e\{\Lambda_T(e) > 0\} = 1$. Thus, $\mathbb{P}^e\{\Lambda_T(x) > 0\} = 1$ for all $x \in E$ and $T \geq W$, $\mathbb{P}^e$-a.s.

$\square$

Proposition (2.2). *Suppose that (E, G) is a homogeneous space and X is equivariant under the action of G. Fix $e \in E$. Suppose that T is a stopping time such that $\mathbb{P}^e[T^2] < \infty$, $\mathbb{P}^e[\Lambda_T(x)]$ is the same for all $x \in E$, and for each pair $(x, y) \in E \times E$, $\mathbb{P}^e[\Lambda_T(gx)\Lambda_T(gy)]$ is the same for all $g \in G$. Then*

$$\mathbb{P}^e[\Lambda_T(x)\Lambda_T(y)] = (\#E)^{-2}\,\mathbb{P}^e[T^2] + \mathbb{P}^e[T],\ (x, y),$$

where , $(x, y) = (\#E)^{-1}\int_0^\infty \left[\mathbb{P}^x\{X_t = y\} + \mathbb{P}^y\{X_t = x\} - 2(\#E)^{-1}\right] dt.$

Proof. As a consequence of a central limit theorem for stationary processes with suitable mixing properties, we find that for any function v on E the asymptotic $\mathbb{P}^e$ distribution of $(\sum_{x\in E} v(x)\Lambda_t(x) - t(\#E)^{-1}\sum_{x\in E} v(x))/\sqrt{t}$ as $t \to \infty$ is Gaussian with mean 0 and variance

$$\sigma^2(m) = \sum_{x\in E}\sum_{y\in E} v(x)v(y),\ (x, y)$$

(cf. Ch 2 of Aldous and Fill (1995)). On the other hand, we know from Proposition (2.1)(i) that $T = e$, $\mathbb{P}^e$-a.s., and the central limit theorem for additive functionals of a regenerative stochastic process (Theorem 3.2 of Asmussen (1987)) yields the same Gaussian asymptotics with the following alternative expression for $\sigma^2(m)$:

$$\begin{aligned}
\mathbb{P}^e[T]\sigma^2(m) &= \mathrm{Var}\left[\sum_{x\in E} v(x)\Lambda_T(x) - T(\#E)^{-1}\sum_{x\in E} v(x)\right] \\
&= \mathrm{Var}\left[\sum_{x\in E} v(x)\Lambda_T(x)\right] + (\#E)^{-2}\left(\sum_{x\in E} v(x)\right)^2 \mathrm{Var}[T] \\
&\quad - 2(\#E)^{-1}\left(\sum_{x\in E} v(x)\right)\mathrm{Cov}\left[\sum_{x\in E} v(x)\Lambda_T(x),\ T\right],
\end{aligned}$$

where $\mathrm{Var}[W] = \mathbb{P}^e[W^2] - \mathbb{P}^e[W]^2$ and $\mathrm{Cov}[V, W] = \mathbb{P}^e[VW] - \mathbb{P}^e[V]\mathbb{P}^e[W]$. Note that for all $x \in E$ we have, by assumption, that $\mathbb{P}^e[\Lambda_T(x)] = (\#E)^{-1}\mathbb{P}^e[T]$ and $\mathbb{P}^e[\Lambda_T(x)T] = (\#E)^{-1}\mathbb{P}^e[T^2]$. Thus,

$$\mathbb{P}^e\left[\left(\sum_{x\in E} v(x)\Lambda_T(x)\right)^2\right] = \left(\sum_{x\in E} v(x)\right)^2 (\#E)^{-2}\,\mathbb{P}^e[T^2] + \mathbb{P}^e[T]\sum_{x\in E}\sum_{y\in E} v(x)v(y),\ (x, y)$$

and the result follows immediately by polarisation.

□

For use in the proof of the next result and later, we need to introduce a little more notation. Adjoin a cemetery state ∂ to $\mathbb{Z}_r$. Given a function $v \geq 0$ on E, let $\mathbb{P}_v^x$ denote the law of the process started at x and killed and sent to ∂ at rate $v(y)$ when the process is in state y. Set $\zeta = \inf\{t \geq 0 : X_t = \partial\}$.

Propostion (2.3). *Suppose that G acts on E, X is equivariant under the action of G, and T, T' are two stopping times. If any three of the random vectors Λ_T, $\Lambda_{T'}$, $\Lambda_{T \vee T'}$, $\Lambda_{T \wedge T'}$ have $\mathbb{P}^e$ distributions that are invariant under the action of G, then the same is true of the remaining random vector.*

Proof. Note that for a stopping time R we have

$$\mathbb{P}^e\left[\exp\left(-\sum_{x \in E} v(x)\Lambda_R(x)\right)\right] = \mathbb{P}^e\left[\exp\left(-\int_0^R v(X_s)\, ds\right)\right] = \mathbb{P}_v^e\{R < \zeta\}.$$

Thus

$$\begin{aligned}
&\mathbb{P}^e\left[\exp\left(-\sum_{x \in E} v(x)\Lambda_{T \wedge T'}(x)\right)\right] \\
&= \mathbb{P}_v^e(\{T < \zeta\} \cup \{T' < \zeta\}) \\
&= \mathbb{P}_v^e\{T < \zeta\} + \mathbb{P}_v^e\{T' < \zeta\} - \mathbb{P}_v^e(\{T < \zeta\} \cap \{T' < \zeta\}) \\
&= \mathbb{P}_v^e\{T < \zeta\} + \mathbb{P}^e\{T' < \zeta\} - \mathbb{P}^e\{T \vee T' < \zeta\} \\
&= \mathbb{P}^e\left[\exp\left(-\sum_{x \in E} v(x)\Lambda_T(x)\right)\right] + \mathbb{P}^e\left[\exp\left(-\sum_{x \in E} v(x)\Lambda_{T'}(x)\right)\right] \\
&\qquad - \mathbb{P}^e\left[\exp\left(-\sum_{x \in E} v(x)\Lambda_{T \vee T'}(x)\right)\right].
\end{aligned}$$

□

To conclude this section, we record the following proposition, which shows that one naive attempt to construct an invariant occupation field succeeds only in trivial cases to produce a stopping time with finite expectation. For $x \in E$, define a *completed hold in* x to be an interval $[\gamma, \delta[$ such that

i) $X_\gamma = x$,

ii) $\gamma = 0$ or $X_{\gamma-} \neq x$,

iii) $X_\delta \neq x$.

For $t \geq 0$, let $N_t(x)$ denote the number of completed holds in x of the form $[\gamma, \delta[$ with $\delta \leq t$. Set

$$\tau = \inf\{t \geq 0 : \text{all } N_t(x), x \in E, \text{ are equal}\}.$$

If $\tau < \infty$, $\mathbb{P}^e$-a.s. for some $e \in E$, then it is obvious that Λ_τ has an invariant distribution for any group acting on E.

Proposition (2.4). *Suppose that (E, G) is a homogeneous space and X is equivariant under the action of G. Fix $e \in E$. Then $\mathbb{P}^e[\tau] < \infty$ if and only if for some $r \in \mathbb{N}$ there exists a bijection J between E and $\mathbb{Z}_r$ such that $J \circ X$ is the completely asymmetric simple random walks that only makes jumps of size $+1$.*

Proof. It is clear that if X is isomorphic to a completely asymmetric random walk, then $\mathbb{P}^e[\tau] < \infty$.

Consider the converse. Suppose that X is not isomorphic to a completely asymmetric random walk and yet $\mathbb{P}^e[\tau] < \infty$. Then, setting $\sigma = \inf\{t \geq 0 : X_t \neq X_0\}$, there exists $y, z \in E$, $y \neq z$, such that $0 < \mathbb{P}^e\{X_\sigma = y\} < 1$ and $0 < \mathbb{P}^e\{X_\sigma = z\} < 1$. Also, if we write $T_1, T_2, \ldots$ for the successive return times to e, then $\tau \in \{T_1, T_2, \ldots\}$, $\mathbb{P}^e$-a.s., by Proposition (2.1)(i).

We claim that there must exist $x \in E$ such that $\mathbb{P}^e\{N_{T_1}(x) = 0\} > 0$. Choose $g, h \in G$ such that $y = ge$ and $z = he$. There is positive $\mathbb{P}^e$ probability that the list of successive states visited by X prior to T_1 is $e, ge, g^2e, \ldots, g^{i-1}e$ (respectively, $e, he, h^2e, \ldots, h^{j-1}e$), where $i = \min\{\ell : g^\ell e = e\}$ (respectively, $j = \min\{\ell : h^\ell e = e\}$). If $E \neq \{e, ge, \ldots, g^{i-1}e\}$ or $E \neq \{e, he, \ldots, h^{j-1}e\}$, then the claim is obvious. Otherwise, $h^{-1}e = g^k e$ for some k Observe that $k = i - 1$ cannot hold, because that, in this instance, would imply $g = h$. Consequently, there is positive $\mathbb{P}^e$ probability that X successively visits just $e, ge, \ldots, g^k e$ prior to T_1, and again the claim follows.

By the strong Markov property, $\{N_{T_n}(x) - N_{T_n}(e)\}_{n \in \mathbb{N}}$ under $\mathbb{P}^e$ is the process of partial sums of a sequence of i.i.d.r.v. that are not a.s. constant. Thus, for $M = \inf\{n \geq 1 : N_{T_n}(x) - N_{T_n}(e) = 0\}$ we have $\mathbb{P}^e[M] = \infty$, and hence $\mathbb{P}^e[T_M] = \infty$. Since $\tau \geq T_M$, this contradicts $\mathbb{P}^e[\tau] < \infty$.

$\square$

Remark. Using ideas similar to those in the above proof and the transience of random walks on $\mathbb{Z}^d$ for $d \geq 3$, it follows that it will typically be the case that $\mathbb{P}^e\{\tau < \infty\} < 1$, but we do not have a necessary and sufficient condition.

3. Construction of a stopping time.

Theorem (3.1). *Suppose that G acts on E and X is equivariant under the action of G. Fix $e \in E$. There is a stopping time U such that $\mathbb{P}^e[\exp(aU)] < \infty$ for sufficiently small $a > 0$ and the $\mathbb{P}^e$ distribution of $(\Lambda_U(x))_{x \in E}$ is invariant under the action of G.*

Proof. Define an equivalence relation on $E \times E$ by declaring that the pairs (x, y) and (x', y') are equivalent if $(x', y') = (gx, gy)$ for some $g \in G$.

Define a *trip* through E to be an ordered list of the elements of E. We will recognise two trips $(z_0, z_1, \ldots, z_{\#E-1})$ and $(z'_0, z'_1, \ldots, z'_{\#E-1})$ to be the same if there exists $0 \leq j \leq \#E - 1$ such that $z'_i = z_{i+j}$ for $0 \leq i \leq \#E - 1$, where the addition in the subscript is performed modulo $\#E$. Given a trip $(z_0, z_1, \ldots, z_{\#E-1})$ and $g \in G$, define the trip $g(z_0, z_1, \ldots, z_{\#E-1})$ as $(gz_0, gz_1, \ldots, gz_{\#E-1})$.

Fix a trip c^* and write C for the orbit of c^* under the action of G. Observe that if $(x, y) \in E \times E$, then the number of trips $(z_0, z_1, \ldots, z_{\#E-1}) \in C$ such that $(x, y) \in \{(z_0, z_1), (z_1, z_2), \ldots, (z_{\#E-1}, z_0)\}$ only depends on the equivalence class of (x, y).

Let $c^1, \ldots, c^m$ be a listing of C. For $1 \leq k \leq m$ choose the representative $(z_0^k, z_1^k, \ldots, z_{\#E-1}^k)$ of c^k so that $z_0^k = e$. Put $z_{\#E}^k = e$.

For $1 \leq k \leq m$ and $1 \leq i \leq \#E$, put

$$T_i^k = \min\{t \geq 0 : X_t = z_i^k\}.$$

Define $S_0^k, \ldots, S_{\#E}^k$ inductively by $S_0^k = T_0^k$ and $S_i^k = T_i^k \circ \theta_{S_{i-1}^k}$ for $1 \leq i \leq \#E$. Define $U^0, \ldots, U^m$ inductively by $U^0 = S_0^1 = T_0^1$ and $U^k = S_{\#E}^k \circ \theta_{U^{k-1}}$ for $2 \leq k \leq m$. Thus $U^0 = 0$ under $\mathbb{P}^e$, and at each U^k, $k \geq 1$, the chain has returned to e after, loosely speaking, passing through E in the order specified by the trip c^k. Set $U = U^m$.

For $1 \le k \le m$, $1 \le i \le \#E$, and $x \in E$, put

$$L_i^k(x) = |\{S_{i-1}^k \circ \theta_{U^{k-1}} \le t < S_i^k \circ \theta_{U^{k-1}} : X_t = x\}|,$$

so that $\Lambda_U = \sum_k \sum_i L_i^k$.

Rewrite this sum as

$$\Lambda_U = \sum_{(x,y)\in E\times E} \quad \sum_{\{(k,i):z_{i-1}^k=x,z_i^k=y\}} L_i^k. \tag{3.1}$$

Note that the summands are independent, and the number of summands in the second sum only depends on the equivalence class of the pair (x, y).

Now for any $g \in G$,

$$\begin{aligned}(\Lambda_U(gw))_{w\in E} &= \sum_{(x,y)\in E\times E} \quad \sum_{\{(k,i):z_{i-1}^k=x,z_i^k=y\}} (L_i^k(gw))_{w\in E} \\ &= \sum_{(x,y)\in E\times E} \quad \sum_{\{(k',i'):z_{i'-1}^{k'}=gx,z_{i'}^{k'}=gy\}} (L_{i'}^{k'}(gw))_{w\in E}. \end{aligned}\tag{3.2}$$

We have observed that

$$\#\{(k,i) : z_{i-1}^k = x, z_i^k = y\} = \#\{(k',i') : z_{i'-1}^{k'} = gx, z_{i'}^{k'} = gy\}.$$

Moreover, by assumption on X, if $z_{i-1}^k = x$, $z_i^k = y$, $z_{i'-1}^{k'} = gx$, and $z_{i'}^{k'} = gy$, then the $\mathbb{P}^e$ distributions of $(L_i^k(w))_{w\in E}$ and $(L_{i'}^{k'}(gw))_{w\in E}$ coincide. Comparing (3.1) and (3.2) (and recalling that the summands are independent) we see that the distribution of Λ_U is invariant under the action of G. The claim regarding the existence of exponential moments is clear from the construction.

□

Remark. The stopping time U_+ given in §1 for the case $E = G = \mathbb{Z}_r$ and X a simple random walk is obtained from the construction in the proof by taking $c^* = (0, 1, \ldots, r-1)$.

4. First return after the cover time.

Let X be as in §2. Define S to be the time of the first visit of X to its starting point after the cover time, W. Write A for the infinitesimal generator of X. We will think of A

as a $(\#E) \times (\#E)$ matrix. Given a function $v \geq 0$ on E, let M_v be the diagonal matrix representing the operation of multiplication by v. For a subset $B \subseteq E$ let A^B denote the infinitesimal generator matrix for X killed on hitting B and let M_v^B denote the diagonal matrix representing multiplication by v restricted to the complement of B (that is, A^B and M_v^B are $(\#E - \#B) \times (\#E - \#B)$ sub-matrices of A and M_v, respectively, constructed by removing the rows and columns that correspond to B).

Theorem (4.1). *For a function $v \geq 0$ on E we have*

$$\mathbb{P}^e\left[\exp\left(-\sum_{x \in E} v(x)\Lambda_S(x)\right)\right] = \sum_{e \notin B \subseteq E} (-1)^{\#B} \frac{(-A^B + M_v^B)^{-1}_{e,e}}{(-A + M_v)^{-1}_{e,e}}.$$

Proof. By definition, $\mathbb{P}^e[\exp(-\sum_{x \in E} v(x)\Lambda_S(x))] = \mathbb{P}^e[\exp(-\int_0^S v(X_s)\, ds]$.

On the one hand, we have by the strong Markov property and the Feynman-Kac formula for Markov chains (see, for example, Pitman (1995b)) that

$$\begin{aligned}
&\mathbb{P}^e\left[\int_S^\infty \exp\left(-\int_0^t v(X_s)\, ds\right) 1(X_t = e)\, dt\right] \\
&= \mathbb{P}^e\left[\exp\left(-\int_0^S v(X_s)\, ds\right)\right] \mathbb{P}^e\left[\int_0^\infty \exp\left(-\int_0^t v(X_s)\, ds\right) 1(X_t = e)\, dt\right] \\
&= \mathbb{P}^e\left[\exp\left(-\int_0^S v(X_s)\, ds\right)\right] (-A + M_v)^{-1}_{e,e}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}^e\left[\int_S^\infty \exp\left(-\int_0^t v(X_s)\, ds\right) 1(X_t = e)\, dt\right] \\
&= \mathbb{P}^e\left[\int_0^\infty \exp\left(-\int_0^t v(X_s)\, ds\right) \prod_{x \in E} 1\left(\int_0^t 1(X_s = x)\, ds > 0\right) 1(X_t = e)\, dt\right] \\
&= \lim_{\alpha \to \infty} \mathbb{P}^e\left[\int_0^\infty \exp(-\int_0^t v(X_s)\, ds) \prod_{x \in E} \left\{1 - \exp(-\alpha \int_0^t 1(X_s = x)\, ds)\right\} 1(X_t = e)\, dt\right] \\
&= \lim_{\alpha \to \infty} \sum_{B \subseteq E} (-1)^{\#B} (-A + M_v + M_{\alpha 1_B})^{-1}_{e,e},
\end{aligned}$$

again by the Feynman-Kac formula.

Combining these two observations, we get

$$\mathbb{P}^e\left[\exp\left(-\int_0^S v(X_s)\,ds\right)\right] = \lim_{\alpha\to\infty}\sum_{B\subseteq E}(-1)^{\#B}\frac{(-A+M_v+M_{\alpha 1_B})^{-1}_{e,e}}{(-A+M_v)^{-1}_{e,e}}$$
$$= \sum_{e\notin B\subseteq E}(-1)^{\#B}\frac{(-A^B+M_v^B)^{-1}_{e,e}}{(-A+M_v)^{-1}_{e,e}}.$$

□

Corollary (4.2). *Suppose that X is a simple random walk on $\mathbb{Z}_r$. Then*

$$\mathbb{P}^0\left[\exp\left(-\sum_{x\in E}v(x)\Lambda_S(x)\right)\right]$$
$$= 1 - \frac{\sum_{i=1}^{r-1}(-A^{\{i\}}+M_v^{\{i\}})^{-1}_{0,0} - \sum_{i=1}^{r-2}(-A^{\{i,i+1\}}+M_v^{\{i,i+1\}})^{-1}_{0,0}}{(-A+M_v)^{-1}_{0,0}}.$$

Proof. In this case, the sum in Theorem (4.1) reduces to a similar sum over subsets $e\notin B\subseteq E$ containing at most 2 points with the coefficients $(-1)^{\#B}$ replaced by suitable combinatorially derived coefficients. When B is empty, a singleton or two adjacent points, the coefficients are, respectively, 1, -1, and 1. When B consists of two non-adjacent points and the arc between them that does not contain e has k points, then the coefficient is

$$\binom{k}{0}-\binom{k}{1}+\binom{k}{2}-\cdots+(-1)^k\binom{k}{k}=0.$$

□

There is an alternative derivation of Corollary (4.2) that the reader might find illuminating. Recall the notation introduced prior to Proposition (2.3)

Applying the strong Markov property at S, we find that

$$\begin{aligned}\mathbb{P}_v^0\{S<\zeta\} =& \frac{\mathbb{P}_v^0\{S<\zeta\}\mathbb{P}_v^0\{X_{\zeta-}=0\}}{\mathbb{P}_v^0\{X_{\zeta-}=0\}}\\ =& \frac{\mathbb{P}_v^0\{S<\zeta,\,X_{\zeta-}=0\}}{\mathbb{P}_v^0\{X_{\zeta-}=0\}}\\ =& \mathbb{P}_v^0\{S<\zeta \mid X_{\zeta-}=0\}.\end{aligned}$$

The last term is the $\mathbb{P}_v^0$ conditional probability that there does not exist a state i such that i is not hit by X but $i+1$ is hit by X before ζ given that $X_{\zeta-}=0$. By the reasoning in Proposition 3 of Pitman (1995b),

$$\mathbb{P}_v^0\{X_t \neq i,\, \forall 0 \leq t < \zeta \mid X_{\zeta-}=0\} = \frac{(-A^{\{i\}}+M_v^{\{i\}})^{-1}_{0,0}}{(-A+M_v)^{-1}_{0,0}}$$

and

$$\mathbb{P}_v^0\{X_t \notin \{i,i+1\},\, \forall 0 \leq t < \zeta \mid X_{\zeta-}=0\} = \frac{(-A^{\{i,i+1\}}+M_v^{\{i,i+1\}})^{-1}_{0,0}}{(-A+M_v)^{-1}_{0,0}},$$

and Corollary (4.2) follows.

For small values of r, it is straightforward to evaluate the expression of Corollary (4.2) using a computer algebra package such as *Mathematica.* We take the holding times in each state to have mean $1/r$. This scaling leads to occupation times that for large r have asymptotically the same distribution as the local times for circular Brownian motion prior to the first return to 0 after the cover time, so it is of interest to compare these results for small r with their Brownian limits, which are further discussed in §5.

Let $\mathbb{P}_{(r)}^0$ denote probabilities for the walk on $\mathbb{Z}_r$ started in state 0. For $r=2$ it is obvious that $\Lambda_S(0)$ and $\Lambda_S(1)$ are i.i.d. exponential random variables with rate 2, so

$$\begin{aligned}
\mathbb{P}_{(2)}^0[\exp(-a\Lambda_S(0))] &= 1-\frac{1}{2}\,a+\frac{1}{4}\,a^2-\frac{1}{8}\,a^3+\frac{1}{16}\,a^4+\mathrm{O}(a)^5\\
\mathbb{P}_{(2)}^0[\exp(-a\Lambda_S(1))] &= \mathbb{P}_{(2)}^0[\exp(-a\Lambda_S(0))].
\end{aligned}$$

For $r=3$ we have

$$\begin{aligned}
\mathbb{P}_{(3)}^0[\exp(-a\Lambda_S(0))] &= 1-\frac{5}{9}\,a+\frac{23}{81}\,a^2-\frac{101}{729}\,a^3+\frac{431}{6561}\,a^4+\mathrm{O}(a)^5\\
\mathbb{P}_{(3)}^0[\exp(-a\Lambda_S(1))] &= 1-\frac{5}{9}\,a+\frac{24}{81}\,a^2-\frac{112}{729}\,a^3+\frac{512}{6561}\,a^4+\mathrm{O}(a)^5\\
\mathbb{P}_{(3)}^0[\exp(-a\Lambda_S(2))] &= \mathbb{P}_{(3)}^0[\exp(-a\Lambda_S(1))],
\end{aligned}$$

so that the one-dimensional marginals of Λ_S are not equal.

For $r=4$ we have

$$\begin{aligned}
\mathbb{P}_{(4)}^0[\exp(-a\Lambda_S(0))] &= 1-\frac{7}{12}\,a+\frac{89}{288}\,a^2-\frac{1081}{6912}\,a^3+\frac{12833}{165888}\,a^4+\mathrm{O}(a)^5\\
\mathbb{P}_{(4)}^0[\exp(-a\Lambda_S(1))] &= 1-\frac{7}{12}\,a+\frac{91}{288}\,a^2-\frac{1133}{6912}\,a^3+\frac{13735}{165888}\,a^4+\mathrm{O}(a)^5\\
\mathbb{P}_{(4)}^0[\exp(-a\Lambda_S(2))] &= 1-\frac{7}{12}\,a+\frac{93}{288}\,a^2-\frac{1197}{6912}\,a^3+\frac{15096}{165888}\,a^4+\mathrm{O}(a)^5\\
\mathbb{P}_{(4)}^0[\exp(-a\Lambda_S(3))] &= \mathbb{P}_{(4)}^0[\exp(-a\Lambda_S(1))].
\end{aligned}$$

Again the one-dimensional marginals of Λ_S are not equal. The results of §5 of Pitman (1995a) imply that as r increases the $\mathbb{P}^0_{(r)}$ distribution of $\Lambda_S(k)$ depends less and less on k as k ranges over $\mathbb{Z}_r$. To be precise,

$$\begin{aligned}\lim_{r\to\infty} \mathbb{P}^0_{(r)}[\exp(-a\Lambda_S(k))] &= \frac{2}{\sqrt{a(2+a)^3}} \tanh^{-1}\sqrt{\frac{a}{2+a}} \\ &= 1 - \frac{2}{3}\,a + \frac{2}{5}\,a^2 - \frac{8}{35}\,a^3 + \frac{8}{63}\,a^4 + \mathrm{O}(a)^5\end{aligned} \tag{4.1}$$

where the convergence is uniform for $k \in \mathbb{Z}_r$, and the limit has an interpretation in terms of Brownian local times which is discussed in the next section. The Laplace transform on the right-hand side of (4.1) can be inverted to conclude that

$$\lim_{r\to\infty} \mathbb{P}^0_{(r)}[\Lambda_S(k) > x] = xK_1(x)\exp(-x), \quad x > 0,$$

where K_1 is the modified Bessel function, and the limit holds uniformly in $x > 0$ and $k \in \mathbb{Z}_r$. Moreover, as we see in the next section, the moments of $\Lambda_S(k)$ converge uniformly in k to those of the limiting distribution.

Before finishing this section, we make some further observations about the time S for the symmetric simple random walk on $\mathbb{Z}_r$. Recall that the stopping time U_+ of §1 is the result of the construction in the proof of Theorem (3.1) when $c^* = (0, 1, 2, \ldots, r-1)$. We can define another stopping time U_- as the result of that construction when $c^* = (0, r-1, r-2, \ldots, 1)$.

Proposition (4.3). *In the above notation,* $S = U_+ \wedge U_-$, $\mathbb{P}^0_{(r)}$*-a.s.*

Proof. First note that under $\mathbb{P}^0_{(r)}$ the stopping time U_+ is the infimum of those times s for which there exists $0 < s_1 < s_2 < \cdots < s_{r-1} < s$ such that $X_{s_i} = i$, $i = 1, \ldots, r-1$ and $X_s = 0$. Similarly, under $\mathbb{P}^0_{(r)}$ the stopping time U_- is the infimum of those times t for which there exists $0 < t_{r-1} < t_{r-2} < \cdots < t_1 < t$ such that $X_{t_i} = i$, $i = r-1, \ldots, 1$ and $X_t = 0$. Clearly, $S \le U_+ \wedge U_-$, so it remains to show that $S \ge U_+ \wedge U_-$.

Define an *excursion interval* to be an interval of the form $[\alpha, \beta[$, where:

i) either $\alpha = 0$ or $X_{\alpha-} \neq 0$,

ii) $X_\alpha = 0$,

iii) $\beta = \inf\{t \ge \gamma : X_t = 0\}$ with $\gamma = \inf\{t \ge \alpha : X_t \neq 0\}$.

Call the corresponding path segment $\{X_t : \alpha \le t < \beta\}$ an *excursion.* Say that the excursion is *positive* (respectively, *negative*) if $X_\gamma = 1$ (respectively, $X_\gamma = r-1$).

The set of states visited by a positive excursion are of the form $\{0, 1, \ldots, j\}$. Moreover, if $X_{\beta-} \neq r-1$, then we can find times $s_1 < \ldots < s_j < t_j < t_{j-1} < \ldots < t_1$ during the excursion interval such that $X_{s_\ell} = X_{t_\ell} = \ell$. An analogous comment holds for negative excursions.

If $[0, S[$ is an excursion interval then we are done, because S is then U_+ or U_-, depending on whether the excursion is positive or negative.

If $[0, S[$ is not an excursion interval, then the union of the set of states visited by the largest (in the sense of visiting the most states) positive excursion during $[0, S[$ and the set of states visited by the largest (in the same sense) negative excursion during $[0, S[$ must be all of $\mathbb{Z}_r$, because two excursions of the same sign visit the same set of states or one visits a subset of the states visited by the other. Suppose first of all that the largest positive excursion during $[0, S[$ occurs before the largest negative excursion during $[0, S[$. Write $[\alpha_+, \beta_+[$ and $[\alpha_-, \beta_-[$ for the associated excursion intervals. Let $\{0, 1, \ldots, j\}$ be the states visited during $[\alpha_+, \beta_+[$ and $\{k, k+1, \ldots, r-1, 0\}$ be the states visited during $[\alpha_-, \beta_-[$. From the observations made above, we have that $k \leq j$ and there exists $\alpha_+ < u_1 < \cdots < u_j < \beta_+ \leq \alpha_- < u_{j+1} < \cdots < u_{r-1} < \beta_-$ such that $u_\ell = \ell$, $\ell = 1, \ldots, r-1$. Thus, $U_+ \leq S$ in this case. Similarly, if the largest negative excursion occurs before the largest positive excursion, then $U_- \leq S$.

□

As we noted above, Λ_S is not in general stationary under $\mathbb{P}^0_{(r)}$, whereas Λ_{U_+} and Λ_{U_-} are both stationary. We have remarked that Λ_{V_+}, Λ_{V_-}, and $\Lambda_{V_\pm} = \Lambda_{V_+ \wedge V_-}$, are stationary, and it might have been tempting (but false!) to conjecture on that basis that for a general equivariant chain if Λ_T and $\Lambda_{T'}$ have group invariant distributions for two stopping times T, T', then $\Lambda_{T \wedge T'}$ also has a group invariant distribution. Recall, however, Proposition (2.3).

5. Brownian counterparts.

In this section, let $\mathbb{P}^0$ govern $X = (X_t,\ t \geq 0)$ as a standard Brownian motion on a circle, and let $\Lambda_t(x)$ denote the corresponding local time at x up to time t. As in §1, we identify the circle with the interval $[0, 1[$ equipped with addition mod 1. With some recycling of notation, define $V_\pm$ to be the first time X makes a complete loop around the circle, beginning and ending at 0. The fact that the local time field $\Lambda_{V_\pm}$ is stationary is derived in Pitman (1995a) from a formula for the Laplace functional

$$\Phi(m) := \mathbb{P}^0\left[\exp\left(-\int \Lambda_{V_\pm}(x)\, m(dx)\right)\right]$$

of a finite measure m on the circle. For m with finite support, say $m = \sum_{u\in F} a_u \delta_u$ where δ_u is a unit mass at u, so that $\Phi\left(\sum_{u\in F} a_u\delta_u\right) := \mathbb{P}^0[\exp(-\sum_{u\in F} a_u \Lambda_{V_\pm}(u))]$, this formula reads

$$\Phi(\sum_{u\in F} a_u\delta_u) = \left(1 + \frac{1}{2}\sum_{A\subseteq F} \Pi(A)\prod_{u\in A}(2a_u)\right)^{-1} \tag{5.1}$$

where $\sum_{A\subset F}$ is a sum over all non-empty subsets A of F, and $\Pi(A)$ is the product of the spacings around the circle between points of A. In particular, for $F = \{u, u+p\}$ for arbitrary u and p in $[0,1[$, and $a, b \geq 0$, the joint Laplace transform of $\Lambda_{V_\pm}(u)$ and $\Lambda_{V_\pm}(u+p)$ at (a,b) is

$$\Phi(a\delta_u + b\delta_{u+p}) = \Big(1 + a + b + 2abp(1-p)\Big)^{-1}$$

Applied to F contained in the set of multiples of $1/r$, formula (5.1) gives the joint Laplace transform of the stationary occupation field derived from the embedded symmetric simple walk on $\mathbb{Z}_r$ as in Proposition (1.1). Similar remarks hold for V_+ and V_- instead of $V_\pm$. Reusing some more notation, let S denote the first return to 0 following the cover time for the circular Brownian motion. Considerable evidence was presented in Pitman (1995a) to support the conjecture that Λ_S is stationary under $\mathbb{P}^0$ with Laplace functional of the form

$$\mathbb{P}^0\left[\exp\left(-\int \Lambda_S(x)\, m(dx)\right)\right] = \frac{\Phi}{1+\Phi}\left(1 + \frac{\Phi}{\sqrt{1-\Phi^2}}\tanh^{-1}\sqrt{1-\Phi^2}\right) \tag{5.2}$$

where $\Phi = \Phi(m)$. For example, this formula is correct when m is supported on two points, which implies that the two-dimensional marginals of the field Λ_S are invariant under circular shifts. For m with one point support (5.2) simplifies to yield the expression (4.1) for the Laplace transform of $\Lambda_S(x)$ for arbitrary $x \in [0,1[$. Formula (5.2) is also correct for m uniform on any sub-interval of the circle, in particular for m uniform on $[0,1[$ with total mass a, when this expression for the Laplace transform for S simplifies to

$$\begin{aligned}\mathbb{P}^0[\exp(-aS)] &= \frac{\sqrt{2a} + \sinh\sqrt{2a}}{(1+\cosh\sqrt{2a})\sinh\sqrt{2a}} \\ &= 1 - \frac{2}{3}a + \frac{13}{45}a^2 - \frac{97}{945}a^3 + \frac{613}{18900}a^4 + \mathrm{O}(a)^5\end{aligned} \tag{5.3}$$

The limit result (4.1) for the symmetric simple random walk on $\mathbb{Z}_r$ follows straightforwardly from (5.3), as S for the embedded random walk converges almost surely to S for the Brownian motion as $r \to \infty$. Moreover, note that $S \le V_\pm$ holds for both the embedded random walk and the Brownian motion, and $V_\pm$ is the same for both the walk and the Brownian motion and has all moments finite. Thus, it follows that all moments of the pre-S occupation times for the random walk converge to moments of pre-S local times for the Brownian motion.

If formula (5.2) does hold for all finite m, it will be a remarkable state of affairs. For Λ_S would then be stationary relative to circular shifts, but any proof of this that proceeded by "path surgery" using features such as the equivariance of X would appear to have a random walk counterpart, and we know from §4 that the random walk analog is false in general.

The method of §§4 and 5 of Pitman (1995a) yields an explicit formula for the Laplace functional of the asymptotic distribution of the stationary occupation field of the symmetric simple random walk on $\mathbb{Z}_r$ at either of the times U_+ and U_- introduced below Proposition (1.1). This asymptotic distribution turns out to be the same for both U_+ and U_-. That is to say, the limiting occupation field for U_+ is invariant with respect to a reversal of direction around the circle. Similar methods allow an exact computation of the Laplace functional of the occupation field of the walk on $\mathbb{Z}_r$ at time U_+. This field too turns out to be reversible, a curious fact which we are unable to prove by path transformation.

The time corresponding to U_+ for the Brownian motion can be defined as follows. Let $B = (B_t,\ t \ge 0)$ be the standard one-dimensional Brownian motion started at 0 obtained by unwrapping the circular Brownian motion X under $\mathbb{P}^0$. Define a continuous increasing process $(M_t^*,\ t \ge 0)$ and a sequence of stopping times $0 = T_0 < T_1 < T_2, \cdots$ inductively as follows. For $n \ge 1$, let T_n be the first time $t > T_{n-1}$ that $(\sup_{T_{n-1} \le s \le t} B_s) - B_t = 1$, and let

$$M_t^* = n - 1 + \sup_{T_{n-1} \le s \le t} B_s - B_t \text{ for } T_{n-1} \le t < T_n$$

Then the Brownian U_+ is the first t such that $M_t^* = 1$. It is easily seen that the values $M_{T_n}^*$ for $n = 1, 2, \cdots$ are the points of a Poisson process with rate 1, so the number of T_n with $n \ge 1$ and $T_n < U_+$ is a Poisson variable with mean 1. The local time process Λ_{U_+} of the circular Brownian motion X now decomposes as the sum of this Poisson random number of local time processes associated with anticlockwise loops of X, and a local time process associated with short excursions of X, as defined in §4 of Pitman (1995a). Results of that paper show that these two components of Λ_{U_+} are independent and stationary, with simple Laplace functionals whose product is the Laplace functional of Λ_{U_+}. To illustrate, this approach yields the formula

$$
\begin{aligned}
\mathbb{P}^0[\exp(-a\Lambda_{U_+}(x)] &= \exp\left(-2\sqrt{\frac{a}{2+a}}\tanh^{-1}\sqrt{\frac{a}{2+a}}\right) \\
&= 1 - a + \frac{5}{6}\,a^2 - \frac{19}{30}\,a^3 + \frac{229}{504}\,a^4 + \mathrm{O}(a)^5
\end{aligned}
\tag{5.4}
$$

This is the Laplace transform of the limit of the $\mathbb{P}^0_{(r)}$ distribution of $\Lambda_{U_+}(k)$ as $r \to \infty$ for arbitrary $k \in \mathbb{Z}_r$, where $\mathbb{P}^0_{(r)}$ governs the symmetric walk on $\mathbb{Z}_r$ with exponential holding times with mean $1/r$, as in §4. Again, all moments also converge. The corresponding limiting $\mathbb{P}^0_{(r)}$ distribution of $U_+/r = (1/r)\sum_{k\in\mathbb{Z}_r}\Lambda_{U_+}(k)$ is the distribution of U_+ for the Brownian motion, which is determined by the Laplace transform

$$
\begin{aligned}
\mathbb{P}^0[\exp(-aU_+)] &= \exp\left(\frac{\sqrt{2a}(1-\cosh(\sqrt{2a}))}{\sinh(\sqrt{2a})}\right) \\
&= 1 - a + \frac{2}{3}\,a^2 - \frac{11}{30}\,a^3 + \frac{451}{2520}\,a^4 + \mathrm{O}(a)^5
\end{aligned}
\tag{5.5}
$$

We conclude by recording a Brownian analog of Proposition (2.2). From Bolthausen (1979) we see for a finite measure m on $[0,1[$ that the asymptotic $\mathbb{P}^0$ distribution of $(\int \Lambda_t(x)\,m(dx) - m([0,1[)t)/\sqrt{t}$ as $t \to \infty$ is Gaussian with mean 0 and variance

$$
\sigma^2(m) = \int\int \gamma(y-x)\,m(dx)\,m(dy),
$$

where $\gamma(v) = \frac{1}{3} - 2v(1-v)$, $0 \le v < 1$. We can argue as in the proof of Proposition (2.2) to obtain the following:

Proposition (5.1). *Suppose that T is a stopping time with $\mathbb{P}^0[T^2] < \infty$ and such that $\mathbb{P}^0[\Lambda_T(x)] = \mu_T$ for all $x \in [0,1[$ and $\mathbb{P}^0[\Lambda_T(x)\Lambda_T(x+p)] = r_T(p)$ for all x and $p \in [0,1[$ for some constant μ_T and some function $r_T(p), 0 \le p < 1$. Then*

$$
\mu_T = \mathbb{P}^0[T];\ \ r_T(0) = \mathbb{P}^0[T^2] + \frac{1}{3}\,\mathbb{P}^0[T] \tag{5.6}
$$

$$
r_T(p) = r_T(0) - 2p(1-p)\mu_T \tag{5.7}
$$

To illustrate, the stationarity of Λ_{U_+} implies that the above identities hold for $T = U_+$. The first two moments μ_{U_+} and $r_{U_+}(0)$ of the common distribution of $\Lambda_{U_+}(x)$ for all $x \in [0,1[$ can be read from (5.4) as

$$
\mu_{U_+} = 1;\ \ r_T(0) = \frac{5}{3}
$$

Now (5.6) implies

$$\mathbb{P}^0[U_+] = 1; \;\; \mathbb{P}^0[U_+^2] = \frac{4}{3}$$

in agreement with the expansion (5.5) As another example, the stationarity of two-dimensional distributions of Λ_S implies that the identities of Proposition (5.1) hold also for $T = S$. From the common Laplace transform of $\Lambda_S(x)$ for for all $x \in [0, 1[$ displayed in (4.1),

$$\mu_S = 1; \;\; r_S(0) = \frac{4}{5}$$

Now (5.6) implies

$$\mathbb{P}^0[S] = \frac{2}{3}; \;\; \mathbb{P}^0[S^2] = \frac{26}{45}$$

in agreement with (5.3). The consequent identity (5.7) for $T = S$ does not seem easy to verify more directly.

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