Coalescing Markov labelled partitions and a continuous sites genetics model with infinitely many types

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Abstract: Let Z be a Borel right process with Lusin state-space E. For any finite set S it is possible to associate with Z a process ζ of coalescing partitions of S with components labelled by elements of E that evolve as copies of Z. It is shown that, subject to a weak duality hypothesis on Z, there is a Feller process X with state-space a certain space of probability measure valued functions on E. The process X has its "moments" defined in terms of expectations for ζ in a manner suggested by various instances of martingale problem duality between coalescing Markov processes and voter model particle systems, systems of interacting Fisher-Wright and Fleming-Viot diffusions, and stochastic partial differential equations with Fisher-Wright noise. Some sample path properties are examined in the special case where Z is a symmetric stable process on $\mathbb{R}$ with index $1 < \alpha \leq 2$. In particular, we show that for fixed $t > 0$ the essential range of the random probability measure valued function X_t is almost surely a countable set of point masses.

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Partitions étiquetées fusionnantes suivant un processus de Markov et un modèle génétique où le positions forment un continu et où il y a une infinité de types

Résumé: Soit Z un processus droit Borelien dont les états forment un espace de Lusin E. On peut, pour tout ensemble S fini, associer à Z un processus ζ de partitions fusionnantes dont les composantes sont étiquettées par des éléments de E, eux-mêmes soumis à évolution à la manière de Z.

On montre que, sous une hypothèse faible de dualité pour Z, il y a un processus de Feller X dont les états sont des fonctions sur E à valeurs mesures de probabilité. Le processus X a des "moments" définis par des espérances pour Z d'une façon déjà suggérée par des exemples divers de dualité de problèmes de martingales entre des processus Markov fusionnants et des systèmes de particules représentant les votes, des systèmes de Fisher-Wright entrelacés et les diffusions de Fleming-Viot ainsi que les équations différentielles partielles stochastiques dont le bruit est de Fisher-Wright.

On examine quelques propriétés des trajectoires pour un cas spécial où Z est un processus stable symétrique d'indice α, $1 < \alpha \leq 2$, à valeurs dans $\mathbb{R}$. Entre autres choses, on montre que pour $t > 0$, fixé, le support essentiel de la fonction aléatoire à valeurs mesures de probabilité X_t est presque sûrement un ensemble dénombrable de masses de Dirac.

1. Introduction

A powerful tool for analysing a number of stochastic systems that are defined as solutions to martingale problems is the notion of *duality* between two martingale problems (cf. §II.3 of [10] or §4.4 of [6]). A particularly successful application of this idea is to be found in the study of the voter model using duality with a system of coalescing random walks (see Ch. V of [10] for a number of results obtained using duality and a comprehensive bibliography).

Shiga [12] noted an analogous duality between systems of delayed coalescing Markov chains and certain systems of interacting Fisher-Wright diffusions. The latter interacting processes arise as diffusion limits for two-type genetics models with populations at a countable set of discrete sites for which there is within site resampling and between site migration. The form of the duality is that multivariate moments for the system of diffusions can be represented as certain expectations for the delayed coalescing Markov chains. Shiga's observation has been particularly useful in studying the phenomenon of cluster formation in such models (see [7] or [3] for recent bibliographies covering papers in this area). Shiga [13] also showed that the natural stochastic partial differential equation analogue of such a system of interacting Fisher-Wright diffusions is dual in a similar way to a system of delayed coalescing Markov processes.

The above two-type genetics models have infinitely-many-types counterparts, and duality for these has been investigated in [9] and [3]. Here the processes at each site are Fleming-Viot diffusions that take values in the set of probability measures on the type-space $[0, 1]$, and once again the processes interact via a migratory drift given by the jump rates of a Markov chain on the site-space. Now the dual process has values that are partitions of some finite set, with each component of the partition labelled by a point in the site-space. The labels evolve as a system of delayed coalescing Markov chains, and when two labels coalesce the corresponding components of the partition are aggregated together to form one component. Once again, the form of the duality is that moment-like expectations for the Fleming-Viot diffusions can be represented as appropriate expectations for the system of labelled partitions.

In all of the above instances, the duality did not play an explicit role in establishing the existence of the process. Rather, existence was obtained using general Markov chain, weak convergence or stochastic differential equation techniques. The duality first entered in when establishing uniqueness of the solution to a martingale problem and deriving properties of the process.

In [7] the use of coalescing systems was more fundamental. There the authors considered a two-type genetics model with continuous site-space, within sites resampling, and between sites migration. The state-space of this process is a suitable space of functions from the site-space into $[0,1]$, with the value of the function at a given site being thought of as the proportion of the "population" at that site that has one of the two types. The process was constructed by explicitly defining a Feller semigroup that had its associated "moments" expressed as expectations for a certain system of delayed coalescing Markov processes. The existence of transition kernels was established using weak convergence arguments beginning with a discrete site-space system of interacting Fisher-Wright diffusions of the sort discussed above. However, the Feller, Chapman-Kolmogorov and strong continuity properties of the transition kernels were established directly from the description in terms of delayed coalescing Markov processes.

It was shown in [7] that the particular class of continuous sites models considered there have space-time rescaling limits that are again Feller processes with semigroups that have similar explicit descriptions in terms of systems of (instantaneously) coalescing Markov processes. Each of the limit processes has the interesting property of being somewhat like a continuous sites "particle system": at any fixed time the value of process lies in a set of $\{0,1\}$ - valued functions. One might expect this from the abovementioned fact that the voter model particle system is dual to a system of instantaneously coalescing random walks. Moreover, it should be the case (as pointed out in [7]) that the limit processes also appear as rescaling limits of suitable long-range, voter-like models.

Our aim in this paper is to show that, subject to a weak duality condition (here duality is used now in the sense of the general theory of Markov processes), any system of coalescing Borel right processes gives rise to a Feller semigroup via the sort of prescription that arose from duality considerations in [7].

Some of our argument is similar to that in [7]. One major difference is that, because of the generality in which we are working, weak convergence arguments are no longer available to establish the existence of transition kernels. Instead, we proceed analytically and base the proof on the solution to the multidimensional Hausdorff moment problem. This state of affairs is somewhat similar to that which occurred in the development of the theory of superprocesses, where existence proofs based on weak convergence ideas were superseded by ones incorporating analytic characterisations of those functions that appear as the Laplace functional of an infinitely divisible random measure (cf. [8]).

Another significant difference is that we work in the infinitely-many-types setting. The type-space in infinitely-many-types models is usually taken to be the interval $[0,1]$. From a modelling perspective, only the measure-theoretic properties of $[0,1]$ are relevant.

In order to make our proofs more transparent, we use instead the Borel-isomorphic space $\{0,1\}^{\mathbb{N}}$. However, our results can easily be translated into ones for $[0,1]$.

The plan of the rest of the paper is as follows. In §2 we discuss the "dual" process of coalescing partitions labelled by points in the site-space that evolve according to some Markov process. In §3 we review some elementary ideas from the theory of vector measures and introduce the space that will be the state-space of the process we are trying to construct. This is a suitable space of functions from the site-space into the set of probability measures on $\{0,1\}^{\mathbb{N}}$. In §4 we state and prove our main theorem on the existence of a Feller semigroup defined in terms of coalescing Markov labelled partitions. We examine the special case of the general construction that arises when the labels come from a symmetric α-stable process on $\mathbb{R}$ with $1 < \alpha \leq 2$ in §5, and prove some results about the clumping behaviour of this model. Finally, we record in an Appendix a couple of facts about Lusin spaces that we are unable to find in the literature.

2. Coalescing Markov labelled partitions

Given a finite set S, let Π^S denote the set of partitions of S. That is, elements of Π^S are subsets $\{A_1, \ldots, A_N\}$ of $\mathcal{P}(S)$ (:= the power set of S) with the property that $\bigcup_i A_i = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Equivalently, we can think of Π^S as the set of equivalence relations on S. We define a partial order on Π^S by declaring that $\pi \leq \pi'$ if π' is a refinement of π, that is, if the components of π are obtained by aggregating together one or more components of π'. Given $\pi = \{A_1, \ldots, A_N\} \in \Pi^S$, put $\eta(\pi) = N$.

Fix another (possibly infinite) set E. An E-labelled partition of S is a subset of $\mathcal{P}(S) \times E$ of the form $\{(A_1, e_{A_1}), \ldots, (A_N, e_{A_N})\}$, with $\{A_1, \ldots, A_N\} \in \Pi^S$ and $e_{A_i} \neq e_{A_j}$ for $i \neq j$. Given $\lambda = \{(A_1, e_{A_1}), \ldots, (A_N, e_{A_N})\} \in \Lambda^S$, put $\alpha(\lambda) = \{A_1, \ldots, A_N\}$ and $\epsilon(\lambda) = (e_A)_{A \in \alpha(\lambda)}$. Let Λ^S denote the set of E-labelled partitions of S.

Given $\pi \in \Pi^S$ and $e = (e_A)_{A \in \pi} \in E^\pi$ such that $e_A \neq e_{A'}$ for $A \neq A'$, put $\lambda(\pi, e) = \{(A, e_A) : A \in \pi\}$. That is, $\lambda(\pi, e)$ is the labelling of π with e.

Denote $\{1, \ldots, n\}$ by $S^{(n)}$. Write $\Pi^{(n)}$ and $\Lambda^{(n)}$ for Π^S and Λ^S when $S = S^{(n)}$. Put $\pi^{(n)}_{\max} = \{\{1\}, \cdots, \{n\}\} \in \Pi^{(n)}$; and for $(e_1, \ldots, e_n) \in E^n$ such that $e_i \neq e_j$ for $i \neq j$, put $\lambda^{(n)}_{\max}(e) = \{(\{1\}, e_1), \ldots, (\{n\}, e_n)\} \in \Lambda^{(n)}$.

Assume now that E is a Lusin space and that (Z, P^z) is a Borel right process on E with semigroup $\{P_t\}_{t \geq 0}$ satisfying $P_t 1 = 1$, $t \geq 0$, so that Z has infinite lifetime. We wish to define an associated Λ^S-valued Borel right process ζ^S that has the following intuitive description. Let $\lambda \in \Lambda^S$. The evolution of ζ^S starting at λ will be such that $\alpha(\zeta^S(t))$ remains unchanged and $\epsilon(\zeta^S(t))$ evolves as a vector of independent copies of Z starting at $\epsilon(\lambda)$ until immediately before two (or more) such labels coincide. At this time, the components of the partition corresponding to the coincident labels are merged into one component. This component is labelled with the common element of E. The evolution then continues in the same way.

It is possible to give a rigorous definition of ζ^S using a "concatenation of processes" construction (cf. §14 of [11]). Alternatively, it is possible to build ζ^S explicitly from N independent copies of Z started at $\epsilon(\lambda)$ by proceeding along the lines of the construction in [7]. Essentially, that latter construction builds the process of labels, and the corresponding partitions can then be added on in a simple, deterministic manner. As either of these constructions is rather straightforward but involves the introduction of a substantial amount of notation, we will omit the details.

We will denote the law of ζ^S starting at λ as P_S^λ. When $S = S^{(n)}$ we write $\zeta^{(n)}$ and $P_{(n)}^\lambda$ for ζ^S and P_S^λ.

Given two finite sets S and T and an injection $\rho : S \to T$, we can define an induced map $R_\rho : \Lambda^T \to \Lambda^S$ as follows. If $\lambda \in \Lambda^T$ is of the form $\{(A_1, e_1), \ldots, (A_n, e_n)\}$, then $R_\rho \lambda = \{(\rho^{-1}(A_i), e_i) : \rho^{-1}(A_i) \neq \emptyset\}$

The following observation is immediate from the definition.

Lemma (2.1). *If S, T, ρ, and R_ρ are as above, then the law of $R_\rho \circ \zeta^T$ under P_T^λ is that of ζ^S under $P_S^{R_\rho \lambda}$.*

Assumption. From now on, we will suppose that there is another Borel right process $\hat{Z}$ with semigroup $\{\hat{P}_t\}_{t \geq 0}$ and a diffuse, Radon measure m on $(E, \mathcal{E})$ such that Z and $\hat{Z}$ are in *weak duality* with respect to m; that is, for all nonnegative Borel functions on f, g on E we have $\int m(de)\, P_t f(e) g(e) = \int m(de)\, f(e) \hat{P}_t g(e)$.

3. The state-space.

We need some elementary ideas from the theory of vector measures. A good reference is [4].

Let $(E, \mathcal{E}, m)$ be the measure space introduced in §2, and let X be a Banach space with norm $\|\cdot\|$. We say that a function $\phi : E \to X$ is *simple* if $\phi = \sum_{i=1}^{k} x_i 1_{E_i}$ for $x_1, \ldots, x_k \in X$ and $E_1, \ldots, E_k \in \mathcal{E}$. We say that a function $\phi : E \to X$ is *m-measurable* if there exists a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} \|\phi_n(e) - \phi(e)\| = 0$ for m-a.e. $e \in E$. The definitions in [4] are given in the case when m is finite, but they make sense in this more general setting. Also, much of the resulting theory holds unchanged, and we will apply without comment results from [4] that are stated for finite m but hold (with trivial modifications to the proof) for our Radon m (or, indeed, for an arbitrary σ-finite measure).

Write K for the compact, metrisable coin-tossing space $\{0,1\}^{\mathbb{N}}$ equipped with the product topology, and let $\mathcal{K}$ denote the corresponding Borel σ-field. Equivalently, $\mathcal{K}$ is the σ-field generated by the cylinder sets.

Write M for the Banach space of finite signed measures on $(K, \mathcal{K})$ equipped with the total variation norm $\|\cdot\|_M$, and let M_1 denote the closed subset of M consisting of probability measures.

Let $L^\infty(m, M)$ denote the space of (equivalence classes of) m-measurable maps $\mu : E \to M$ such that $\operatorname{ess\,sup}\{\|\mu(e)\|_M : e \in E\} < \infty$, and equip $L^\infty(m, M)$ with the obvious norm to make it a Banach space. Let Ξ denote the closed subspace of $L^\infty(m, M)$ consisting of (equivalence classes of) maps with values in M_1.

Write C for the Banach space of continuous functions on K equipped with the usual supremum norm $\|\cdot\|_C$. Let $L^1(m, C)$, denote the Banach space of (equivalence classes of) m-measurable maps $\mu : E \to C$ such that $\int m(de)\, \|\mu(e)\|_C < \infty$, and equip $L^1(m, C)$ with the obvious norm to make it a Banach space. Then $L^1(m, C)$ is the Banach space of Bochner integrable C-valued functions on E (cf. Theorem II.2.2 of [4]).

From the discussion at the beginning of §IV.1 in [4] and the fact that M is isometric to the dual space of C under the pairing $(\nu, y) \mapsto \langle \nu, y \rangle = \int \nu(dk)\, y(k)$, we see that $L^\infty(m, M)$ is isometric to a closed subspace of the dual of $L^1(m, C)$ under the pairing $(\mu, x) \mapsto \int m(de)\, \langle \mu(e), x(e) \rangle$.

It is not true that $L^\infty(m, M)$ is isometric to the whole of the dual of $L^1(m, C)$. From Theorem IV.1.1 of [4] this would be the case if and only if M had the *Radon-Nikodym property* with respect to m. If κ is coin-tossing measure on K, then M contains $L^1(\kappa)$ as a closed, separable subspace. By the remarks following Definition III.1.3 of [4] we see that $L^1(\kappa)$ fails to have the Radon-Nikodym property, and hence, by Theorem III.3.2 of [4], the same is true of M.

From Corollary V.4.3 and Theorem V.5.1 of [5] we see that, as $L^1(m, C)$ is separable, Ξ equipped with the relative weak* topology is a compact, metrisable space.

For a finite set T, let M^T (respectively, C^T) denote the Banach space of finite signed measures (respectively, continuous functions) on the Cartesian product K^T with the usual norm $\|\cdot\|_{M^T}$ (respectively, $\|\cdot\|_{C^T}$). With a slight abuse of notation, write $\langle\,\cdot\,,\,\cdot\,\rangle$ for the pairing between these two spaces. Following our usual convention, we will write $M^{(n)}$ and $C^{(n)}$ when $T = \{1, \ldots, n\}$.

Given $\phi \in L^1(m^{\otimes T}, C^T)$, define $I_T(\,\cdot\,;\phi) \in C(\Xi)$ by

$$I_T(\mu;\phi) = \int m^{\otimes T}(de)\,\langle \bigotimes_{t\in T} \mu(e_t), \phi(e)\rangle.$$

When $T = \{1, \ldots, n\}$ we write $I_n(\,\cdot\,;\phi)$. Of course, $I_T(\,\cdot\,;\phi)$ is always of the form $I_n(\,\cdot\,;\phi')$ for $n = |T|$ and a suitable ϕ', but this more general notation will be useful in what follows.

Lemma (3.1). *The linear subspace spanned by the constant functions and functions of the form $I_n(\,\cdot\,;\phi)$ with $\phi = \psi \otimes \chi$, $\psi \in L^1(m^{\otimes n}) \cap C(E^n)$ and $\chi \in C^{(n)}$ is dense in $C(\Xi)$.*

Proof. The subspace in question is an algebra that contains the constants. The result will follow from the Stone-Weierstrass theorem if we can show that the subspace separates points of Ξ. However, by definition of $L^1(m, C)$ and Lemma (A.2) in the Appendix, functions of the form $\phi = \sum_{i=1}^k \psi_i \otimes \chi_i$, with $\psi_i \in L^1(m) \cap C(E)$ and $\chi_i \in C$, are dense in $L^1(m, C)$, and hence the set of linear functions $I_1(\,\cdot\,;\phi)$ for ϕ of this type separates points.

□

4. Statement and proof of the main result

In order to complete our preparation, we need a little more notation. Given a finite set S, partitions $\pi, \pi' \in \Lambda^S$ such that $\pi \leq \pi'$, and $k = (k_A)_{A\in\pi} \in K^\pi$, define $\gamma(k;\pi',\pi) \in K^{\pi'}$ by setting, for each $A' \in \pi'$ such that $A' \subset A \in \pi$, $\gamma(k;\pi',\pi)_{A'} = k_A$. For example, if $S = S^{(4)} = \{1, 2, 3, 4\}$, $\pi = \{\{1, 3\}, \{2, 4\}\}$, and $\pi' = \pi^{(4)}_{\max} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, then

$\gamma((k_{\{1,3\}}, k_{\{2,4\}}); \pi', \pi) = (k'_{\{1\}}, k'_{\{2\}}, k'_{\{3\}}, k'_{\{4\}})$ where $k'_{\{1\}} = k'_{\{3\}} = k_{\{1,3\}}$ and $k'_{\{2\}} = k'_{\{4\}} = k_{\{2,4\}}$.

Further, for $\lambda, \lambda' \in \Lambda^S$ with $\alpha(\lambda) \le \alpha(\lambda')$ and $\mu \in \Xi$, define the probability measure $\Gamma(\mu; \lambda', \lambda)$ on $(K^{\lambda'}, \mathcal{K}^{\lambda'})$ to be the push-forward of the product measure $\bigotimes_{(A,e)\in\lambda} \mu(e)$ under the map $\gamma(\,\cdot\,; \alpha(\lambda'), \alpha(\lambda))$. For example, if $S = S^{(4)} = \{1,2,3,4\}$, $\alpha(\lambda) = \{\{1,3\},\{2,4\}\}$, and $\alpha(\lambda') = \pi^{(4)}_{\max} = \{\{1\},\{2\},\{3\},\{4\}\}$, then for a bounded Borel function $(k'_{\{1\}}, k'_{\{2\}}, k'_{\{3\}}, k'_{\{4\}}) \mapsto F(k'_{\{1\}}, k'_{\{2\}}, k'_{\{3\}}, k'_{\{4\}})$ we have

$$\Gamma(\mu; \lambda', \lambda)(F) = \int (\mu(e_{\{1,3\}}) \otimes \mu(e_{\{2,4\}}))(dk_{\{1,3\}}, dk_{\{2,4\}})\, F(k_{\{1,3\}}, k_{\{2,4\}}, k_{\{1,3\}}, k_{\{2,4\}}).$$

Given a finite set S, $\lambda \in \Lambda^S$, $t \ge 0$, and $\mu \in \Xi$, define the probability measure $A \mapsto q(A; S, \lambda, t, \mu)$ on $(K^\lambda, \mathcal{K}^\lambda)$ to be $A \mapsto P^\lambda_S[\Gamma(\mu; \lambda, \zeta^S(t))(A)]$.

Recall that a probability kernel P on Ξ is *Feller* if $PF \in C(\Xi)$ when $F \in C(\Xi)$, and a Markov semigroup $\{Q_t\}_{t\ge0}$ on Ξ is *Feller* if each kernel Q_t is Feller in the above sense and $\lim_{t\downarrow0} Q_t F = F$ in $C(\Xi)$ for $F \in C(\Xi)$ (that is, $\{Q_t\}_{t\ge0}$ is *strongly continuous*).

Theorem (4.1). *There exists a unique, Feller, Markov semigroup $\{Q_t\}_{t\ge0}$ on Ξ such that for all $\phi \in L^1(m^{\otimes n}, C^{(n)})$, $n \in \mathbb{N}$, we have*

$$\int Q_t(\mu, d\nu) I_n(\nu; \phi) = \int m^{\otimes n}(de)\, q(\phi(e)(\cdot); S^{(n)}, \lambda^{(n)}_{\max}(e), t, \mu), \tag{4.1}$$

where the integrand is interpreted as 0 on the null set of $(e_1, \ldots, e_n)$ such that $e_i = e_j$ for some pair (i,j). Consequently, there is a Hunt process, $(X, \mathbb{P}^x)$, with state-space Ξ and transition semigroup $\{Q_t\}_{t\ge0}$.

Proof. We break the proof into a number of steps that we identify as we proceed.

(i) Well-definedness. We need to check that the right-hand side of (4.1) doesn't depend on the choice of representative for the equivalence class of μ. For this we need to make a few observations.

It follows from the duality hypothesis that if $\psi \in L^1(m^{\otimes\pi})$ and D is a $m^{\otimes\pi}$-null Borel subset of E^π, then

$$\int m^{\otimes\pi}(de)\, \psi(e)(\bigotimes_{A\in\pi} P_t(e_A, \cdot))(D) = \int m^{\otimes\pi}(de)\, 1_D(e)(\bigotimes_{A\in\pi} \hat{P}_t(e_A, \cdot))(\psi) = 0,$$

and hence the finite signed measure $\int m^{\otimes \pi}(de)\, \psi(e) \bigotimes_{A\in\pi} P_t(e_A,\cdot)$ is absolutely continuous with respect to $m^{\otimes \pi}$. Therefore, by definition of ζ^S, if $\pi, \pi' \in \Pi^S$ with $\pi \le \pi'$ and $\psi \in L^1(m^{\otimes \pi'})$, then the finite signed measure

$$\int m^{\otimes \pi'}(de')\, \psi(e') P_S^{\lambda(\pi',e')}\{\epsilon(\zeta^S(t)) \in de;\, \alpha(\zeta^S(t)) = \pi\}$$

is absolutely continuous with respect to $m^{\otimes \pi}$ (recall that $\lambda(\pi,' e')$ is the labelling of the partition π' with the vector e').

Denote the Radon-Nikodym derivative of this latter measure by $\Theta_t(\pi', \pi)\psi$. Note that $\Theta_t(\pi', \pi) : L^1(m^{\otimes \pi'}) \to L^1(m^{\otimes \pi})$ is a bounded linear operator with norm at most 1.

Thus, for $\phi = \psi \otimes \chi$, where $\psi \in L^1(m^{\otimes n})$ and $\chi \in C^{(n)}$, we have

$$\begin{aligned}\int m^{\otimes n}(de)\, q(\phi(e)(\cdot); S^{(n)}, \lambda^{(n)}_{\max}(e), t, \mu)\\ = \sum_{\pi\in\Pi^{(n)}} I_\pi(\mu; (\Theta_t(\pi^{(n)}_{\max}, \pi)\psi) \otimes (\chi \circ \gamma(\,\cdot\,; \pi^{(n)}_{\max}, \pi)).\end{aligned} \tag{4.2}$$

Consider an arbitrary $\phi \in L^1(m^{\otimes n}, C^{(n)})$. Define $\tilde\psi = \|\phi\|_{C^{(n)}} \in L^1(m^{\otimes n})$, and put $\tilde\phi = \tilde\psi \otimes 1 \in L^1(m^{\otimes n}, C^{(n)})$. For any $\mu, \mu' \in \Xi$ we have, by (4.2), that

$$\begin{aligned}&\left|\int m^{\otimes n}(de)\, q(\phi(e)(\cdot); S^{(n)}, \lambda^{(n)}_{\max}(e), t, \mu) - \int m^{\otimes n}(de)\, q(\phi(e)(\cdot); S^{(n)}, \lambda^{(n)}_{\max}(e), t, \mu')\right|\\ &\le \int m^{\otimes n}(de)\, q(\tilde\phi(e)(\cdot); S^{(n)}, \lambda^{(n)}_{\max}(e), t, |\mu - \mu'|)\\ &= \sum_{\pi\in\Pi^{(n)}} I_\pi(|\mu-\mu'|; (\Theta_t(\pi^{(n)}_{\max}, \pi)\tilde\psi) \otimes 1).\end{aligned}$$

(Here, of course, $|\mu - \mu'|$ is typically not a member of Ξ, and we are extending in the obvious way the functions q and I_π defined on Ξ to all of $L^\infty(m, M)$.) In particular, if μ and μ' both belong to the same equivalence class, then the rightmost member above is 0, as required.

We remark at this point that it can be the case that two different functions $\phi \in L^1(m^{\otimes n}, C^{(n)})$ and $\phi' \in L^1(m^{\otimes n'}, C^{(n')})$, $n, n' \in \mathbb{N}$, are such that $I_n(\,\cdot\,;\phi) = I_{n'}(\,\cdot\,;\phi')$. Hence, there appears to be a potential ambiguity in (4.1): the left hand side should be the same for all choices of n and ϕ that lead to the same element of $C(\Xi)$, whereas it appears that, *a priori*, the right hand side can depend on the particular choice of n and ϕ. We show below in part (ii) of the proof that this ambiguity is not present.

(ii) Existence of measures. We next show that for each $\mu \in \Xi$ and $t \ge 0$ there exists a Borel probability measure $Q_t(\mu,\cdot)$ on Ξ that satisfies (4.1). It suffices to show that on

some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there is a Ξ-valued random variable V such that for all $\phi \in L^1(m^{\otimes \ell}, C^{(\ell)})$, $\ell \in \mathbb{N}$, we have

$$\mathbb{P}[\int m^{\otimes \ell}(de)\langle V(e)^{\otimes \ell}, \phi(e)\rangle] = \int m^{\otimes \ell}(de)\, q(\phi(e)(\cdot); S^{(\ell)}, \lambda_{\max}^{(\ell)}(e), t, \mu), \tag{4.3}$$

because then we can define $Q_t(\mu, \cdot)$ to be the distribution of V. We note that (4.3) will certainly be enough to establish that the possible ambiguity mentioned in part (i) does not occur.

Let $\mathcal{B}$ be a countable ring of sets of finite m-measure that generates the σ-field $\mathcal{E}$, and $\mathcal{G}$ be the field of sets generated by the cylinder sets in K. We begin with the claim that on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ it is possible to construct a family of random variables $\{Y_{B,G}\}_{B \in \mathcal{B}, G \in \mathcal{G}}$ such that $Y_{B,G}$ takes values in $[0, m(B)]$ for all $B \in \mathcal{B}$, $G \in \mathcal{G}$, and

$$\mathbb{P}[\prod_{i=1}^{\ell} Y_{B_i,G_i}] = \int m^{\otimes \ell}(de) \bigotimes_{i=1}^{\ell} 1_{B_i}(e) q(\prod_{i=1}^{\ell} G_i; S^{(\ell)}, \lambda_{\max}^{(\ell)}(e), t, \mu) \tag{4.4}$$

for $B_1, \ldots, B_\ell \in \mathcal{B}$ and $G_1, \ldots, G_\ell \in \mathcal{G}$.

In order to establish (4.4), let $(D_1, H_1), (D_2, H_2), \ldots$ be an enumeration of $\mathcal{B} \times \mathcal{G}$ (note that it is not necessarily the case that both $D_i \neq D_j$ and $H_i \neq H_j$ for $i \neq j$). By Kolmogorov's extension theorem, it suffices for (4.4) to show that for each $k \in \mathbb{N}$ there exist random variables $W_1, \ldots, W_k \in [0, 1]$ such that

$$\begin{aligned}\mathbb{P}[\prod_{i=1}^{k} W_i^{n_i}] = \int & m^{\otimes |n|}(de) \bigotimes_{i=1}^{k} (\frac{1_{D_i}}{m(D_i)})^{\otimes n_i}(e) q(\prod_{i=1}^{k} H_i^{n_i}; S^{(|n|)}, \lambda_{\max}^{(|n|)}(e), t, \mu) \\ &:= F(n_1, \ldots, n_k),\end{aligned}$$

where $n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ and $|n| := n_1 + \cdots + n_k$.

By the solution of the multidimensional Hausdorff moment problem (cf. Proposition 4.6.11 of [1]), we need to check for all $n, q \in \mathbb{N}_0^k$ that

$$\sum_{0 \leq p \leq n} (-1)^{|p|} \binom{n}{p} F(q + p) \geq 0, \tag{4.5}$$

where $\leq$ is the usual coordinatewise partial order on $\mathbb{N}_0^k$ and $\binom{n}{p} := \prod_i \binom{n_i}{p_i}$.

Put $\bar{E} = \prod_{i=1}^{k} E^{n_i + q_i}$. From Lemma (2.1) the left hand side of (4.5) is just

$$\int_{\bar{E}} \bigotimes_{i=1}^{k} m^{\otimes (n_i + q_i)}(de) \bigotimes_{i=1}^{k} (\frac{1_{D_i}}{m(D_i)})^{\otimes (n_i + q_i)}(e) q(J; S, \lambda(e), t, \mu),$$

where

$$S = \bigcup_{i=1}^{k} \{(i,1), \ldots, (i, n_i + q_i)\},$$

$$\lambda(e) = \bigcup_{i=1}^{k} \{(\{(i,1)\}, e_{i,1}), \ldots, (\{(i, n_i+q_i)\}, e_{i,n_i+q_i})\},$$

and

$$J = \prod_{i=1}^{k} (J_{i,1} \times \cdots \times J_{i,n_i+q_i}),$$

with

$$J_{i,j} = \begin{cases} H_i, & \text{if } 1 \le j \le q_i, \\ K \backslash H_i, & \text{if } q_i + 1 \le j \le n_i + q_i, \end{cases}$$

and so (4.5) certainly holds.

If $B, D \in \mathcal{B}$ and $G, H \in \mathcal{G}$, with $B \cap D = \emptyset$ and $G \cap H = \emptyset$, then computations using (4.4) show that

$$\mathbb{P}[(Y_{B \cup D, G} - Y_{B,G} - Y_{D,G})^2] = 0,$$

$$\mathbb{P}[(Y_{B, G \cup H} - Y_{B,G} - Y_{B,H})^2] = 0,$$

and

$$\mathbb{P}[(Y_{B,K} - m(B))^2] = 0.$$

We may thus suppose that the construction of $\{Y_{B,G}\}_{B \in \mathcal{B}, G \in \mathcal{G}}$ is such that the equalities

$$Y_{B \cup D, G} = Y_{B,G} + Y_{D,G},$$

$$Y_{B, G \cup H} = Y_{B,G} + Y_{B,H}, \tag{4.6}$$

and

$$Y_{B,K} = m(B)$$

hold identically.

For $n \in \mathbb{N}$, let $\mathcal{K}_n$ denote the sub-σ-field of $\mathcal{K}$ generated by cylinder sets of the form $A_1 \times \cdots \times A_n \times \{0,1\} \times \{0,1\} \times \cdots$. We can identify $\mathcal{K}_n$ in the obvious way with the σ-field of all subsets of $\{0,1\}^n$. Write $M(\{0,1\}^n)$ and $C(\{0,1\}^n)$), respectively, for the Banach spaces of finite signed measures and continuous functions on $\{0,1\}^n$. Of course, $M(\{0,1\}^n)$ is isometric to $\ell_1(\mathbb{R}^{2^n})$ and $C(\{0,1\}^n)$ is isometric to $\ell_\infty(\mathbb{R}^{2^n})$. Write $M_1(\{0,1\}^n)$ for the subset of $M(\{0,1\}^n)$ consisting of probability measures.

We see from (4.6) that for each $\omega \in \Omega$ the map $1_B \otimes 1_G \to Y_{B,G}(\omega)$, $B \in \mathcal{B}$, $G \in \mathcal{K}_n$, extends to a unique, positivity preserving, linear functional with norm 1 on $L^1(m, C(\{0,1\}^n))$

(where we again stress that we are identifying sets in $\mathcal{K}_n$ with subsets of $\{0,1\}^n$). Furthermore, this functional assigns the value $m(B)$ to the function $1_B \otimes 1$, $B \in \mathcal{B}$. Consequently, for each $\omega \in \Omega$ there is an element $U_n(\omega)$ of $L^\infty(m, M(\{0,1\}^n)) = L^1(m, C(\{0,1\}^n))^*$ such that $\int m(de)\, 1_B(e)(U_n(e)(\omega))(G) = Y_{B,G}(\omega)$ for $B \in \mathcal{B}$ and $G \in \mathcal{K}_n$, and $U_n(\omega)$ has a representative that takes values in $M_1(\{0,1\}^n)$. A monotone class argument shows that U_n is a $L^\infty(m, M(\{0,1\}^n))$ - valued random variable if we equip $L^\infty(m, M(\{0,1\}^n))$ with the Borel σ-field arising from the weak* topology.

The sequence $\{U_n\}_{n\in\mathbb{N}}$ is consistent in the sense that, for $m \times \mathbb{P}$-a.e. $(e,\omega) \in E$ and all $n' < n$, if we we compose the natural projection from $M(\{0,1\}^n)$ onto $M(\{0,1\}^{n'})$ with $U_n(e)(\omega)$, then we obtain $U_{n'}(e)(\omega)$.

For $n \in \mathbb{N}$, $e \in E$ and $\omega \in \Omega$, define $V_n(e)(\omega) \in M_1$ by setting

$$V_n(e)(\omega)(\prod_{i=1}^\infty A_i) = (U_n(e)(\omega))(\prod_{i=1}^n A_i)\, \kappa(\prod_{i=n+1}^\infty A_i).$$

It is clear that V_n is a Ξ-valued random variable.

By the consistency property noted above, for $m \times \mathbb{P}$-a.e. $(e,\omega) \in E \times \Omega$ the sequence of probability measures $\{V_n(e)(\omega)\}_{n\in\mathbb{N}}$ converges in the weak* topology on M_1 as $n \to \infty$ to the unique probability measure that coincides with $U_n(e)(\omega)$ on $\mathcal{K}_n$. Hence, by dominated convergence, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every and $\phi \in L^1(m, C)$ we have that the sequence $\{\int m(de)\langle V_n(e)(\omega), \phi(e)\rangle\}_{n\in\mathbb{N}}$ is convergent. Therefore, the sequence $\{V_n(\omega)\}_{n\in\mathbb{N}}$ converges in Ξ to a point $V(\omega)$. In particular, V is a Ξ-valued random variable. Moreover, for $\mathbb{P}$-a.e. $\omega \in \Omega$ the function $V(\omega)$ has the property that for m-a.e. $e \in E$ the value $V(e)(\omega) \in M_1$ is the unique probability measure that coincides with $U_n(e)(\omega)$ on $\mathcal{K}_n$.

With $B_1, \ldots, B_\ell \in \mathcal{B}$ and $G_1, \ldots, G_\ell \in \mathcal{G}$ as above, set $\psi = \bigotimes_{i=1}^\ell 1_{B_i} \in L^1(m^{\otimes \ell})$ and $\chi = \bigotimes_{i=1}^\ell 1_{G_i} \in C^{(\ell)}$. By construction, we have

$$\mathbb{P}[\int m^{\otimes\ell}(de)\langle V(e)^{\otimes\ell}, \psi \otimes \chi(e)\rangle] = \int m^{\otimes\ell}(de)\, \psi(e) q(\chi; S^{(\ell)}, \lambda^{(\ell)}_{\max}(e), t, \mu)$$

Linear combinations of functions of the same form as ψ (respectively, χ) are dense in $L^1(m^{\otimes\ell})$ (respectively, $C^{(\ell)}$), and so (4.3) holds, as required.

(iii) Uniqueness. It is immediate from Lemma (3.1) and a monotone class argument that for each $\mu \in \Xi$ and $t \geq 0$ there is at most one probability measure $Q_t(\mu, \cdot)$ satisfying (4.1).

(iv) Feller property. We now show that if $F \in C(\Xi)$, then $\mu \mapsto \int Q_t(\mu, d\nu)F(\nu)$ is also an element of $C(\Xi)$. By Lemma (3.1), it suffices to check the special case of $F = I_n(\,\cdot\,;\phi)$, where $\phi = \psi \otimes \chi$, with $\psi \in L^1(m^{\otimes n})$ and $\chi \in C^{(n)}$, but this is immediate from (4.2).

One consequence of the Feller property is, of course, that $\mu \mapsto \int Q_t(\mu, d\nu)G(\nu)$ is Borel for G bounded and Borel; and so $Q_t(\cdot,\cdot)$ is a kernel on Ξ for each $t \geq 0$.

(v) Semigroup property. Noting Lemma (3.1) and (4.2), the semigroup property of the kernels $\{Q_t\}_{t\geq 0}$ follows from the two observations that for $s, t \geq 0$ and $\pi \leq \pi' \leq \pi'' \in \Pi^{(n)}$ we have

$$\gamma(\,\cdot\,;\pi'',\pi') \circ \gamma(\,\cdot\,;\pi',\pi) = \gamma(\,\cdot\,;\pi'',\pi)$$

and, by the Markov property of $\zeta^{(n)}$,

$$\sum_{\pi \leq \tilde{\pi} \leq \pi''} \Theta_s(\pi'',\tilde{\pi})\Theta_t(\tilde{\pi},\pi) = \Theta_{s+t}(\pi'',\pi).$$

(vi) Strong continuity. Given what we have already shown, in order to show that $\lim_{t\downarrow 0} Q_t F = F$ in $C(\Xi)$ for $F \in C(\Xi)$, it suffices by standard semigroup arguments (cf. the Remark after Theorem I.9.4 in [2]) to show that $\lim_{t\downarrow 0} Q_t F(\mu) = F(\mu)$ for each $\mu \in \Xi$. By Lemma (3.1), it further suffices to consider the case $F = I_n(\,\cdot\,;\phi)$, where $\phi = \psi \otimes \chi$ with $\psi \in L^1(m^{\otimes n}) \cap C(E^n)$, $\chi \in C^{(n)}$, and both ψ and χ are nonnegative.

By definition of $\zeta^{(n)}$, the total variation distance between the distribution of $\zeta^{(n)}(t)$ under $P_{(n)}^{\lambda^{(n)}_{\max}(e)}$ and the push-forward of the probability measure $\bigotimes_{i=1}^n P_t(e_i,\cdot)$ by the function $\lambda(\pi^{(n)}_{\max},\cdot)$ is bounded above by $P_{(n)}^{\lambda^{(n)}_{\max}(e)}\{\alpha(\zeta^{(n)}(t)) \neq \pi^{(n)}_{\max}\}$ (that is, by the probability that a coalescence has not occurred by time t). This probability converges to 0 as $t \downarrow 0$.

We are thus left with showing that

$$\lim_{t\downarrow 0} \int m^{\otimes n}(de)\,\psi(e) \int \bigotimes_{i=1}^n P_t(e_i, df_i)G(f) = \int m^{\otimes n}(de)\,\psi(e)G(e), \tag{4.7}$$

where we put $G(e) = \langle \bigotimes_i \mu(e_i), \chi\rangle$. By the duality hypothesis,

$$\int m^{\otimes n}(de)\,\psi(e) \int \bigotimes_{i=1}^n P_t(e_i, df_i)G(f) = \int m^{\otimes n}(de)\,G(e) \int \bigotimes_{i=1}^n \hat{P}_t(e_i, df_i)\,\psi(f),$$

and (4.7) follows.

(vii) Existence of a Hunt process. The existence of a Hunt process with transition semigroup $\{Q_t\}_{t\geq 0}$ is immediate from general theory (see, for example, Theorem I.9.4 of [2]).

$\square$

Remarks. (a) An inspection of the above proof shows that a similar result will hold if the processes ζ^S are replaced in the definition of Q_t by certain other Markov systems of coalescing labelled partitions. This will be the case provided that the new systems have the following properties. Firstly, the consistency condition Lemma (2.1) should hold. Secondly, the set of the labels at a fixed time should have the distribution of a subset of a collection of independent copies of Z (so that an analogue of (4.2) holds). Finally, the total variation distance between the distribution of the labels at time t and that of a collection of independent copies of Z should converge to 0 as $t \downarrow 0$ (cf. the proof of strong continuity). For example, one could have the components of the partition coalesce at a rate proportional to a "collision local time" between the associated labels in a manner analogous to that considered in [7].

(b) Fix $k \in \mathbb{N}$ and let $G_1, \ldots, G_k$ be a partition of K into non-empty sets that are both open and closed (that is, sets with a continuous indicator function). Such a partition exists for all k. Let $\Sigma = \{(p_1, \ldots, p_k) \in [0,1]^k : p_1 + \cdots + p_k = 1\}$ denote the standard k-simplex. Define $L : M_1 \to \Sigma$ by $L\nu = (\nu(G_1), \ldots, \nu(G_k))$ and define a process $\{\tilde{X}_t\}_{t \geq 0}$ with state-space the subset of $L^\infty(m, \mathbb{R}^k)$ consisting of Σ-valued functions by $\tilde{X}_t(e) = L(X_t(e))$. It is easy to verify Dynkin's well-known sufficient condition for a function of a Markov process to be Markov and conclude that $\tilde{X}$ is a Feller process. The process $\tilde{X}$ is the k-types analogue of our infinitely-many-types model. In particular, when $k = 2$ the process $\{\bar{X}_t\}_{t \geq 0}$ with state-space the subset of $L^\infty(m)$ consisting of $[0,1]$-valued functions defined by $\bar{X}_t(e) = (\tilde{X}_t(e))_1$ is also a Feller process. The "cluster process" of [7] is a particular instance of this latter construction.

Open Problem. When Z belongs to the class of Lévy processes considered in [7], the sort of weak convergence arguments used there in the two-type case show that the process X has continuous sample-paths. The same result should hold (for similar reasons) when Z is a nice enough process on $\mathbb{R}$. It would be interesting to know general necessary and sufficient conditions on Z for the path continuity of X.

5. The stable case.

Let $((Z_1, Z_2), P^{(z_1, z_2)})$ be the Cartesian product of the right process (Z, P^z) with itself. It follows from a variance calculation using (4.1) that the Hunt process X evolves deterministically if and only if

$$m \otimes m\{(z_1, z_2) \in E^2 : P^{(z_1,z_2)}\{\exists\, t \geq 0 : Z_1(t) = Z_2(t)\} \neq 0\} = 0. \tag{5.1}$$

When (5.1) holds, for $\mu \in \Xi$ we have $\mathbb{P}^\mu$-a.s. that $X_t = \mu_t$ for all $t \geq 0$, where $\mu_t \in \Xi$ is the unique point that satisfies, for $\psi \in L^1(m)$ and $\chi \in C$, $I_1(\mu_t; \psi \otimes \chi) = I_1(\mu; \psi_t \otimes \chi)$, with ψ_t the Radon-Nikodym derivative $(\int m(de)\, \psi(e) P_t(e, df))/m(df)$.

A particularly interesting example of a non-deterministic evolution is the case when Z is a symmetric stable process on $\mathbb{R}$ with index $1 < \alpha \leq 2$. (Of course, Z is in weak duality to itself under m = Lebesgue measure.) For the remainder of this section we will consider this special case. It is not difficult to check using the scaling properties of Z and (4.1) that for $c > 0$ the law of the process $\{X_{c^\alpha t}(c\,\cdot)\}_{t \geq 0}$ under $\mathbb{P}^\mu$ coincides with the law of $\{X_t\}_{t\geq 0}$ under $\mathbb{P}^{\mu(c\,\cdot)}$ (cf. the proof of Proposition 5 in [7]). In particular, if $\mu = 1 \otimes \beta$, $\beta \in M_1$, then the laws of $\{X_{c^\alpha t}(c\,\cdot)\}_{t\geq 0}$ and $\{X_t\}_{t \geq 0}$ under $\mathbb{P}^\mu$ coincide.

The group of translations on $\mathbb{R}$ induces a group of shift maps $\{\tau_x\}_{x \in \mathbb{R}}$ on Ξ by $(\tau_x \mu)(e) = \mu(e + x)$. Suppose that $\mathcal{M}$ is a probability measure on Ξ that is stationary and ergodic with respect to this group of shifts. An argument using the above scaling relations and the L^2 ergodic theorem shows that as $c \to \infty$ the law of $\{X_{c^\alpha t}(c\,\cdot)\}_{t\geq 0}$ under $\mathbb{P}^{\mathcal{M}}$ converges to the law of $\{X_t\}_{t \geq 0}$ under $\mathbb{P}^{1 \otimes \beta}$, where $\beta \in M_1$ is defined by

$$\int \mathcal{M}(d\mu) I_1(\mu; \psi \otimes \chi) = I_1(1 \otimes \beta; \psi \otimes \chi)$$

for $\psi \in L^1(m)$ and $\chi \in C$ (cf. the proof of Theorem 6(i) in [7]).

In genetics terminology, the following Proposition (5.1) states that, at a given time, m-a.e. site has a population that is purely one of a countable set of types.

On the other hand, if $\mathcal{M}$ is as above and β is diffuse, then it follows from Lemma (5.2) below and the pointwise ergodic theorem that the sequence of random probability measures $\{N^{-1} \int_{[-N/2,N/2]} m(de)\, X_t(e)\}_{N \in \mathbb{N}}$ converges $\mathbb{P}^{\mathcal{M}}$-a.s. to β in the weak* topology on M_1, and so globally no particular type is present with positive density.

Proposition (5.1). *For $\mu \in \Xi$ and $t > 0$ fixed, $\mathbb{P}^\mu$-a.s. the probability measure $X_t(e)$ is a point mass for m-a.e. $e \in \mathbb{R}$. Moreover, $\mathbb{P}^\mu$-a.s. there exists a countable set $S \subset K$ such that for m-a.e. $e \in \mathbb{R}$, $X_t(e) = \delta_k$ for some $k \in S$.*

Proof. Consider the first claim. Let $\mathcal{G}^{(2)}$ be the countable field of subsets of K^2 generated by sets of the form $G_1 \times G_2$, where G_1 and G_2 are cylinder sets in K. For $G \in \mathcal{G}^{(2)}$ put

$\chi_G = 1_G \in C^{(2)}$, and define $\bar{\chi}_G \in C$ by $\bar{\chi}_G(k) = \chi_G(k,k)$. Observe that $\beta \in M_1$ is a point mass if and only if $\langle \beta \otimes \beta, \chi_G \rangle = \langle \beta, \bar{\chi}_G \rangle$ for all $G \in \mathcal{G}^{(2)}$.

Let $\mathcal{B}$ be a countable ring of sets of finite m-measure that generates the Borel σ-field on $\mathbb{R}$. For $B \in \mathcal{B}$ and $\epsilon > 0$, define $\psi_B^\epsilon \in L^1(m^{\otimes 2})$ by $\psi_B^\epsilon(x,y) = (2\epsilon)^{-1} 1\{|x-y| \le \epsilon\} 1_B(y)$.

Observe that

$$\lim_{z_1 \to z_2} P^{(z_1,z_2)}\{\exists\, 0 \le s \le t : Z_1(s) = Z_2(s)\} = 1.$$

It follows from (4.1) that

$$\begin{aligned}\lim_{\epsilon \downarrow 0} \mathbb{P}^\mu[I_2(X_t; \psi_B^\epsilon \otimes \chi_G)^2] &= \lim_{\epsilon \downarrow 0} \mathbb{P}^\mu[I_4(X_t; (\psi_B^\epsilon \otimes \chi_G)^{\otimes 2})] \\ &= \mathbb{P}^\mu[I_2(X_t; (1_B \otimes \bar{\chi}_G)^{\otimes 2})],\end{aligned}$$

$$\begin{aligned}\lim_{\epsilon \downarrow 0} \mathbb{P}^\mu[I_2(X_t; \psi_B^\epsilon \otimes \chi_G) I_1(X_t; 1_B \otimes \bar{\chi}_G)] &= \lim_{\epsilon \downarrow 0} \mathbb{P}^\mu[I_3(X_t; (\psi_B^\epsilon \otimes \chi_G) \otimes (1_B \otimes \bar{\chi}_G))] \\ &= \mathbb{P}^\mu[I_2(X_t; (1_B \otimes \bar{\chi}_G)^{\otimes 2})],\end{aligned}$$

and, of course,

$$\mathbb{P}^\mu[I_1(X_t; 1_B \otimes \bar{\chi}_G)^2] = \mathbb{P}^\mu[I_2(X_t; (1_B \otimes \bar{\chi}_G)^{\otimes 2})].$$

Thus,

$$\lim_{\epsilon \downarrow 0} \mathbb{P}^\mu[(I_2(X_t; \psi_B^\epsilon \otimes \chi_G) - I_1(X_t; 1_B \otimes \bar{\chi}_G))^2] = 0.$$

Note also that $\mathbb{P}^\mu$-a.s.

$$\lim_{\epsilon \downarrow 0} I_2(X_t; \psi_B^\epsilon \otimes \chi_G) = \int_B m(de)\, \langle X_t(e)^{\otimes 2}, \chi_G \rangle$$

by the Lebesgue differentiation theorem. By definition,

$$I_1(X_t; 1_B \otimes \bar{\chi}_G) = \int_B m(de)\, \langle X_t(e), \bar{\chi}_G \rangle.$$

Therefore, $\mathbb{P}^\mu$-a.s. for m-a.e. $e \in \mathbb{R}$ we have

$$\langle X_t(e) \otimes X_t(e), \chi_G \rangle = \langle X_t(e), \bar{\chi}_G \rangle,$$

for all $G \in \mathcal{G}^{(2)}$, as required.

Now consider the second claim. By the Pettis measurability theorem (see Theorem II.1.2 of [4]), there exists a (random) m-null set $N \subset \mathbb{R}$ such that the set $X_t(\mathbb{R} \backslash N)$ is separable in the norm (that is, total variation) topology on M. A set of point masses is separable in the total variation topology if and only if it is countable, and the result follows from the first claim.

□

Open Problem. It follows from Fubini's theorem that the behaviour described in Proposition (5.1) for a fixed time $t > 0$ occurs $\mathbb{P}^{\mu}$-a.s. at set of times $t > 0$ with full Lebesgue measure. It is natural to inquire if $\mathbb{P}^{\mu}$-a.s. the behaviour occurs at all times $t > 0$.

Lemma (5.2). *If $\mathcal{M}$ is a stationary, ergodic probability measure on Ξ, then so is $\mathcal{M}Q_t$ for each $t \geq 0$.*

Proof. Let $\{\sigma_x\}_{x\in\mathbb{R}}$ be the group of operators on $L^1(m^{\otimes n}, C^{(n)})$ defined by

$$(\sigma_x\phi)(e_1, \ldots, e_n) = (e_1 + x, \ldots, e_n + x).$$

It follows from (4.1) that

$$Q_t I_n(\,\cdot\,; \sigma_{-x}\phi) = Q_t(I_n(\,\cdot\,; \phi) \circ \tau_x) = (Q_t I_n(\,\cdot\,; \phi)) \circ \tau_x \tag{5.2}$$

for all $\phi \in L^1(m^{\otimes n}, C^{(n)})$. Thus, by Lemma (3.1) and a monotone class argument, $Q_t(F \circ \tau_x) = (Q_t F) \circ \tau_x$ for all bounded Borel functions F, and the stationarity of $\mathcal{M}Q_t$ follows from the stationarity of $\mathcal{M}$.

Turning to the ergodicity claim, we need to show for all functions $F, F' \in L^2(\mathcal{M}Q_t)$ that

$$\lim_{N\to\infty} N^{-1} \int_{[0,N]} m(dx)\, \mathcal{M}Q_t((F \circ \tau_x)F') = (\mathcal{M}Q_t F)(\mathcal{M}Q_t F').$$

As the continuous functions are dense in $L^2(\mathcal{M}Q_t)$, it suffices by Lemma (3.1) to take $F = I_n(\,\cdot\,; \phi)$ and $F' = I_{n'}(\,\cdot\,; \phi')$ for $\phi \in L^1(m^{\otimes n}, C^{(n)})$ and $\phi' \in L^1(m^{\otimes n'}, C^{(n')})$, $n, n' \in \mathbb{N}$.

Note that for each $z_2 \in \mathbb{R}$

$$\lim_{|z_1|\to\infty} P^{(z_1,z_2)}\{\exists\, 0 \leq s \leq t : Z_1(s) = Z_2(s)\} = 0. \tag{5.3}$$

As

$$Q_t((I_n(\,\cdot\,; \phi) \circ \tau_x) I_{n'}(\,\cdot\,; \phi')) = Q_t I_{n+n'}(\,\cdot\,; (\sigma_{-x}\phi) \otimes \phi'),$$

it follows from (4.1) and (5.3) that for each $\mu \in \Xi$

$$\lim_{x\to\infty} |(Q_t I_{n+n'}(\,\cdot\,; (\sigma_{-x}\phi) \otimes \phi'))(\mu) - (Q_t I_n(\,\cdot\,; \sigma_{-x}\phi))(\mu)(Q_t I_{n'}(\,\cdot\,; \phi'))(\mu)| = 0.$$

The result now follows from (5.2) and the ergodicity of $\mathcal{M}$.

$\square$

Appendix: Lusin spaces

Lemma (A.1). *Suppose that m is a Radon measure on a Lusin space E. There exists a bounded, continuous, strictly positive function f such that $\int m(de)\, f(e) < \infty$.*

Proof. By definition, each point $x \in E$ has an open neighbourhood V_x such that $m(V_x) < \infty$. As E is a separable metric space, Lindelöf's theorem implies that there exists a countable subcollection of $\{V_x\}_{x \in E}$, say $\{W_i\}_{i \in \mathbb{N}}$, such that $E = \bigcup_i W_i$. The function 1_{W_i} is lower semicontinuous and so there exists an increasing sequence of continuous, nonnegative functions $\{f_{ij}\}_{j \in \mathbb{N}}$ such that $1_{W_i} = \sup_j f_{ij}$ (cf. the remark at the beginning of §A2 of [11]). It suffices to take $f = \sum_{i,j} c_{ij} f_{ij}$, where $c_{ij} > 0$ and $\sum_{i,j} c_{ij}(1 \vee \int m(de)\, f_{ij}(e)) < \infty$.

$\square$

Lemma (A.2). *Suppose that m is a Radon measure on a Lusin space E. The continuous integrable functions are dense in $L^1(m)$.*

Proof. Let f be the function guaranteed by Lemma (A.1). Consider $g \in L^1(m)$. As $\lim_{n \to \infty} \|((g \wedge nf) \vee (-nf)) - g\|_1 = 0$, it suffices to prove that $(g \wedge nf) \vee (-nf)$ is the limit in $L^1(m)$ of a sequence of continuous, integrable functions for each $n \in \mathbb{N}$. Any bounded, Borel function is a limit in m-measure of a sequence of bounded, continuous functions (for example, by a monotone class theorem such as Theorem A0.6 of [11]). Let $\{h_k\}_{k \in \mathbb{N}}$ be a sequence of bounded, continuous functions that converges in m-measure to $(g \wedge nf) \vee (-nf)$. Then $\lim_k \|((h_k \wedge nf) \vee (-nf)) - ((g \wedge nf) \vee (-nf))\|_1 = 0$, as required.

$\square$

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