Partition structures derived from Brownian motion and stable subordinators

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Abstract

Explicit formulae are obtained for the distribution of various random partitions of a positive integer n, both ordered and unordered, derived from the zero set M of a Brownian motion by the following scheme: pick n points uniformly at random from $[0, 1]$, and classify them by whether they fall in the same or different component intervals of the complement of M. Corresponding results are obtained for M the range of a stable subordinator and for bridges defined by conditioning on $1 \in M$. These formulae are related to discrete renewal

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theory by a general method of discretizing a subordinator using the points of an independent homogeneous Poisson process.

Keywords: composition, excursion, local time, random set, renewal.

1 Introduction

A *partition of* n is an unordered collection of positive integers with sum n, usually coded by the vector of counts $(m_j, 1 \leq j \leq n)$, where m_j is the number of j's in the partition. The *number of components* of the partition is then $\sum m_j$, while $\sum j m_j = n$. A *random partition of* n, is a random variable π_n with values in the set of partitions of n. Kingman (1978) introduced the concept of a *partition structure*, that is a sequence $(\mathbb{P}_n,\ n = 1, 2, \ldots)$ of distributions for random partitions π_n of n, which is *consistent* in the following sense: if n objects are partitioned into subsets with sizes given by π_n, and an object is deleted uniformly at random, independently of π_n, the partition of the $n-1$ remaining objects has component sizes distributed according to $\mathbb{P}_{n-1}$. Kingman (1982) and Aldous (1985) interpreted this concept in terms of an *exchangeable random partition* of the set of positive integers $\mathbb{N}$, whose restriction Π_n to the set $\mathbb{N}_n$ of integers $\{1, \ldots, n\}$ has the following property: given π_n, the induced partition of n, Π_n is uniformly distributed over all partitions of the set $\mathbb{N}_n$ with component sizes dictated by π_n. For π_n with counts $(m_j, 1 \leq j \leq n)$, the number of such partitions of $\mathbb{N}_n$ is

$$N(m_1, \ldots, m_n) := \frac{n!}{\prod_{j=1}^{n} (j!)^{m_j} m_j!}. \tag{1}$$

Let M be a random closed subset of $[0,1]$, for example the zero set of a random process $B = (B_t, 0 \leq t \leq 1)$ with continuous paths, when the interval components of M^c, the open complement of M in $[0,1]$, will be called *excursion intervals*. Note that this allows a final *meander interval* of the form $(G_1, 1]$, where G_1 is the last zero of B before time 1, to be included among the excursion intervals. Let U_1, $U_2, \ldots$ be a sequence of i.i.d. uniform $[0,1]$ random variables, independent of M. Define a random equivalence relation $\sim$ on $\mathbb{N}$ by $i \sim j$ iff $i = j$ or U_i and U_j fall in the same interval component of M^c. The collection of $\sim$-equivalence classes is then an exchangeable random partition of $\mathbb{N}$. To paraphrase Kingman's (1982) representation theorem:

every partition structure can be associated with an exchangeable random partition of $\mathbb{N}$ obtained by this construction from some random closed subset M of $[0,1]$. See Aldous (1985) for an elegant proof.

This paper presents explicit formulae for various probabilities associated with random partitions induced by the zero set of Brownian motion and Brownian bridge. The portion of results regarding partition structures can be read from recent work by Perman *et al.* (1992) and Pitman (1995). But this paper goes further to investigate features of the time ordering of components of the various random partitions, which involves more than just the partition structure. Formulae for the partition structure are then recovered by appropriate summations over *compositions of* n, that is to say ordered partitions of n. Preliminary forms of some of the results involving compositions appear in work of Pitman and Yor (1992) and Aldous and Pitman (1994). Gnedin (1996) develops the notion of a *composition structure*, and establishes a representation of such structures in terms of a random closed set M as above.

Let $(B_t, t \geq 0)$ with $B_0 = 0$ be a reflecting Brownian motion on $[0, \infty)$, or more generally a recurrent Bessel process of dimension δ where $0 < \delta < 2$. See Revuz and Yor (1994) for background. Let $\alpha = (2-\delta)/2$. It is known (Molchanov and Ostrovski 1969) that the zero set of B is the the closure of the range of a stable(α) subordinator inverse to the local time process of B at zero. In the Brownian case ($\delta = 1, \alpha = 1/2$), this result goes back to Lévy (1939). The structure of the zero set of B for general α with $0 < \alpha < 1$ plays a fundamental role in distributional limit theorems in renewal theory (Dynkin 1961, Lamperti 1962).

Suppose B is defined on a probability space $(\Omega, \mathcal{F}, P)$. Let P_0 defined on the same space govern $(B_t, 0 \leq t \leq 1)$ as a Bessel bridge:

$$P_0(\,\cdot\,) := P(\,\cdot\,|B_1 = 0).$$

For a real number x, let $[x]_0 := 1$, $[x]_n := x(x+1)\ldots(x+n-1)$, $n = 1, 2, \ldots$.

Proposition 1 *Fix n. For $1 \leq j \leq n$ let M_j be the number of excursion intervals of B that contain exactly j of n points $U_1, \ldots, U_n$, assumed independent of B and uniformly distributed on $[0,1]$. Let $(m_j, 1 \leq j \leq n)$ be a count vector with $\sum_j jm_j = n$ and $\sum_j m_j = k$. Then for B the Bessel process* *of dimension* $2-2\alpha$

$$P(M_j = m_j, 1 \le j \le n) = N(m_1, \ldots, m_n)\frac{(k-1)!\alpha^{k-1}}{(n-1)!}\prod_j([1-\alpha]_{j-1})^{m_j} \quad (2)$$

while for B *the Bessel bridge of dimension* $2-2\alpha$

$$P_0(M_j = m_j, 1 \le j \le n) = N(m_1, \ldots, m_n)\frac{k!\alpha^k}{[\alpha]_n}\prod_j([1-\alpha]_{j-1})^{m_j} \quad (3)$$

A remarkable fact emerges from this calculation which does not seem at all obvious intuitively:

Corollary 2 *Let* $K_n := \sum_j M_j$*, the number of components of the random partition of* n*. For every* $1 \le k \le n$*, the conditional distribution of the random partition of* n *given* $(K_n = k)$ *is the same for* B *a Bessel process as for* B *a Bessel bridge of the same dimension.*

Expressions for the exact distribution of K_n in the two cases can be obtained by summing the above formulae over all partitions of n into k components. Alternatively, it follows from the results presented here that in both cases (K_n) is a Markov chain with simple inhomogeneous transition probabilities, and the distribution of K_n can be described by a recursion using the forwards equations. Only in the Brownian case $\alpha = 1/2$ is there much simplification:

Corollary 3 *For the partition structure derived from the zeros of Brownian motion, for* $1 \le k \le n$*,*

$$P(K_n = k) = \binom{2n-k-1}{n-1}2^{k+1-2n} \quad (4)$$

whereas for Brownian bridge

$$P_0(K_n = k) = \frac{k(n-1)!}{[\frac{3}{2}]_{n-1}}\binom{2n-k-1}{n-1}2^{k+1-2n} \quad (5)$$

Comparison of Corollary 3 and Exercise III.10.10 of Feller (1968) shows that K_n for Brownian motion has the same distribution as $\tilde{K}_n$ defined to be the number of visits to the origin strictly before time $2n$ (counting the visit at time 0) for a simple symmetric random walk on the integers. Similarly, K_n for Brownian bridge has the same distribution as $\tilde{K}_n$ given that the walk returns to zero at time $2n$. In Section 2 these coincidences are explained to some extent by an interpretation in terms of discrete renewal theory of the random partitions of n generated by a Bessel process or bridge.

The asymptotic behavior of K_n for large n involves the local time of B at zero up to time 1, that is the random variable S defined by the formula

$$S := , (1-\alpha)^{-1} \lim_{\epsilon \to 0} \epsilon^{\alpha} \#\{i : P_{(i)} > \epsilon\} \text{ a.s.} \tag{6}$$

both for the Bessel process and Bessel bridge of dimension $2-2\alpha$, where $P_{(i)}$ is the length of the ith longest component interval of M^c.

Proposition 4 *Both for the Bessel process and the Bessel bridge of dimension $2-2\alpha$,*

$$\lim_{n\to\infty} \frac{K_n}{n^{\alpha}} = S \ \textit{a.s.} \tag{7}$$

It is known (Molchanov and Ostrovski 1969) that the P distribution of S is the Mittag Leffler distribution with moments $E(S^p) = , (p+1)/, (p\alpha+1)$, $p > -1$, and that $P_0(S \in ds) = , (\alpha+1) s P(S \in ds)$. (In the Brownian case $\alpha = 1/2$ this is not the usual normalization of local time. Rather $S = \sqrt{2}L$, where the P distribution of L is that of the absolute value of a standard normal variable, and the P_0 distribution of L is Rayleigh.)

The asymptotic behavior of the sizes of the large components of the partition of n is dictated by the law of large numbers: if $N_{(i)n}$ is the size of the ith largest component in the partition of n derived from an arbitrary random closed subset M of $[0,1]$ as considered earlier, then

$$\lim_{n\to\infty} \frac{N_{(i)n}}{n} = P_{(i)} \text{ a.s.} \tag{8}$$

For the Bessel process or Bessel bridge, it is known (Kingman 1975, Pitman and Yor 1995) that

$$\lim_{i\to\infty} i^{1/\alpha} P_{(i)} = (S/, (1-\alpha))^{1/\alpha} \text{ a.s.} \tag{9}$$

Proposition 4 follows from (9) by conditioning on $(P_{(1)}, P_{(2)}, \ldots)$ and applying results of Karlin (1967) .

Since Corollary 2 amounts to the fact that the distribution of the random vector $(N_{(i)n}, i = 1, \ldots, k)$ given $K_n = k$ is the same for the Bessel bridge and the Bessel process, that corollary is the exact discrete analog of the next one, which follows from it via Proposition 4.

Corollary 5 (Pitman and Yor 1992) *The conditional joint distribution of the ranked excursion interval lengths $P_{(i)}$ given S, the local time at 0 up to time 1, is the same for the Bessel process as for the Bessel bridge of the same dimension.*

Features of the joint distribution of the ranked excursion lengths $P_{(i)}$ derived from a Bessel process or Bessel bridge have been studied by a number of authors. See Pitman and Yor (1995) for a recent survey. This distribution is more difficult to describe explicitly than the closely related Poisson-Dirichlet distribution, for which see Kingman (1975), Watterson (1976).

For $(P_{(1)}, P_{(2)}, \ldots)$ with the Poisson-Dirichlet distribution with parameter $\theta > 0$, there is the following simpler description of a size-biased random permutation $P_1, P_2, \ldots$ of the ranked lengths $P_{(1)}, P_{(2)}, \ldots$:

$$P_n = (1 - W_1) \cdots (1 - W_{n-1}) W_n \qquad (n = 1, 2, \ldots) \tag{10}$$

where $W_1, W_2, \ldots$ are i.i.d with beta$(1, \theta)$ distribution. This distribution of $(P_1, P_2, \ldots)$ is what Ewens (1988) calls the GEM model, after Griffiths, Engen and McCloskey. See also Hoppe (1986, 1987), Donnelly (1986), Donnelly and Joyce (1989). In the present framework, a size-biased random permutation of the $P_{(i)}$ derived from excursion intervals can be constructed as follows: let P_1 be the length of I_1, the excursion interval containing U_1; inductively, let P_{j+1} be the length of the excursion interval I_{j+1} containing the first U_i that does not fall in $I_1 \cup \cdots \cup I_j$. The analog of the GEM description in the Bessel set up is provided by the following result:

Proposition 6 (Perman *et al.* 1992) *Fix α with $0 < \alpha < 1$. For a real number q with $q > -1$, let P^q be the probability with density proportional to S^q relative to the probability P that makes B a Bessel process of dimension $2 - 2\alpha$, where S is the local time of B at 0 up to time 1. And for $q > -2$ let P_0^q be derived similarly from the corresponding Bessel bridge law P_0. For* *each $\theta > -\alpha$, the joint law of the ranked interval lengths $P_{(1)}, P_{(2)}, \dots$ is the same under $P^{\frac{\theta}{\alpha}}$ as under $P_0^{\frac{\theta}{\alpha}-1}$. Under either of these laws, the size-biased random permutation $P_1, P_2, \dots$ of the interval lengths admits the description* (10) *for independent W_n with* $\mathrm{beta}(1-\alpha, \theta+n\alpha)$ *distributions.*

Note that by letting $\alpha \to 0$ for fixed $\theta > 0$, the joint law of the size-biased permutation $P_1, P_2, \dots$ for the (α, θ) model in Proposition 6 converges to the joint law for the GEM model. It is shown in Pitman (1995) that this construction yields a family of random partition structures, indexed by two parameters α and θ, with an explicit sampling formula that reduces to formulae (2) and (3) in the cases $0 < \alpha < 1$, $\theta = 0$ and $\theta = \alpha$, and to the Ewens sampling formula in case $\alpha = 0$, $\theta > 0$. See also Pitman (1996) and Kerov (1995) .

While Proposition 1 can be derived from this analysis of the size-biased random permutation of the interval lengths generated by the Bessel process or Bessel bridge, this argument ignores interesting features of the time ordering of intervals. The approach taken here is to derive Proposition 1 by analysis of the composition of n induced by the time ordering of the intervals. This brings out the connections with renewal theory mentioned earlier which do not seem to generalize to the two-parameter set up.

2 Compositions

With the set up for Proposition 1, fix n and let

$$U_{(1)} < U_{(2)} < \cdots < U_{(n)} \tag{11}$$

denote the order statistics of n independent uniform $[0,1]$ variables $U_1, \cdots, U_n$, called the *sample points*, assumed independent of B under both P and P_0. In the notation of Proposition 1, let $K_n = \sum_j M_j$. That is to say, K_n is the number of distinct excursion intervals of B discovered by the n sample points. Given $K_n = k$, define N_j for $1 \le j \le k$ to be the number of sample points that fall in the jth of these k excursion intervals, where the excursion intervals are ordered by their starting times. Note that by definition, $N_j \ge 1$ for $1 \le j \le k$, and $\sum_{j=1}^k N_j = n$. This section describes the distribution of the composition of n defined by the random sequence $(N_1, \dots, N_{K_n})$ of random length K_n, both for the Bessel process and the Bessel bridge. Proposition

1 is then deduced by summing probabilities from this distribution over all compositions corresponding to a given partition of n.

Define a sequence of $n-1$ indicator variables $Z_{ni}, 1 \le i \le n-1$, in terms of B and the first n sample points, as follows:

$$Z_{ni} := 1\{B_t = 0 \text{ for some } U_{(i)} < t < U_{(i+1)}\}. \tag{12}$$

Since Z_{ni} is also the indicator of the event that $N_1 + \cdots + N_j = i$ for some $1 \le j < K_n$, the random sequence $(Z_{ni}, 1 \le i \le n-1)$ is just a recoding of $(N_j, 1 \le j \le K_n)$. It is convenient to set

$$Z_{n0} := 1\{B_t = 0 \text{ for some } 0 < t < U_{(1)}\}. \tag{13}$$

Then $Z_{n0} = 1$, and $K_n = \sum_{i=0}^{n-1} Z_{ni}$, both P and P_0 a.s.. Also, let

$$Z_{nn} := 1\{B_t = 0 \text{ for some } U_{(n)} < t < 1\}. \tag{14}$$

Then $Z_{nn} = 1$ P_0 a.s., but

$$P(Z_{nn} = 1) = P(U_{(n)} < G(1)) = E[G(1)^n] = [\alpha]_n/n! \tag{15}$$

where $G(1)$ is the last zero of B before time 1, and the last equality is obtained from the distribution of $G(1)$, which is known (Dynkin 1961) to be beta$(\alpha, 1-\alpha)$. Unlike Z_{ni} for $i < n$, the indicator variable Z_{nn} is not determined by the composition of n defined by $(N_j, 1 \le j \le K_n)$. Rather, Z_{nn} indicates which of the following two cases obtains: either the last N_{K_n} sample points fall in a complete excursion interval (when $Z_{nn} = 1$, call it the *final complete excursion case*) or the last N_{K_n} sample points fall in the meander interval $(G(1), 1]$ (when $Z_{nn} = 0$, call it the *final meander case*).

The joint law of the $n+1$ indicator variables $Z_{n0}, Z_{n1}, \ldots, Z_{nn}$ will now be obtained using the standard representation of uniform order statistics in terms of a Poisson process. Let $\tau_0 = 0$, $\tau_n = \eta_1 + \cdots + \eta_n$ where $\eta_1, \eta_2, \ldots$ is a sequence of independent exponential variables with mean 1, supposed defined on the same probability space $(\Omega, \mathcal{F}, P)$ as the Bessel process B, and independent of B. For $i = 0, 1, \ldots$ let

$$Z_i = 1\{B_t = 0 \text{ for some } \tau_i < t < \tau_{i+1}\}. \tag{16}$$

Lemma 7 *Under P governing B as a Bessel process of dimension $2-2\alpha$, the joint distribution of the indicators $Z_{n0}, Z_{n1}, \ldots, Z_{nn}$ defined by* (12)–(14) *is identical to the joint distribution of indicators $Z_0, Z_1, \ldots, Z_n$ defined by* (16)*. These Z_i are renewal indicators:*

$$Z_i = 1(S_m = i \text{ for some } m = 0, 1, 2, \ldots) \tag{17}$$

where $S_0 = 0$, $S_m = X_1 + \cdots + X_m$ for a sequence of i.i.d. positive integer valued r.v.'s X_i with distribution

$$P(X_i = k) = (-1)^{k-1}\binom{\alpha}{k} = \frac{\alpha[1-\alpha]_{k-1}}{k!} \tag{18}$$

Under P_0 governing B as a Bessel bridge of dimension $2-2\alpha$, the joint distribution of $Z_{n0}, Z_{n1}, \ldots, Z_{nn}$ is identical to the P conditional joint distribution of $Z_0, Z_1, \ldots, Z_n$ given $Z_n = 1$.

Proof. The first sentence is immediate from the stability of the zero set of B under scaling, and the standard fact that the joint law of $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ is identical to that of $\tau_1/\tau_{n+1},\ \tau_2/\tau_{n+1}, \ldots,\ \tau_n/\tau_{n+1}$. The second sentence is due to the strong Markov property of the Bessel process at the right end of each excursion interval that contains at least one of the times τ_i, since X_j is just the number of τ_i that fall in the jth such excursion interval. The formula

$$P(Z_n = 1) = \frac{[\alpha]_n}{n!} = (-1)^n\binom{-\alpha}{n} \tag{19}$$

follows from (15), and yields the generating function

$$\sum_{n=0}^{\infty} P(Z_n = 1)x^n = (1-x)^{-\alpha} \tag{20}$$

Let $F(x) = \sum_{k=1}^{\infty} P(X_i = k)x^k$. The standard formula of discrete renewal theory (Feller 1968 XIII.3)

$$\sum_{n=0}^{\infty} P(Z_n = 1)x^n = (1-F(x))^{-1} \tag{21}$$

implies

$$F(x) = 1 - (1-x)^{\alpha} \tag{22}$$

which amounts to (18). Finally, the statement for the Bessel bridge is implied by (i) and (ii) below, which are consequences of the fact that if $\mathcal{G}$ is a σ-field independent of B and τ is a positive $\mathcal{G}$-measurable random variable, then the process $(B(uG(\tau))/\sqrt{G(\tau)}, 0 \leq u \leq 1)$ is a Bessel bridge independent of the σ-field generated by $\mathcal{G}$ and $G(\tau)$. See Revuz and Yor (1994, Exercise (3.8) of Ch. XII).

(i) Under P given $Z_n = 1$, that is to say, given $\tau_n < G(\tau_{n+1})$ where $G(\tau_{n+1})$ is the last zero of B before τ_{n+1}, the process $(B(uG(\tau_{n+1}))/\sqrt{G(\tau_{n+1})}, 0 \leq u \leq 1)$ is a Bessel bridge.

(ii) Under P given $Z_n = 1$, this bridge is independent of the $\tau_i/G(\tau_{n+1})$ for $1 \leq i \leq n$, which are jointly distributed like the $U_{(i)}, 1 \leq i \leq n$. $\square$

The generating function (20) appears in Feller (1971, XII, (8.11)), in connection with the renewal process of ladder epochs of a real valued random walk $\tilde{S}_n$ with $P(\tilde{S}_n > 0) = \alpha$ for all n. See also Theorem 4 of Section XII.7 of Feller (1971). It follows that the distribution of X_i displayed in (18) with probability generating function (22) is identical to the distribution of $\min\{n : \tilde{S}_n > 0\}$. For $\alpha = 1/2$ this is also the distribution of half the return time to zero for a simple symmetric random walk on the integers (Feller 1968, XIII, (4.4)). This distribution (18) on $\{1, 2, \ldots\}$ with parameter $0 \leq \alpha \leq 1$ has appeared in other contexts (Mandelbrot 1956, Pillai and Jayakumar 1995). Multiplying this distribution by a positive parameter λ gives the the family of Lévy measures corresponding to the two-parameter family of discrete stable distributions on $\{0, 1, 2, \ldots\}$ characterized by Steutel and Van Harn (1979, Theorem 3.2). It is easily seen that the probability distribution (18) on $\{1, 2, \ldots\}$ is uniquely characterized by the property that the discrete hazard probabilities are of the form

$$P(X = k | X \geq k) = \alpha/k, \quad k = 1, 2, \ldots,$$

for some constant α. That is

$$P(X = k) = \frac{\alpha}{k} P(X \geq k), \quad k = 1, 2, \ldots, \tag{23}$$

which is a key formula in the following calculations. (The case $\alpha = 1$ is degenerate: $P(X = 1) = 1$). Formula (23) is the discrete analogue of the fact, exploited in Pitman and Yor (1992), that the stable (α) Lévy measure

Λ is characterized among all Lévy measures on $(0,\infty)$ by the identity

$$\frac{\Lambda(dt)}{dt} = \frac{\alpha}{t}\Lambda(t,\infty), \quad t>0. \tag{24}$$

In the renewal set up let $J_n = \sum_{i=0}^{n-1} Z_i$, representing the number of renewals in $\{0,\dots,n-1\}$. And define the age variable A_n by

$$A_n := n - \max\{k : k<n, Z_k = 1\} = n - (X_1 + X_2 + \cdots + X_{J_n - 1}). \tag{25}$$

Proposition 8 *Let K_n be the number of distinct excursion intervals of B that contain at least one of the n sample points picked uniformly at random on $[0,1]$, independently of B. Given $K_n = k$, let N_j for $1 \le j \le k$ be the number of sample points that fall in the jth of these k excursion intervals, where the excursion intervals are ordered by their starting times. Let Z_{nn} as in (14) be the indicator of the event that the last such excursion interval is a complete excursion interval (not the meander interval). For the Bessel process of dimension $2-2\alpha$, the joint law of*

$$(K_n, N_1, \dots, N_{K_n - 1}, N_{K_n}, Z_{nn}) \tag{26}$$

is identical to the joint law of

$$(J_n, X_1, \dots, X_{J_n - 1}, A_n, Z_n) \tag{27}$$

derived from the renewal process in Lemma 7. That is to say,

$$P(K_n = k, N_i = n_i, 1 \le i \le k) = n_k \alpha^{k-1} \prod_{i=1}^{k} \frac{[1-\alpha]_{n_i - 1}}{n_i!} \tag{28}$$

$$P(Z_{nn} = 1 \mid K_n = k, N_i = n_i, 1 \le i \le k) = \frac{\alpha}{n_k}. \tag{29}$$

For the Bessel bridge of dimension $2-2\alpha$, the joint law of $(K_n, N_1, \dots, N_{K_n})$ is identical to the conditional joint law of $(J_n, X_1, \dots, X_{J_n})$ given a renewal at time n:

$$P_0(K_n = k, N_i = n_i, 1 \le i \le k) = \frac{n!}{[\alpha]_n} \alpha^k \prod_{i=1}^{k} \frac{[1-\alpha]_{n_i - 1}}{n_i!}. \tag{30}$$

Proof. This follows immediately from Lemma 7, (19) and (23). $\square$

Proposition 1 can now be deduced from Proposition 8, using the following elementary lemma:

Lemma 9 *Suppose that $(N_1, \ldots, N_{K_n})$ is a sequence of positive integer random variables of random length K_n, with $N_1 + \cdots + N_{K_n} = n$, such that for each $k = 1, \ldots, n$ with $P(K_n = k) > 0$, the conditional distribution of $(N_1, \ldots, N_k)$ given $(K_n = k)$ is exchangeable. That is, for all $k = 2, \ldots, n$ and $n_i \geq 1$ with $\sum_{i=1}^k n_i = n$,*

$$P(K_n = k, N_i = n_i, 1 \leq i \leq k) = s(n_1, \ldots, n_k)$$

for some symmetric function $s(n_1, \ldots, n_k)$. Let $M_j = \#\{i : N_i = j\}$. Then

$$P(M_j = m_j, 1 \leq j \leq n) = \frac{k!}{\prod_j m_j!} \tilde{s}(m_1, \ldots, m_n)$$

where $k = \sum_j m_j$ and $\tilde{s}(m_1, \ldots, m_n)$ is the common value of $s(n_1, \ldots, n_k)$ for every sequence $(n_1, \ldots, n_k)$ with $\#\{i : n_i = j\} = m_j$.

Proof of Proposition 1. Formula (3) in the bridge case follows by application of the lemma to the P_0 distribution of $(N_1, \ldots, N_{K_n})$ displayed in (30). Note that (30) can be rewritten

$$P_0(K_n = k, N_i = n_i, 1 \leq i \leq k) = P(X_i = n_i, 1 \leq i \leq k) \frac{n!}{[\alpha]_n} \tag{31}$$

where the X_i are i.i.d. with common distribution as in (18). The derivation of the formula (2) for the unconditioned Bessel process is complicated by the fact that $(N_1, \ldots, N_{K_n})$ is not exchangeable in this case. Instead of (31), from (28) and (18),

$$P(K_n = k, N_i = n_i, 1 \leq i \leq k) = P(X_i = n_i, 1 \leq i \leq k) \frac{n_k}{\alpha}. \tag{32}$$

To obtain $P(M_j = m_j, 1 \leq j \leq n)$, this asymmetric function of $(n_1, \ldots, n_k)$ must be added over the $k!/\prod_j m_j!$ sequences $(n_1, \ldots, n_k)$ with the prescribed frequencies $(m_1, \ldots, m_n)$. The key to this calculation is the well known fact that if $X_1, \ldots, X_k$ are k exchangeable random variables, and A is an event in the exchangeable σ-field of $X_1, \ldots, X_k$, then

$$E(X_k | X_1 + \cdots + X_k = n, A) = \frac{n}{k}. \tag{33}$$

Applied to the i.i.d. sequence $X_1, X_2, \ldots$ at hand, this shows that

$$\sum n_k P(X_i = n_i, 1 \leq i \leq k) = \frac{n}{k} P(\#\{i : X_i = j\} = m_j, 1 \leq j \leq n),$$

where the sum is over all sequences $(n_1, \ldots, n_k)$ with the prescribed frequencies $(m_1, \ldots, m_n)$. In view of (31), this allows summation of the terms in (32) to yield

$$P(M_j = m_j, 1 \leq j \leq n) = \frac{1}{\alpha} \frac{n}{k} \frac{[\alpha]_n}{n!} P_0(M_j = m_j, 1 \leq j \leq n). \tag{34}$$

Now (2) follows from (3). □

The nature of the special contribution of the last count N_{K_n} to the partition of n for the unconditioned Bessel process is clarified by Proposition 11 below. This proposition follows at once from Proposition 8 and the next lemma. Part (i) of the lemma is an elementary discrete analog of a corresponding result for subordinators (Pitman and Yor 1992, Th. 7.1). Part (ii) can be formulated in that setting in terms of exchangeable increments.

Lemma 10 *Let $(\tilde{N}_1, \ldots, \tilde{N}_{J_n}) = (X_1, \ldots, X_{J_n - 1}, A_n)$ be the random composition of n derived from an i.i.d. sequence of positive integer valued random variables X_i, with $J_n = \min\{k : X_1 + \cdots + X_k \geq n\}$,*

$$A_n = n - (X_1 + X_2 + \ldots + X_{J_n - 1}).$$

For $1 \leq j \leq n$ let $\tilde{M}_j = \#\{i : 1 \leq i \leq J_n, \tilde{N}_i = j\}$. Then

(i)

$$P(A_n = a | \tilde{M}_j = m_j, 1 \leq j \leq n) = \frac{h(a) m_a}{\sum_{j=1}^n h(j) m_j} \tag{35}$$

where $h(j) := P(X_i \geq j)/P(X_i = j)$, $j = 1, 2, \ldots$, and it is assumed that $P(X_i = j) > 0$ for all j.

(ii) *Conditionally given* ($\tilde{M}_j = m_j, 1 \le j \le n$, *and* $A_n = j$), *with* $\sum_j m_j = k$, $(X_1, \ldots, X_{k-1})$ *has the exchangeable joint distribution of a uniformly distributed random permutation of* m_i^- *values equal to* i, $i = 1, \ldots, n$, *where* $m_i^- := \begin{cases} m_i \text{ for } i \ne j \\ m_j - 1 \text{ for } i = j \end{cases}$

In particular, the lemma shows that A_n is an unbiased pick from the given counts, for every n and all possible counts, iff $h(j)$ is constant, that is to say the common X distribution is geometric (p) for some p. And A_n is a size-biased pick from the given counts iff the common X distribution is as in (18) for some $0 < \alpha < 1$.

Proposition 11 *Let* $(N_1, \ldots, N_{K_n})$ *be derived from the Bessel process of dimension* $2 - 2\alpha$ *as in Proposition 8.*

(i) *conditionally given the partition of* n, *the last count* N_{K_n} *is a size-biased choice from the unordered set of integers with sum* n:

$$P(N_{K_n} = a | M_j = m_j, 1 \le j \le n) = a m_a / n, \quad 1 \le a \le n$$

(ii) *conditionally given the partition of* n *and* N_{K_n} *with* $K_n = k$, *the joint distribution of* $(N_1, \ldots, N_{k-1})$ *is exchangeable.*

Let $P_{(1)} \ge P_{(2)} \ge \ldots$ denote the ranked lengths of all the excursion intervals of the unconditioned Bessel process $(B_t, 0 \le t \le 1)$. The meander length $\mu := 1 - G(1)$ appears as one of these lengths, while all other lengths correspond to complete excursion intervals contained in $(0, 1)$. Pitman and Yor (1992) showed that for each $i = 1, 2, \ldots$

$$P(\mu = P_{(i)} | P_{(1)}, P_{(2)}, \ldots) = P_{(i)}. \tag{36}$$

That is to say, given all the excursion lengths including the meander length, the meander length is picked by size-biased sampling. See also Pitman-Yor (1996) for a generalization of this result. Part (i) of Proposition 11 is a discrete analog of (36), with a partition of n, corresponding to ranked sequence of positive integers with sum n, instead of a ranked sequence of positive real numbers with sum 1. The last count in the discrete scheme is only indirectly related to the meander length μ. If $U_{(n)} > G(1)$, the last count N_{K_n} gives the number of uniform order statistics to fall in the meander interval, but if $U_{(n)} < G(1)$, then N_{K_n} gives the number of order statistics to fall in some complete excursion before the meander interval. So it does not seem possible to derive the discrete result from its continuous analog (36). However, (36) can be deduced from the discrete result by a limiting argument, using the fact that $P(U_{(n)} > G(1)) \to 1$ as $n \to \infty$.

3 Discretization of a Subordinator

This section presents some general calculations for a subordinator, which, in the case of a stable subordinator of index α, are relevant to the study of the various random partitions induced by the zero set of the Bessel process of dimension $2 - 2\alpha$.

Let $(T_s, s \geq 0)$ be a subordinator, that is an increasing process with stationary independent increments. See Fristedt (1974) for background. Assume for simplicity that T has no drift component, and Lévy measure Λ with infinite total mass. So

$$E[\exp(-\lambda T_s)] = \exp[-s\Psi(\lambda)] \tag{37}$$

where

$$\Psi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\Lambda(dt). \tag{38}$$

In the stable (α) case with $0 < \alpha < 1$, the Laplace exponent is $\Psi(\lambda) = c\lambda^\alpha$ for some constant $c > 0$, which corresponds to the Lévy measure

$$\Lambda(dt) = \frac{c\alpha}{\Gamma(1-\alpha)}\, t^{-\alpha-1}dt \qquad (t > 0). \tag{39}$$

Fix $\lambda > 0$. Let $\tau_0 = 0$, and let $\tau_1, \tau_2, \ldots$ be the points of a $PP(\lambda)$, that is a Poisson process with rate λ on $(0, \infty)$, assumed independent of (T_s). Define a sequence of indicator random variables $Z_0, Z_1, \ldots$ by

$$Z_n = 1(T_s \in [\tau_n, \tau_{n+1}) \text{ for some } s \geq 0). \tag{40}$$

That is, $Z_n = 1$ if the regenerative random set of $(0, \infty)$ defined by the range of (T_s) has at least one point between times τ_n and τ_{n+1}, and $Z_n = 0$ if not. Note that $Z_0 = 1$.

Proposition 12 *The sequence $(Z_0, Z_1, Z_2, \ldots)$ is a discrete renewal process:*

$$Z_n = 1(S_m = n \text{ for some } m = 0, 1, 2, \ldots) \tag{41}$$

where $S_0 = 0$, $S_m = X_1 + \ldots + X_m$ for a sequence of i.i.d. positive integer valued random variables $X_1, X_2, \ldots$ with distribution

$$P(X_i = k) = (-1)^{k-1} \frac{\lambda^k}{k!} \frac{\Psi^{(k)}(\lambda)}{\Psi(\lambda)} \qquad (k = 1, 2, \ldots) \tag{42}$$

where

$$\Psi^{(k)}(\lambda) = (-1)^{k-1} \int_0^\infty t^k e^{-\lambda t} \Lambda(dt) \tag{43}$$

is the kth derivative at λ of the Laplace exponent $\Psi(\lambda)$. Consequently,

$$\sum_{k=1}^{\infty} z^k P(X_i = k) = 1 - \frac{\Psi(\lambda(1-z))}{\Psi(\lambda)} \tag{44}$$

$$\sum_{n=0}^{\infty} z^n P(Z_n = 1) = \frac{\Psi(\lambda)}{\Psi(\lambda(1-z))}. \tag{45}$$

Proof. This follows by the method of creating a big Poisson point process by marking the Poisson point process of jumps of the subordinator by the times, measured from the left end of each jump, of any points of the $PP(\lambda)$ that appear in that jump interval. See Greenwood and Pitman (1980), or Rogers and Williams (1987,VI.49) for details of this construction. Define X_k to be the number of points of the $PP(\lambda)$ that appear in the kth jump interval of (T_s) that contains at least one point of the $PP(\lambda)$. Then on the one hand formula (41) for Z_n in terms $X_1, X_2, \ldots$ is true a.s. because the assumptions on T imply that every point of the $PP(\lambda)$ falls a.s. in some jump interval of T. On the other hand, the X_i are i.i.d. with the stated distribution, due to standard facts about Poisson processes (see Kingman 1993). The formula (44) for the generating function of the X_i follows easily, and yields (45) via (21). □

Let $H_1, H_2, \ldots$ denote the successive jump lengths of the subordinator that are hit by the $PP(\lambda)$. So X_k is the number of points of the $PP(\lambda)$ in the interval of length H_k in the complement of the range of $(T_s, s \geq 0)$. Let $G_1, G_2, \ldots$ denote the lengths of the successive sub-intervals of $(0, \infty)$ that remain when all these jump intervals are deleted. So $(0, \infty)$ is partitioned into consecutive intervals of lengths.

$$G_1, H_1, G_2, H_2, \ldots$$

such that the range of T is confined to the union of the G-intervals, and the points of the $PP(\lambda)$ all appear in the union of the H-intervals. The proof of Proposition 12 is easily developed further to establish the following:

Corollary 13 *The sequence $(G_1, G_2, \ldots)$ is i.i.d., as is the sequence of pairs $((H_1, X_1), (H_2, X_2), \ldots)$. The G-sequence is independent of the (H, X)-sequence, with each G_i distributed according to an infinitely divisible law with Laplace transform*

$$E(e^{-\eta G_i}) = \frac{\Psi(\lambda)}{\Psi(\lambda+\eta)} \qquad (\eta > 0). \tag{46}$$

Conditionally given all the X_i, the H_i are conditionally independent, with

$$P(H_i \in dt | X_i = k) = \frac{t^k e^{-\lambda t}\Lambda(dt)}{(-1)^{k-1}\Psi^{(k)}(\lambda)} \tag{47}$$

while the unconditional distribution of the H_i is

$$P(H_i \in dt) = \frac{(1-e^{-\lambda t})\Lambda(dt)}{\Psi(\lambda)}.$$

In the stable (α) case, simple calculations show that

$$G_i \sim \text{Gamma } (\alpha, \lambda), \tag{48}$$

that is to say

$$P(G_i \in dt) = , (\alpha)^{-1}\lambda^{\alpha} t^{\alpha-1} e^{-\lambda t} dt \qquad (t > 0).$$

The distribution of the X_i is given by (18), and for $k = 1, 2, \ldots$

$$(H_i | X_i = k) \sim \text{Gamma } (k-\alpha, \lambda). \tag{49}$$

Suppose the subordinator $(T_s, s \geq 0)$ is the process inverse to the local time process $(S_t, t \geq 0)$ associated with a point 0 in the state space of a Markov process B starting at $B_0 = 0$. For example, B could be a Brownian motion on $\mathbb{R}$ or a Bessel process of dimension $2-2\alpha$ as in previous sections. Assume B is such that bridges and excursions of B from 0 back to 0 over time t admit a clear definition for every $t > 0$. Then, according to the theory of Markovian bridges and excursions (see e.g. Getoor and Sharpe 1982, Fizsimmons *et al.* 1993), on the interval of length G_i the process B moves according to a bridge of length G_i. On the interval of length H_i the process B makes an excursion of length H_i. And given all the lengths G_i and H_i, these bridges and excursions are independent processes with the prescribed lengths. In particular, in case B is a BM or Bessel process, the operation of standardizing these bridges and excursions to have length one by Brownian scaling produces a sequence of independent standard bridges and a sequence of independent standard excursions which are independent both of each other and of the G and H sequences. Moreover the entire path of B can be recovered from these independent objects by an obvious concatenation.

As a final remark, conditionally given $X = k$, the places of the k points of the $PP(\lambda)$ in the interval of length H_k are distributed like the order statistics of k independent random variables that are uniformly distributed on the interval of length H_k.

4 Interval Partitions

Consider again the set up for Propositions 1 and 8, with K_n the number of distinct excursion intervals of B discovered by the sample of n points. Given $K_n = k$, N_j for $1 \le j \le k$ is the number of sample points that fall in the jth of these k excursion intervals, where the excursion intervals are ordered by their starting times. For $0 \le t \le 1$ let

$$\begin{aligned} G(t) &= \text{ time of last zero of } B \text{ before } t \\ D(t) &= \text{ time of next zero of } B \text{ after } t. \end{aligned}$$

Consider first the bridge case under P_0, so $B_0 = B_1 = 0$, and for M the zero set of B, each component interval of the complement of M relative to $[0,1]$ corresponds to a complete excursion of B away from 0. Given $K_n = k$, for $j = 1, \ldots, k$ define ϵ_j to be the length of the jth *excursion interval*, which contains N_j sample points:

$$\epsilon_j = D(U_{(N_1+\ldots+N_j)}) - G(U_{(N_1+\ldots+N_j)}),$$

and define $\beta_1, \ldots, \beta_{k+1}$ to be the lengths of the successive *bridge intervals* in the complement of the union of the k excursion intervals discovered by the sample points:

$$\begin{aligned}
\beta_1 &= G(U_{(N_1)}) \\
\beta_j &= G(U_{(N_1+\cdots+N_{j+1})}) - D(U_{(N_1+\cdots+N_j)}), \quad 2 \leq j \leq k \\
\beta_{k+1} &= 1 - D(U_{(n)}).
\end{aligned}$$

The terminology reflects the following consequence of the established theory of diffusion bridges and excursions (Rogers and Williams 1987, Fizsimmons *et al.* 1993): conditionally given $K_n = k$, $\epsilon_1, \ldots, \epsilon_k$, and $\beta_1, \ldots, \beta_{k+1}$, the Bessel bridge B decomposes into an alternating concatenation

$$\text{bridge/excursion/} \cdots \text{/bridge}$$

of $k+1$ independent bridges and k independent excursions of the prescribed lengths, independently of $N_1, \ldots, N_k$. In the case B is reflecting Brownian motion and $n = 1$, so there is one excursion interval of length ϵ_1 straddling U_1, and two remaining bridges of lengths β_1 and β_2, Aldous and Pitman (1994) exploited this tripartite decomposition, and showed the joint law of $(\beta_1, \epsilon_1, \beta_2)$ in this case is the exchangeable Dirichlet $(1/2, 1/2, 1/2)$ law. Proposition 15 below is a generalization of this result.

The joint law of random variables $Y_1, \ldots Y_k$ is called *Dirichlet with parameters* $\alpha_1, \ldots, \alpha_k$, if $0 \leq Y_i \leq 1$, $\sum_1^k Y_i = 1$, and for $0 \leq y_i \leq 1$ with $\sum_1^k y_i = 1$, the random vector $(Y_1, \ldots, Y_{k-1})$ has joint density at $(y_1, \ldots, y_{k-1})$ proportional to $\prod_1^k y_i^{\alpha_i - 1}$. To display clearly the correspondence between variables and parameters, a Dirichlet law for $(Y_1, \ldots Y_k)$ will be indicated by a table as in the statement of the following standard result (see e.g. Wilks 1962).

Lemma 14 *Let $Y_i = T_i/T$ where $T_1, \ldots T_k$ are independent gamma variables with common scale parameter and shape parameters $\alpha_1, \ldots, \alpha_k$, and $T = \sum_1^k T_i$. Then*

the law of	Y_1	Y_2	$\cdots$	Y_k
is Dirichlet	α_1	α_2	$\cdots$	α_k

Proposition 15 *Under the Bessel bridge law P_0, conditional on $K_n = k$ and $N_i = n_i, 1 \le i \le k$,*

the law of	ϵ_1	$\ldots$	ϵ_k	β_1	$\ldots$	β_{k+1}
is Dirichlet	$n_1 - \alpha$	$\ldots$	$n_k - \alpha$	α	$\ldots$	α

Proof. This follows easily by the method used to prove Lemma 7, using Corollary 13, (48),(49), and Lemma 14. □

In the unconditioned case the description of the joint law of the various interval lengths given $N_1, \ldots, N_{K_n}$ involves two cases. Let $\mu = 1 - G(1)$, the length of the final *meander interval* $[G(1), 1]$.
Either **(final complete excursion case):** $U_{(n)} > G(1)$, in which case $G(U_{(n)}) = G(1)$, $1 - G(U_{(n)}) = \mu$, and the largest N_{K_n} sample points fall in the meander interval of length μ.
Or **(final meander case):** $U_{(n)} < G(1)$, in which case $G(U_{(n)}) < G(1)$, and the largest N_{K_n} sample points fall in a complete excursion interval, of length ϵ_{K_n}, that is separated from the final meander interval by a bridge interval of length β_{K_n+1}, so

$$1 - G(U_{(n)}) = \epsilon_{K_n} + \beta_{K_n+1} + \mu.$$

Note that the random variable Z_{nn} appearing in Proposition 8 is the indicator of the final complete excursion case. The account of Proposition 8 is now completed as follows:

Proposition 16 *For the Bessel process of dimension $2 - 2\alpha$ conditionally given $K_n = k$, $N_i = n_i$, $1 \le i \le k$, and the final complete excursion case,*

the law of	β_1	$\cdots$	β_{k+1}	ϵ_1	$\cdots$	ϵ_k	μ
is Dirichlet	α	$\cdots$	α	$n_1 - \alpha$	$\cdots$	$n_k - \alpha$	$1 - \alpha$

whereas given $K_n = k$, $N_i = n_i$, $1 \le i \le k$, and the final meander case,

the law of	β_1	$\cdots$	β_k	ϵ_1	$\cdots$	ϵ_{k-1}	μ
is Dirichlet	α	$\cdots$	α	$n_1 - \alpha$	$\cdots$	$n_{k-1} - \alpha$	$n_k + 1 - \alpha$

Proof. Following the proof of Propositions 12 and 15, and using notation introduced in (27), the only additional ingredient is this: given that $Z_n = 1$, the meander length $\tau_{n+1} - G(\tau_{n+1})$ is distributed like the time till the first mark in a marked excursion interval, with density proportional to

$\lambda e^{-\lambda t}\Lambda(t,\infty)$, independently of $Z_1,\ldots,Z_{n-1}$, and all bridge and excursion interval lengths identified before time $G(\tau_{n+1})$. In the stable(α) case, this distribution is gamma $(1-\alpha,\lambda)$. Similarly, given that $J_n = k$, $X_i = n_i$, $1 \le i \le k$, and $Z_n = 0$, the meander length $\tau_{n+1} - G(\tau_{n+1})$ is distributed like the time till the (n_k+1)th mark in intervals with at least n_k+1 marks, with density proportional to $\lambda^{n_k+1}t^{n_k}e^{-\lambda t}\Lambda(t,\infty)$, independently of all bridge and excursion interval lengths identified before time $G(\tau_{n+1})$. In the stable(α) case, this distribution is gamma $(n_k+1-\alpha,\lambda)$. Proposition 16 now follows by an argument parallel to the proof of Proposition 15. □

Corollary 17 *Let R_n denote the length of the complement in $[0,1]$ of the union of excursion intervals containing $U_1,\ldots,U_n$. Then for the Bessel process of dimension $2-2\alpha$,*

$$\text{the } P \text{ conditional distribution of } R_n \text{ given } K_n = k \text{ is } \mathrm{beta}(k\alpha, n-k\alpha) \tag{50}$$

whereas for the Bessel bridge of dimension $2-2\alpha$,

$$\text{the } P_0 \text{ conditional distribution of } R_n \text{ given } K_n = k \text{ is } \mathrm{beta}(k\alpha+\alpha, n-k\alpha). \tag{51}$$

Proof. The result in the bridge case follows immediately from Proposition 15 and the addition rule for components with a Dirichlet distribution. For the free Bessel process, after conditioning on $K_n = k$,

$$R_n = \beta_1 + \ldots + \beta_k + (\beta_{k+1}+\mu)Z_{nn},$$

so application of Proposition 16 and the addition rule for Dirichlet components yields

$$(R_n | K_n = k, Z_{nn} = 0) \stackrel{d}{=} \mathrm{beta}(k\alpha, n+1-k\alpha)$$

$$(R_n | K_n = k, Z_{nn} = 1) \stackrel{d}{=} \mathrm{beta}(k\alpha+1, n-k\alpha),$$

It will be shown below that

$$P(Z_{nn} = 1 | K_n = k) = \frac{k\alpha}{n} \tag{52}$$

Now (50) follows at once from the following elementary fact, familiar to Bayesian statisticians, applied with $a = k\alpha$, $b = n - k\alpha$:

$$\text{beta}(a,b) = \frac{a}{a+b}\text{beta}(a+1,b) + \frac{b}{a+b}\text{beta}(a,b+1) \tag{53}$$

where the right side is the mixture of the two beta distribution with weights $a/(a+b)$ and $b/(a+b)$. It only remains to verify (52), which can be done as follows:

$$\begin{aligned} P(Z_{nn} = 1 | K_n = k) &= E\left(\frac{\alpha}{N_k} \,\middle|\, K_n = k \right) \\ &= \frac{P_0(K_n = k)}{P(K_n = k)} \frac{[\alpha]_n}{n!} \\ &= \frac{\alpha k (n-1)!}{[\alpha]_n} \frac{[\alpha]_n}{n!} \end{aligned}$$

where the first equality is due to (29), the second to (32) and (31), and the third follows by summation from (2) and (3). □

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