A New Duality Relation for Random Walks

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Abstract

For the random walk on the nonnegative integers with reflecting barrier it is shown that the right tails of the probability of the first return from state 0 to state 0 are simple transition probabilities of a "dual"random walk which is obtained from the original process by interchanging the one step probabilities. A combinatorical and analytical proof are presented and extensions and relations to other concepts of duality in the literature are discussed. [1]

1 Introduction

Let $\mathbf{P}_0$ govern a reflecting random walk (X_n) on the nonnegative integers $\{0, 1, 2, \cdots\}$ started at $X_0 = 0$ with one-step upward transition probabilities p_i and one-step downward transition probabilities q_i where $q_0 = 0$, and $p_i + q_i = 1$ for all $i \geq 0$. Define $T_0 = \inf\{n > 0 : X_n = 0\}$ as the time of first return. Similary, let $\tilde{\mathbf{P}}_0$ govern (X_n) as a corresponding *dual* random walk with transition probabilities defined by switching p_i and q_i for $i \geq 1$. In this paper

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we will show by different methods that the $\mathbf{P}_0$ distribution of T_0 is related to the ordinary transition probabilities of $\tilde{\mathbf{P}}_0$.

Theorem 1.1. *With the notation of the previous paragraph the probabilities* $\mathbf{P}_0$ *and* $\tilde{\mathbf{P}}_0$ *satisfy*

$$\mathbf{P}_0(T_0 = 2n) \;=\; \tilde{\mathbf{P}}_0(X_{2n-2} = 0) - \tilde{\mathbf{P}}_0(X_{2n} = 0) \qquad (n = 1, 2, 3, \cdots), \tag{1}$$

or equivalently

$$\mathbf{P}_0(T_0 > 2n) = \tilde{\mathbf{P}}_0(X_{2n} = 0) \qquad (n = 0, 1, 2, \cdots). \tag{2}$$

Elementary arguments allow this formula to be re-expressed in a number of different ways. Let $L_m = \max\{k : 0 \le k \le m, X_k = 0\}$. By the Markov property of $\mathbf{P}_0$

$$\mathbf{P}_0(L_{2n} = k) = \mathbf{P}_0(X_k = 0)\ \mathbf{P}_0(T_0 > 2n - k) \qquad (0 \le k \le 2n) \tag{3}$$

which allows (2) to be generalized to

$$\tilde{\mathbf{P}}_0(L_{2n} = k) = \mathbf{P}_0(L_{2n} = 2n - k) \qquad (0 \le k \le 2n) \tag{4}$$

In other words: for each n

$$\textit{the } \tilde{\mathbf{P}}_0 \textit{ distribution of } L_{2n} \textit{ equals the } \mathbf{P}_0 \textit{ distribution of } 2n - L_{2n} \tag{5}$$

If $p_i = 1/2$ for all i, $\mathbf{P}_0 = \tilde{\mathbf{P}}_0$ governs (X_n) as if $X_n = |S_n|$ for a simple symmetric random walk (S_n) on the integers started at $S_0 = 0$. Then (2) reduces to a well known relation for this random walk which Feller [5] derives from the reflection principle, and (5) reduces to the symmetry about n of the discrete arcsine distribution of L_{2n}.

Section 2 shows how these identities amount to an identity of Wall [23] for continued fraction expansions of corresponding generating functions, whose probabilistic interpretation was indicated by Gerl [7]. A different proof is based on a path bijection argument and presented in Section 3. Section 4 compares this duality relation with a general form of duality between reflecting and absorbing barrier processes introduced by Siegmund [20]. In Section 5 we consider the duality (1) for random walks on the nonnegative integers with holding probabilities which may be strictly positive. According to the spectral theory for random walks on the nonnegative integers of Karlin-McGregor [11], for $\mathbf{Q}_0$ governing such a walk, $\mathbf{Q}_0(X_{2n} = 0)$ is the nth moment of a probability distribution on the interval $[-1, 1]$. These moments are related to the canonical moments of the spectral measure and to continued fractions via its Stieltjes transform. We derive an integral representation for the first return probabilities which is used to prove (2) and to show that the existence of $\tilde{\mathbf{Q}}_0$ governing a dual walk such that an analog of (1) holds is equivalent to the assumption that all holding probabilities of $\mathbf{Q}_0$ vanish. If this property is satisfied we characterize for a given $\mathbf{Q}_0$ all probabilities $\tilde{\mathbf{Q}}_0$ such that (1) is satisfied. A final section contains miscellaneous further remarks.

2 A proof based on Wall's identity

It is well known that the generating functions

$$P_0(z) = \sum_n \mathbf{P}_0(X_{2n} = 0)z^n; \quad F_0(z) = \sum_n \mathbf{P}_0(T_0 = 2n)z^n \tag{6}$$

are related by

$$F_0(z) = 1 - 1/P_0(z) \ . \tag{7}$$

I.J. Good [8] used (7) and a simple recursive argument to obtain expressions for $P_0(z)$ and $F_0(z)$ as continued fractions. In particular, Good obtained

$$P_0(z) = \frac{1\,|}{|\,1} - \frac{q_1 z\,|}{|\,1} - \frac{p_1 q_2 z|}{|\,1} - \frac{p_2 q_3 z|}{|\,1} - \cdots \tag{8}$$

Continued fractions of this form were considered already by Wall [22, 23], who showed that for $\tilde{P}_0(z)$ defined by (8) with $\tilde{p}_i = q_i$ instead of p_i and $\tilde{q}_i = p_i$ instead of q_i, there is the identity ([23] (75.3))

$$P_0(z)\tilde{P}_0(z) = \frac{1}{1-z} \tag{9}$$

Or, in terms of $F_0(z)$ and $\tilde{F}_0(z) = 1 - 1/\tilde{P}_0(z)$, (see [23] (75.6))

$$\tilde{F}_0(z) = \frac{F_0(z) - z}{F_0(z) - 1} \tag{10}$$

As observed by Gerl [7], the above formulae express the generating functions for the dual walk

$$\tilde{P}_0(z) = \sum_n \tilde{\mathbf{P}}_0(X_{2n} = 0)z^n; \quad \tilde{F}_0(z) = \sum_n \tilde{\mathbf{P}}_0(T_0 = 2n)z^n. \tag{11}$$

in terms of those of the original walk. From these relations the duality (1) in Theorem 1.1 follows easily by using (7) and comparing coefficients in the corresponding power series of $F_0(z)$ and $\tilde{P}_0(z)$.

The above argument shows that the probabilistic identity (1) is easily derived from Wall's identity (9), and vice-versa. For completeness, we now recall Wall's proof of (9). Let

$$\frac{A_n(z)}{B_n(z)} = \frac{1\,|}{|\,1} + \frac{q_1 z\,|}{|\,1} + \frac{p_1 q_2 z|}{|\,1} + \cdots + \frac{p_{n-2} q_{n-1} z|}{|\quad 1}$$

and

$$\frac{C_n(z)}{D_n(z)} = \frac{1\,|}{|\,1} + \frac{p_1 z\,|}{|\,1} + \frac{q_1 p_2 z|}{|\,1} + \cdots + \frac{q_{n-2} p_{n-1} z|}{|\quad 1}$$

denote the nth approximant for the generating function $P_0(-z)$ and $\tilde{P}_0(-z)$. Using the recurrence relations (see [23], (1.4))

$$\begin{aligned} A_{n+1}(z) &= A_n(z) + p_{n-1} q_n z A_{n-1}(z) \ , \quad A_1(z) = 1 \ , \quad A_0(z) = 0 \\ D_{n+1}(z) &= D_n(z) + q_{n-1} p_n z D_{n-1}(z) \ , \quad D_1(z) = 1 \ , \quad D_0(z) = 1 \end{aligned}$$

we readily derive (by induction)

$$(1+z)A_n(z) = q_{n-1}zD_{n-1}(z) + D_n(z)$$

and similary

$$B_n(z) = q_{n-1}zC_{n-1}(z) + C_n(z)\ .$$

Combining these two equations yields

$$\begin{aligned}(1+z)A_n(z)C_{n-1}(z) - B_n(z)D_{n-1}(z) &= D_n(z)C_{n-1}(z) - D_{n-1}(z)C_n(z)\\ &= k_n z^{n-1}\end{aligned}$$

where $k_n = (-1)^n \prod_{j=1}^{n-1} q_{j-1}p_j$ and the last line follows from [23], p.16. By the Euler Minding formuals (see Perron [17], p. 5) $B_n(z)$ and $D_{n-1}(z)$ are polynomials of degree $n-1$ and $n-2$ with positive coefficients and $B_n(0) = D_{n-1}(0) = 1$. Thus we obtain for all $z > 0$

$$\left|\frac{A_n(z)}{B_n(z)}\frac{C_{n-1}(z)}{D_{n-1}(z)} - \frac{1}{1+z}\right| = \left|\frac{k_n z^{n-1}}{(1+z)B_n(z)D_{n-1}(z)}\right| \leq |z|^{n-1}$$

which implies for all $0 < z < 1$ that

$$P_0(-z)\tilde{P}_0(-z) = \lim_{n\to\infty}\frac{A_n(z)}{B_n(z)}\lim_{n\to\infty}\frac{C_{n-1}(z)}{D_{n-1}(z)} = \frac{1}{1-z} \tag{12}$$

Now both sides of (12) are analytic functions in the cut plane $\mathbb{C}\setminus[1,\infty)$ (see Jones and Thron [10]) and the relation (9) follows.

3 Combinatorial proof of the duality relation

The assertion of Theorem 1.1 will be proved in the form (2). For fixed n each side of (2) is a polynomial in $q_1,\cdots,q_n$:

$$\mathbf{P}_0(X_{2n}=0) = F(q_1,\cdots,q_n)$$

say, is the sum over all paths from $(0,0)$ to $(0,2n)$ of products of conditional probabilities along paths in a diagram, call it $D(F,n)$, as in the middle left of Figure 1 for $n=3$. Similarly

$$\tilde{\mathbf{P}}_0(T_0 > 2n) = G(q_1,\cdots,q_n)$$

say, is a sum of products over paths starting at $(0,0)$ and ending at any of the n points $(n+1,n+1),(n+2,n),\cdots(2n,2)$, in a diagram, call it $D(G,n)$, as shown in the top right of Figure 1. The problem is to show $D(F,n)\cong D(G,n)$, where for two diagrams D and D', each defined by a collection of directed segments with associated factors e.g. 1, q_i or $p_i = 1-q_i$ for some i, and each inducing a polynomial in $(q_1,q_2,\cdots)$ by a sum of products over paths, $D\cong D'$ means the two polynomials are identical. Let $D'(G,n)$ denote the diagram with the same shape as $D(F,n)$, as in the middle right panel of Figure 1 for $n=3$, obtained from $D(G,n)$ as follows: move the initial path segment with a factor of 1 from the left end of $D(G,n)$ to the right end, insert $n-1$ additional segments with factors of 1 to join up the loose ends on the downsloping right side of the $D(G,n)$, and reverse the direction of paths. There is an obvious one-to-one product preserving correspondence between paths in $D(G,n)$ and $D'(G,n)$, so $D(G,n) \cong D'(G,n)$. For a general n the diagram $D'(G,n)$ has the same set of paths as in $D(F,n)$, but with the factors in $D(F,n)$ modified as follows:

- on segment $(i,i) \to (i+1,i+1)$, $1 \leq i \leq n-1$, replace p_i by 1;
- elsewhere replace p_i by p_{i+2};
- on segment $(2k,0) \to (2k+1,1)$, $1 \leq k \leq n-1$, replace 1 by p_2;

Note that the factors q_i on down-segments appear in the same places in both $D(F,n)$ and $D'(G,n)$. Only the factors on up-segments are different. In either diagram $D(F,n)$ or $D'(G,n)$, call a path through the diagram from $(0,0)$ to $(0,2n)$, combined with a choice of 1 or $-q_i$ for each up-segment with a factor p_i, an *expanded path*. The set of all expanded paths in $D(F,n)$ defines an *expanded diagram*, call it $D^*(F,n)$, in which each up-segment in $D(F,n)$ labelled by a p_i is replaced by two distinct up-segments labelled 1 and $-q_i$. Let $D^*(G,n)$ be derived similarly from $D'(G,n)$. By construction $D(F,n) \cong D^*(F,n)$ and $D(G,n) \cong D'(G,n) \cong D^*(G,n)$. The argument is completed by the following proposition, which shows $D^*(F,n) \cong D^*(G,n)$.

In either diagram $D^*(F,n)$ or $D^*(G,n)$, call a sequence of $2n$ factors derived from an expanded path a *selection* from the expanded diagram. Note that the selection determines the expanded path, since each factor 1 or $-q_i$ indicates an up-segment, and each q_i a down-segment.

Proposition. *For each selection from either $D^*(F,n)$ or $D^*(G,n)$, there is a unique corresponding selection from the other diagram with the same sequence of factors modulo signs, and such corresponding selections have identical products.*

Granted the correspondence between selections modulo signs, the equality of corresponding products follows easily: corresponding selections have same number of factors 1, hence the same number of - factors, because for every selection in either diagram, the number 1's plus the number of - factors equals n.

For $m \leq n$ the expanded diagram $D^*(F,m)$ is obtained in an obvious way by restriction of the paths in $D^*(F,n)$ to the region $\{(i,k) : i+k \leq 2m\}$. The diagrams $D^*(G,m)$ and $D^*(G,n)$ are similarly related. This can be seen for $m=1$ or 2 and $n=3$ in Figure 1.

The correspondence between $D^*(F,1)$ and $D^*(G,1)$ is trivial, since these two diagrams are identical: $F(q_1) = G(q_1) = q_1$. The correspondence between selections in $D^*(F,2)$ and $D^*(G,2)$ can be seen from Figure 1 to be as follows: there are 3 selections in each expanded diagram with corresponding products:

$$\begin{array}{rcl} \text{Product from } D^*(F,2) & & \text{Product from } D^*(G,2) \\ (1)(q_1)(1)(q_1) & = & (1)(q_1)(1)(q_1) \\ (1)(1)(q_2)(q_1) & = & (1)(1)(q_2)(q_1) \\ (1)(-q_1)(q_2)(q_1) & = & (1)(q_1)(-q_2)(q_1) \end{array}$$

Thus $F(q_1, q_2) = G(q_1, q_2) = q_1^2 + q_2 q_1 - q_1^2 q_2$.

For $n = 3$ it can be seen from Figure 1 that there are 13 selections in each expanded diagram with corresponding products:

$$\begin{array}{rcl}
\text{Product from } D^*(F,3) & & \text{Product from } D^*(G,3) \\
(1)(1)(1)(q_3)(q_2)(q_1) & = & (1)(1)(1)(q_3)(q_2)(q_1) \\
(1)(1)(-q_2)(q_3)(q_2)(q_1) & = & (1)(1)(q_2)(-q_3)(q_2)(q_1) \\
(1)(1)(q_2)(1)(q_2)(q_1) & = & (1)(1)(q_2)(1)(q_2)(q_1) \\
(1)(1)(q_2)(-q_1)(q_2)(q_1) & = & (1)(1)(q_2)(q_1)(-q_2)(q_1) \\
(1)(1)(q_2)(q_1)(1)(q_1) & = & (1)(1)(q_2)(q_1)(1)(q_1) \\
(1)(-q_1)(q_2)(1)(q_2)(q_1) & = & (1)(q_1)(-q_2)(1)(q_2)(q_1) \\
(1)(-q_1)(q_2)(-q_1)(q_2)(q_1) & = & (1)(q_1)(-q_2)(q_1)(-q_2)(q_1) \\
(1)(-q_1)(-q_2)(q_3)(q_2)(q_1) & = & (1)(q_1)(-q_2)(-q_3)(q_2)(q_1) \\
(1)(-q_1)(1)(q_3)(q_2)(q_1) & = & (1)(q_1)(1)(-q_3)(q_2)(q_1) \\
(1)(-q_1)(-q_2)(q_3)(q_2)(q_1) & = & (1)(q_1)(-q_2)(-q_3)(q_2)(q_1) \\
(1)(q_1)(1)(q_1)(1)(q_1) & = & (1)(q_1)(1)(q_1)(1)(q_1) \\
(1)(q_1)(1)(1)(q_2)(q_1) & = & (1)(q_1)(1)(1)(q_2)(q_1) \\
(1)(q_1)(1)(-q_1)(q_2)(q_1) & = & (1)(q_1)(1)(q_1)(-q_2)(q_1)
\end{array}$$

The sum of these 13 products is $F(q_1, q_2, q_3) = G(q_1, q_2, q_3)$.

Proof of the correspondence for general n. Assume, for simplicity of notation but without loss of generality, that $0 < q_i < 1$ for every i, and that the q_i are all distinct. This allows a selection to be treated as a sequence of real numbers rather than a sequence of symbolic factors. Let $(r_1, \cdots, r_{2n})$ denote a selection from $D^*(F, n)$ and let $(x_0, x_1, \cdots, x_{2n})$ be the corresponding lattice path. Note that $(x_0, x_1, \cdots, x_{2n})$ is a path in $D(F, n)$ rather than $D^*(F, n)$. So $(r_1, \cdots, r_{2n})$ determines $(x_0, x_1, \cdots, x_{2n})$ but not vice-versa. To be precise, $(x_0, x_1, \cdots, x_{2n})$ is a sequence of non-negative integers with $x_0 = x_{2n} = 0$ and $(r_1, \cdots, r_{2n})$ is a sequence of real numbers such that each r_k equals $1, q_x$, or $-q_x$ for some $1 \le x \le n$, and for each $1 \le k \le 2n - 1$

$$r_{k+1} = \left\{ \begin{array}{ll} 1 & \text{if } x_k = 0 \\ 1, -q_{x_k}, \text{ or } q_{x_k} & \text{if } x_k > 0 \text{ and } k + x_k \le 2n \\ q_{x_k} & \text{if } k + x_k = 2n \end{array} \right\} \tag{13}$$

The path (x_k) can be recovered inductively from (r_k) by

$$x_{k+1} = \left\{ \begin{array}{ll} x_k + 1 & \text{if } r_{k+1} = 1 \text{ or } -q_{x_k} \\ x_k - 1 & \text{if } r_{k+1} = q_{x_k} \end{array} \right\} \tag{14}$$

Similarly, a sequence $(s_1, \cdots, s_{2n})$ is a selection from $D^*(G, n)$ associated with the lattice path $(y_0, y_1, \cdots, y_{2n})$ in $D(G, n)$ if and only if $y_0 = y_{2n} = 0, y_m \ge 0$ for $1 \le m \le 2n$, each s_k

equals $1, q_y$, or $-q_y$ for some $1 \leq y \leq n$, and for each $1 \leq k \leq 2n-1$

$$s_{k+1} = \left\{ \begin{array}{ll} 1 & \text{if } 0 \leq k \leq n-1 \text{ and } y_k = k \\ 1, -q_{y_k+2}, \text{ or } q_{y_k} & \text{if } y_k > 0 \text{ and } k + x_k \leq 2n \\ 1 \text{ or } -q_2 & \text{if } 2 \leq k \leq 2n-2 \text{ and } y_k = 0 \\ q_{y_k} & \text{if } k + y_k = 2n \end{array} \right\} \tag{15}$$

and the path (y_k) can be recovered inductively from (s_k) by

$$y_{k+1} = \left\{ \begin{array}{ll} y_k + 1 & \text{if } s_{k+1} = 1 \text{ or } -q_{y_k+2} \\ y_k - 1 & \text{if } s_{k+1} = q_{y_k} \end{array} \right\} \tag{16}$$

It is claimed that
(i) for each selection $(r_1, \cdots, r_{2n})$ in $D^*(F,n)$, there exists a unique selection $(s_1, \cdots, s_{2n})$ in $D^*(G,n)$ such that $|s_k| = |r_k|$ for every $1 \leq k \leq 2n$;
(ii) each selection $(s_1, \cdots, s_{2n})$ in $D^*(G,n)$ can be so obtained from some selection $(r_1, \cdots, r_{2n})$ in $D^*(F,n)$;
(iii) if (x_k) and (y_k) are the lattice paths associated with such (r_k) and (s_k) then $x_k - y_k$ equals either 0 or 2 for every $0 \leq k \leq 2n$.
Claims (i) and (iii) will be established by induction over k for $1 \leq k \leq 2n$. Note that in (i) the only issue is existence: uniqueness is obvious from (15) because in every case the options for s_{k+1} have distinct absolute values.

Claim (ii) is shown by a similar induction in the reverse direction, decrementing k from $2n$ to 1.

Fix an arbitrary selection (r_k) with path (x_k) in $D^*(F,n)$, and make the following:
Inductive Hypothesis for (i) and (iii): *There is a unique possible sequence $(s_1, \cdots, s_k)$ for the first k factors of a selection in $D^*(G,n)$ with $|s_j| = |r_j|$ for every $1 \leq j \leq k$, and if $(y_1, \cdots, y_k)$ is initial portion of the lattice path associated with $(s_1, \cdots, s_k)$ via (16), then $x_k - y_k \in \{0, 2\}$.*

This is trivial for $k = 1$. Assuming the inductive hypothesis for k, consider how s_{k+1} is determined by x_k, y_k and r_{k+1} via (13) and (15). There are six possible cases, according to whether $x_k - y_k = 0$ or 2, and whether r_{k+1} equals 1, q_{x_k} or $-q_{x_k}$. Each case is described by a row in the following table. In each row the assumed values of $x_k - y_k$ and r_{k+1} are given in the first and second columns. These values, combined with the basic rules (13), (14), (15), (16) and $|s_{k+1}| = |r_{k+1}|$, then force the values of $x_{k+1} - x_k, s_{k+1}, y_{k+1} - y_k$ and $x_{k+1} - y_{k+1}$ as indicated. In each case, there is a unique possible choice of s_{k+1} according to the rules of $D^*(G,n)$ such that $|s_{k+1}| = |r_{k+1}|$, and this choice of s_{k+1} implies $y_{k+1} - x_{k+1} \in \{0, 2\}$. This is the inductive hypothesis with $k+1$ instead of k.

$x_k - y_k$	r_{k+1}	$x_{k+1} - x_k$	s_{k+1}	$y_{k+1} - y_k$	$x_{k+1} - y_{k+1}$
0	1	1	1	1	0
2	1	1	1	1	2
0	q_{x_k}	-1	q_{y_k}	-1	0
2	q_{x_k}	-1	$-q_{y_k+2}$	1	0
0	$-q_{x_k}$	1	q_{y_k}	-1	2
2	$-q_{x_k}$	1	$-q_{y_k+2}$	1	2

To check a row of the table, it must be verified that the value of s_{k+1} indicated in the 4th column with $|s_{k+1}| = |r_{k+1}|$ is actually available according to the rules of $D^*(G,n)$. (As mentioned before, uniqueness of such an s_{k+1} is obvious). The increments $x_{k+1} - x_k$ and $y_{k+1} - y_k$ are then forced by (14) and (16), and $x_{k+1} - y_{k+1}$ is found from

$$x_{k+1} - y_{k+1} = (x_k - y_k) + (x_{k+1} - x_k) + (y_{k+1} - y_k)$$

The cases work out as follows, according to the values of $(x_k - y_k, r_{k+1})$:

- $(0,1)$ or $(2,1)$. That is, $r_{k+1} = 1$, which implies $k + x_k < 2n$, hence $k + y_k < 2n$. So from (15), 1 is available as a choice for s_{k+1}.
- $(0, q_{x_k})$. The availability of the choice $r_{k+1} = q_{x_k}$ implies $y_k = x_k > 0$. The set of possible choices for s_{k+1} is then

  $$\text{either } \{1, q_{y_k}\},\ \{1, -q_{y_k+2}, q_{y_k}\}, \text{ or } \{q_{y_k}\},$$

  depending on (k, y_k). Whatever this set of choices, there is a choice with $|s_{k+1}| = |r_{k+1}|$, namely $s_{k+1} = q_{y_k}$.
- $(2, q_{x_k})$. Since $x_k \le k$ and $k + x_k \le 2n$ for all paths (x_k) in $D(F,n)$, $y_k < k$ and $k + y_k < 2n$. The set of possible choices for s_{k+1} is therefore

  $$\text{either } \{1, -q_{y_k+2}, q_{y_k}\} \text{ if } y_k > 0, \text{ or } \{1, -q_{y_k+2}\} \text{ if } y_k = 0.$$

  Either way, there is a choice with $|s_{k+1}| = |r_{k+1}|$, namely $s_{k+1} = -q_{y_k+2} = -q_{x_k}$.
- $(0, -q_{x_k})$. The choice of $r_{k+1} = -q_{x_k}$ implies $x_k > 0$ and $k + x_k < 2n$. The set of possible choices for s_{k+1} is therefore $\{1, -q_{x_k+2}, q_{x_k}\}$. The choice with $|s_{k+1}| = |r_{k+1}|$ is $s_{k+1} = q_{x_k}$.
- $(2, -q_{x_k})$. Then $y_k + k < 2n$. The set of possible choices for s_{k+1} is either $\{1, -q_{x_k}\}$ if $y_k = 0$ or $\{1, -q_{x_k}, q_{x_k-2}\}$ if $y_k > 0$. The choice with $|s_{k+1}| = |r_{k+1}|$ is $s_{k+1} = -q_{x_k}$.

Remark. As a check on the above correspondence, it can be seen by direct calculation that in either $D^*(F,n)$ or $D^*(G,n)$ the number of expanded paths is

$$\sum_{u=0}^{n-1} \frac{n-u}{n+u} \binom{n+u}{n} 2^u \tag{17}$$

The coefficient of 2^u represents the number of different paths in $D(F,n)$, or in $D(G,n)$, that have u up-segments with factors of the form p_i for some i. Each such path contributes 2^u selections in the expanded diagram. In $D(G,n)$ the coefficient can be evaluated using the ballot theorem. In $D(F,n)$ the same coefficent is found using Theorem 4 in Chapter III of Feller [5].

4 Comparison with Siegmund Duality

It is natural to compare the duality relation in Theorem 1.1 between two random walks, call it *Wall's duality*, with the general form of duality between reflecting and absorbing barrier processes introduced by Siegmund [20], and studied also by Cox and Rösler [3], Clifford-Sudbury [1], Diaconis-Fill [4]. Let $(\mathbf{Q}_x)$ be the family of distributions governing (X_n) as a Markov chain with statespace $[0,\infty)$ or $\{0,1,2\cdots\}$ indexed by the starting state $x \geq 0$. Another such family of distributions $(\hat{\mathbf{Q}}_x)$ is said to be dual to $(\mathbf{Q}_x)$ in Siegmund's sense if for all states y and z,

$$\mathbf{Q}_y(X_n \leq z) = \hat{\mathbf{Q}}_z(X_n \geq y)$$

Taking $y = 0$ shows this identity can only hold if $\mathbf{Q}_0$ makes 0 an absorbing state. As shown by Siegmund, if this relation holds for $n = 1$ then it holds also for every n, by repeated application of the Chapman-Kolmogorov equations. For $(\mathbf{Q}_x)$ governing the absorbing nearest neighbour random walk, with one-step probabilities p_x, q_x and r_x for jumps up, down, and holds, and $r_0 = 1$, it is known and easily checked that there is a Siegmund dual $(\hat{\mathbf{Q}}_x)$ governing a partially reflecting walk with one-step probabilities

$$\hat{q}_x = p_x; \quad \hat{r}_x = 1 - (p_x + q_{x+1}); \quad \hat{p}_x = q_{x+1}$$

provided $p_x + q_{x+1} \leq 1$. The latter constraint is usually regarded as an artifact of discrete time, as both continuous time birth-death processes and diffusions with absorption at 0 have Siegmund duals without such constraints. Also, as noted in Section 5 of Cox and Rösler [3], when $r_x = 0$ for all x this constraint can be worked around by watching the chain only every two steps. Let $(\mathbf{Q}_x)$ be derived from the reflecting walk $(\mathbf{P}_x)$ as in the introduction by making state 0 absorbing. Assume for simplicity that q_x is a non-increasing function of x. Then $(\mathbf{Q}_x)$ has a dual walk $(\hat{\mathbf{Q}}_x)$ that is partially reflecting at 0 with transition probabilities $\hat{q}_x, \hat{r}_x, \hat{p}_x$ as above. Duality gives

$$\mathbf{Q}_1(X_n \leq 0) = \hat{\mathbf{Q}}_0(X_n \geq 1)$$

which, by taking complements, gives

$$\mathbf{P}_0(T_0 > n+1) = \mathbf{P}_1(T_0 > n) = \mathbf{Q}_1(X_n > 0) = \hat{\mathbf{Q}}_0(X_n = 0)$$

Compare with the dual of (2), which gives

$$\mathbf{P}_0(T_0 > 2n) = \tilde{\mathbf{P}}_0(X_{2n} = 0)$$

So both methods yield a dual chain such that the right tails of the $\mathbf{P}_0$ distribution of T_0 are simple transition probabilities of the dual. But the two dual chains are slightly different, even in the simplest case $q_i = q$ for all i. Then the Siegmund dual of the absorbing walk is the partially reflecting q-up, p-down walk with the representation, familiar from queuing theory, as $S_n - \min_{0\leq k \leq n} S_k$ where S_n is a simple q-up, p-down walk on the integers. Wall's dual in this case is the same as Siegmund's away from 0, but with sure reflection from 0.

There seem to be two differences between Wall's duality and Siegmund's, the second apparently more important than the first:

1. The Wall dual of a reflecting walk is another reflecting walk, whereas the Siegmund dual of a reflecting walk is a walk absorbed at 0;

2. Wall's duality only involves the walks started in the particular state 0, whereas Siegmund's involves two processes starting from general positions.

A natural question about Wall's duality is whether it can be extended to more processes, for example walks with non-zero holding probabilities r_x, or continuous time birth-death processes. The generating functions $P_0(z)$ and $F_0(z)$ for such walks admit continued fraction representations (see e.g. Good [8], Flajolet [6]), as do corresponding Laplace transforms for birth-death processes ([16], Jones and Magnus [9], [10]). We will give a partial answer to the first problem in the following section.

Perhaps Wall's duality can cast in terms of the general notion of duality of two Markov processes formulated in Section II.3 of Liggett [15], of which Siegmund's duality is a special case. However, to do so it is first necessary to discover some generalization of Wall's duality involving the two walks with arbitrary starting states.

5 More on Wall's duality

In this Section we discuss an approach via spectral theory (see [11]) to the duality in order to investigate if and how Wall's duality can be transferred to more general random walks on the nonnegative integers. Our results are limited to random walks started at $X_0 = 0$ but indicate some of the difficulties in generalizing Wall's duality. Throughout this section let the probability $\mathbf{Q}_0$ govern a random walk (X_n) on the nonnegative integers started at $X_0 = 0$ with one step transition probabilities p_j, q_j and r_j for jumps up, down, and holds, where $p_j > 0$, $q_{j+1} > 0$, $p_j + q_j + r_j \leq 1$ $(j = 0, 1, 2, \ldots)$. The probability $\tilde{\mathbf{Q}}_0$ with transition probabilities $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{r}_j$ is called *dual* to $\mathbf{Q}_0$ if and only if

$$\mathbf{Q}_0(T_0 = n) \;=\; \tilde{\mathbf{Q}}_0(X_{n-2} = 0) - \tilde{\mathbf{Q}}_0(X_n = 0) \tag{18}$$

holds for all $n = 2, 3, \ldots$. As pointed out in the previous discussion a dual of a probability $\mathbf{Q}_0$ with $r_j = 0$, $q_0 = 0$ is obtained by switching the transition probabilities p_j and q_j for $j \geq 1$. We now address the question of characterizing all dual probabilities $\tilde{\mathbf{Q}}_0$ [in the sense of (18)] corresponding to a given probability $\mathbf{Q}_0$.

It is shown in [11] that the n-step transition probabilities of the random walk (X_n) with governing probability $\mathbf{Q}_0$ can be represented as

$$\mathbf{Q}_0(X_n = j) \;=\; \frac{\int_{-1}^{1} x^n Q_j(x) d\psi(x)}{\int_{-1}^{1} Q_j^2(x) d\psi(x)} \tag{19}$$

where ψ is the spectral measure of the probability $\mathbf{Q}_0$ (with support contained in the interval $[-1, 1]$) and $Q_i(x)$ is a polynomial of degree i, recursively defined by

$$(x - r_i)Q_i(x) = q_i Q_{i-1}(x) + p_i Q_{i+1}(x) \tag{20}$$

whenever $p_i > 0$, and $Q_{-1}(x) = 0$, $Q_0(x) = 1$. If $n < j$ it follows from (19) that $\mathbf{Q}_0(X_n = j) = 0$ which shows that the polynomials $Q_i(x)$ are orthogonal with respect to the distribution $d\psi(x)$ on the interval $[-1, 1]$. From (19) and [8] we obtain a continued fraction expansion for the Stieltjes transform of the spectral measure ψ

$$\begin{aligned} P_0(z) &= \sum_{n=0}^{\infty} \mathbf{Q}_0(X_n = 0) z^n = \int_{-1}^{1} \frac{d\psi(x)}{1 - zx} \\ &= \frac{1 \mid}{\mid 1 - r_0 z} - \frac{p_0 q_1 z^2 \mid}{\mid 1 - r_1 z} - \frac{p_1 q_2 z^2 \mid}{\mid 1 - r_2 z} - \cdots . \end{aligned} \tag{21}$$

The spectral measure ψ is determined by its moments $c_1, c_2, \ldots$. In what follows it turns out to be useful to switch from the ordinary moments to the so *called canonical moments* of ψ which are obtained as follows. It is shown in [23], p. 263, that the Stieltjes transform of ψ can also be written as

$$P_0(z) = \frac{1 \mid}{\mid 1 + (1 - 2\zeta_1) z} - \frac{4\zeta_1\zeta_2 z^2 \mid}{\mid 1 + (1 - 2\zeta_2 - 2\zeta_3) z} - \frac{4\zeta_3\zeta_4 z^2 \mid}{\mid 1 + (1 - 2\zeta_4 - 2\zeta_5) z} - \cdots$$

where $\zeta_j = (1 - m_{j-1}) m_j$ $(j \geq 1)$, $m_0 = 1$, $\zeta_0 = 0$. The quantities m_j vary in the interval $[0, 1]$ and are in one to one correspondence with the ordinary moments $c_1, c_2, \ldots$ (see Skibinsky [21]). We call m_j the jth canonical moment of the distribution ψ. If i_0 is the first index such that $m_{i_0} \in \{0, 1\}$ then ψ is supported at a finite number of points and the continued fraction expansion of $P_0(z)$ terminates. If $m_i \in (0, 1)$ for all $i \geq 1$, then ψ has infinite support and we put formally $i_0 = \infty$.

Observing (21) we see that the canonical moments of the spectral measure ψ can be determined from the probabilities of $\mathbf{Q}_0$ by solving recursively the equations

$$\begin{aligned} -1 + 2\zeta_{2j} + 2\zeta_{2j+1} &= r_j \qquad (0 \leq j < i_0) \\ 4\zeta_{2j-1}\zeta_{2j} &= p_{j-1} q_j . \qquad (0 \leq j < i_0) \end{aligned} \tag{22}$$

For example, the first two canonical moments of ψ can be expressed in terms of the transition probabilities as

$$m_1 = \frac{1 + r_0}{2}, \quad m_2 = \frac{p_0 q_1}{1 - r_0^2} .$$

Let $\tilde{\psi}$ denote the measure corresponding to the sequence of canonical moments $\tilde{m}_j = 1 - m_j$ $(j \geq 1)$, then $\tilde{\psi}$ is called the *dual* of the spectral measure ψ of the probability $\mathbf{Q}_0$.

Note that $\tilde{\psi}$ is not necessarily the spectral measure of a probability $\tilde{\mathbf{Q}}_0$ governing the random walk (X_n). By the previous discussion it follows that this property is only satisfied if one can find transition probabilities $\tilde{p}_j, \tilde{r}_j$ and $\tilde{q}_j$ corresponding to $\tilde{\mathbf{Q}}_0$ that solve the equations

$$\begin{aligned} -1 + 2\tilde{\zeta}_{2j} + 2\tilde{\zeta}_{2j+1} &= \tilde{r}_j \qquad (0 \leq j < i_0) \\ 4\tilde{\zeta}_{2j-1}\tilde{\zeta}_{2j} &= \tilde{p}_{j-1} \tilde{q}_j \qquad (0 \leq j < i_0) \end{aligned} \tag{23}$$

for all $j \geq 0$, where $\tilde{\zeta}_j = (1 - \tilde{m}_{j-1})\tilde{m}_j$ $(j \geq 1)$, $\tilde{\zeta}_0 = 0$.

Theorem 5.1. *Let ψ denote the spectral measure of the probability $\mathbf{Q}_0$ and $\tilde{\psi}$ the dual of ψ, then for all $n \geq 2$*

$$\mathbf{Q}_0(T_0 = n) \;=\; \int_{-1}^{1} (1 - x^2) x^{n-2} d\tilde{\psi}(x) \;.$$

Proof. From Lau [14] it follows that the Stieltjes transform of the measure $(1 - x^2)d\tilde{\psi}(x)$ is given by

$$\begin{aligned} H(z) &= \int_{-1}^{1} \frac{(1 - x^2)d\tilde{\psi}(x)}{1 - zx} \\ &= \frac{4\tilde{\gamma}_1\tilde{\gamma}_2 \quad |}{|1 + (1 - 2\tilde{\gamma}_2 - 2\tilde{\gamma}_3)z} - \frac{4\tilde{\gamma}_3\tilde{\gamma}_4 z^2 \quad |}{|1 + (1 - 2\tilde{\gamma}_4 - 2\tilde{\gamma}_5)z} - \cdots \end{aligned}$$

where $\tilde{\gamma}_j = \tilde{m}_{j-1}(1 - \tilde{m}_j) = (1 - m_{j-1})m_j = \zeta_j$ and $\tilde{\gamma}_1 = \zeta_1$. Now (21) and (22) give

$$P_0(z) = \left[1 - r_0 z - z^2 H(z)\right]^{-1}$$

and from (7) we obtain

$$F_0(z) \;=\; r_0 z + z^2 H(z) \;=\; r_0 z + z^2 \sum_{n=0}^{\infty} \int_{-1}^{1} (1 - x^2) x^n d\tilde{\psi}(x) z^n$$

which implies the assertion of the theorem.

Example 5.2. Consider the situation of Section 1, where $\mathbf{P}_0$ governs (X_n) with $p_0 = 1$ and $r_j = 0$, $p_j + q_j = 1$ $(j = 0, 1, 2, \ldots)$. It is easy to see that in this case the the solution of (22) is

$$m_{2j-1} = \frac{1}{2}\,, \quad m_{2j} = q_j \qquad (j = 1, 2, \ldots)$$

and the dual $\tilde{\psi}$ of the spectral measure has canonical moments $\tilde{m}_{2j-1} = \frac{1}{2}$, $\tilde{m}_{2j} = p_j$. Thus there is a one to one correspondence between the transition probabilities of $\mathbf{P}_0$ and $\tilde{\mathbf{P}}_0$ and the canonical moments of the distributions ψ and $\tilde{\psi}$. Observing (23) we obtain that $\tilde{\psi}$ is the spectral measure of a probability $\tilde{\mathbf{P}}_0$ where the corresponding transition probabilities satisfy $\tilde{r}_j = 0$ $(j \geq 0)$ and $\tilde{p}_{j-1}\tilde{q}_j = q_{j-1}p_j$ $(j \geq 1)$. By the special choice $\tilde{p}_0 = 1$ it follows that $\tilde{\psi}$ is the spectral measure of a random walk governed by $\tilde{\mathbf{P}}_0$ which is obtained from $\mathbf{P}_0$ by interchanging p_j and q_j for $j \geq 1$. This is Wall's duality described and investigated in Section 1 - 3.

Theorem 5.3. *Let $\mathbf{Q}_0$ govern (X_n) started at $X_0 = 0$ with one step transition probabilities p_j, q_j and r_j for jumps up, down, and holds.*
a) *There exists a probability $\tilde{\mathbf{Q}}_0$ governing (X_n) such that the duality (18) holds for $\mathbf{Q}_0$ and $\tilde{\mathbf{Q}}_0$ if and only if $r_j = 0$ for all $j = 0, 1, 2, \ldots$.*

b)*If* $r_j = 0$ *for all* $j = 0, 1, 2, \ldots$ *define*

$$S_n \;=\; 1 + \sum_{k=1}^{n} p_0 (\prod_{j=1}^{k} \frac{p_j}{q_j}) Q_k(1) Q_{k+1}(1) \quad , \quad S_{-1} = 0 \ , \quad S_0 = 1 \tag{24}$$

and $S = \lim_{n\to\infty} S_n$ *(*$= \infty$ *if the sum diverges). For every* $t \in [0, \frac{1}{S}]$ *the probability* $\tilde{\mathbf{Q}}_0^t$ *with transition probabilities* $\tilde{r}_n(t) = 0$,

$$\tilde{q}_n(t) \;=\; 1 - \tilde{p}_n(t) \;=\; \frac{1 - tS_{n-2}}{1 - tS_{n-1}} \cdot \frac{Q_{n+1}(1)}{Q_n(1)} p_n \quad , \quad \tilde{q}_0(t) = 1 - \tilde{p}_0(t) = t$$

is a dual of $\mathbf{Q}_0$ *[in the sense of (18)]. Especially, the dual of the probability* $\mathbf{Q}_0$ *is unique if and only if the sum in (24) diverges.*

Proof. a] Let ψ denote the spectral measure of $\mathbf{Q}_0$ and $\tilde{\psi}$ the corresponding dual spectral measure. By the Karlin McGregor representation (19) and the determinacy of the Hausdorff moment problem it follows from Theorem 5.1 that the assertion can be proved by showing that $\tilde{\psi}$ is the spectral measure of some probability $\tilde{\mathbf{Q}}_0$ if and only if $r_j = 0$ for all $j = 0, 1, 2, \ldots$.

Now let m_j and $\tilde{m}_j = 1 - m_j$ denote the canonical moments of ψ and $\tilde{\psi}$, respectively and assume that $\tilde{\psi}$ is the spectral measure of $\tilde{\mathbf{Q}}_0$ with transition probabilities $\tilde{p}_j$, $\tilde{q}_j$ and $\tilde{r}_j$. Obviously a measure is a spectral measure of a random walk if and only if $r_j \geq 0$ for all $j \geq 0$. Thus we have from (22) and (23)

$$0 \leq r_0 = -1 + 2m_1 = 2\tilde{m}_1 - 1 = -\tilde{r}_0 \leq 0$$

which implies $m_1 = \frac{1}{2}$, $r_0 = \tilde{r}_0 = 0$. Assume that $r_i = \tilde{r}_i = 0$, $m_{2i+1} = \tilde{m}_{2i+1} = \frac{1}{2}$ for all $0 \leq i \leq j - 1$, then

$$\begin{aligned} 0 \leq r_j \;&=\; -1 + 2\zeta_{2j} + 2\zeta_{2j+1} \;=\; -(1 - m_{2j})(1 - 2m_{2j+1}) \\ &=\; -(1 - m_{2j})(2\tilde{m}_{2j+1} - 1) \;=\; -\frac{1 - m_{2j}}{1 - \tilde{m}_{2j}} \tilde{r}_j \;\leq\; 0 \end{aligned}$$

which proves $r_j = \tilde{r}_j = 0$ and $m_{2j+1} = \tilde{m}_{2j+1} = \frac{1}{2}$. Thus, if $\tilde{\psi}$ is the spectral measure of a random walk governed by $\tilde{\mathbf{Q}}_0$, then $r_j = \tilde{r}_j = 0$ for all $j = 0, 1, 2, \ldots$.

Conversely assume that $\mathbf{Q}_0$ satisfies $r_j = 0$ for all $j = 0, 1, 2, \ldots$, then it is easy to see that $m_{2j-1} = \tilde{m}_{2j-1} = \frac{1}{2}$ for all j and the proof of the existence of a probability $\tilde{\mathbf{Q}}_0$ with spectral measure $\tilde{\psi}$ reduces to the problem of finding transition probabilities $\tilde{p}_j$, $\tilde{q}_j$ ($\tilde{r}_j = 0$) such that

$$\tilde{p}_{j-1}\tilde{q}_j = (1 - \tilde{m}_{2j-2})\tilde{m}_{2j} \tag{25}$$

($\tilde{m}_0 = 0$) holds for all $j \geq 1$. Observing $m_{2j-1} = \frac{1}{2}$ it follows by an induction argument that the solution of (22) is given by

$$m_{2j} = 1 - \tilde{m}_{2j} = \frac{Q_{j-1}(1)}{Q_j(1)} q_j \quad , \quad 1 - m_{2j} = \tilde{m}_{2j} = \frac{Q_{j+1}(1)}{Q_j(1)} p_j$$

where the polynomials $Q_j(x)$ are defined in (20). Now (25) can be rewritten as

$$\tilde{p}_{j-1}\tilde{q}_j = q_{j-1}p_j \frac{Q_{j-2}(1)Q_{j+1}(1)}{Q_{j-1}(1)Q_j(1)} \quad , \quad \tilde{p}_0\tilde{q}_1 = \frac{Q_2(1)}{Q_1(1)}p_1 \ , \tag{26}$$

and putting $\tilde{p}_0 = 1$ it follows that there is at least one solution of (26), namely

$$\tilde{p}_j \ = \ \frac{Q_{j-1}(1)}{Q_j(1)}q_j \quad , \quad \tilde{p}_0 = 1 \ . \tag{27}$$

Thus $\tilde{\psi}$ is the spectral measure of $\tilde{\mathbf{Q}}_0$ with transition probabilities (27) ($\tilde{r}_j = 0$, $\tilde{q}_j = 1 - \tilde{p}_j$) and the assertion a] follows.

b] By the discussion in the proof of the converse part in a] we have to find all solutions of (26). Now the right hand side of (26) is a chain sequence with minimal parameter sequence

$$\tilde{m}_{2j} = 1 - m_{2j} = \frac{Q_{j+1}(1)}{Q_j(1)}p_j \qquad\qquad (j = 0, 1, 2, \ldots)$$

(see Chihara [2]) and therefore the problem of solving (26) reduces to the problem of finding all parameter sequences of the given chain sequence $\{(1 - \tilde{m}_{2j-2})\tilde{m}_{2j}\}_{j=1}^{\infty}$. But this problem is solved in [2] and therefore all solutions of (26) are obtained from $\tilde{q}_0(t) = t$ and

$$\tilde{q}_n(t) \ = \ \frac{1 - tS_{n-2}}{1 - tS_{n-1}} \cdot \frac{Q_{n+1}(1)}{Q_n(1)}p_n \qquad (n = 1, 2, 3, \ldots)$$

where $0 \le t \le \frac{1}{S}$ (note that there is a typo in (3.8) of [2]). This proves assertion b].

We will conclude this Section with a discussion of the situation in Section 1 where $\mathbf{P}_0$ governs the random walk with transition probabilities $r_x = 0$, $p_0 = 1$, $p_x = 1 - q_x$. Observing (20) it follows that $Q_k(1) = 1$ ($k \ge 0$) and, by Theroem 5.3, a dual of $\mathbf{P}_0$ in the sense of (18) is unique if and only if the dual $\tilde{\mathbf{P}}_0$ obtained from $\mathbf{P}_0$ by switching the p_x and q_x for $x \ge 1$ is recurrent. By an application of a result in [7] we thus obtain the following result.

Corollary 5.4. *Let $\mathbf{P}_0$ govern a reflecting random walk (X_n) on the nonnegative integers $\{0, 1, 2, \cdots\}$ started at $X_0 = 0$ with one-step upward transition probabilities p_i and one-step downward transition probabilities q_i where $q_0 = 0$, and $p_i + q_i = 1$ for all $i \ge 0$. The dual of $\mathbf{P}_0$ in the sense of (18) is unique if and only if the random walk governed by $\mathbf{P}_0$ is transient or null recurrent.*

Example 5.5. Let $\mathbf{P}_0$ govern the random walk with transition probabilities $p_0 = 1$, $p_x = p$ ($x \ge 1$), then it is easy to see that

$$S_n = \frac{1 - \rho^{n+1}}{1 - \rho} \quad (n \ge 0) \ , \quad S_{-1} = 0$$

where $\rho = p/q$. If $p \ge q$ we have $S = \infty$ (the random walk is transient or null recurrent) and there exists only one dual probability of $\mathbf{P}_0$ obtained by switching the p_x and q_x (as we did in Section 1). If $p < q$ (the random walk is positive recurrent) we have $S = (1-\rho)^{-1}$ and it follows that for every $t \in [0, 1-\rho]$ the probability $\tilde{\mathbf{Q}}_0^t$ with transition probabilities

$$\tilde{p}_n(t) = 1 - \frac{1-\rho-t(1-\rho^{n-1})}{1-\rho-t(1-\rho^n)}p\ ,\quad \tilde{p}_0(t) = 1-t\ ,\quad \tilde{p}_1(t) = 1 - \frac{p}{1-t}$$

is a dual of $\mathbf{P}_0$. Note that the special choice $t = (1-\rho)$ gives the probability $\tilde{\mathbf{Q}}_0$ with transition probailities $\tilde{p}_0 = p/q$, $\tilde{p}_x = p$ $(x \geq 1)$ as a dual of $\mathbf{P}_0$ while the choice $t = 0$ gives the dual probability $\tilde{\mathbf{P}}_0$ considered in Section 1 - 3.

6 Miscellaneous Remarks.

1. As an example where probabilities can be calculated much more easily on one side of the identity (2) than on the other, fix $0 < p < 1$, and let $\mathbf{P}_0$ govern the walk (X_n) with transition probabilities

$$p_x = \frac{p^{x+1}+q^{x+1}}{p^x+q^x};\qquad q_x = 1-p_x \quad (x = 1, 2, \cdots)$$

By Theorem 5.3 there exists a unique dual $\tilde{\mathbf{P}}_0$ of $\mathbf{P}_0$ [in the sense of (18)] which is obtained by switching the transition probabilities p_x and q_x for $x \geq 1$. Then for $n = 0, 1, 2, \cdots$

$$\tilde{\mathbf{P}}_0(T_0 > 2n) = \mathbf{P}_0(X_{2n} = 0) = (pq)^n \binom{2n}{n} \tag{28}$$

$$\tilde{\mathbf{P}}_0(X_{2n} = 0) = \mathbf{P}_0(T_0 > 2n) = 1 - \sum_{m=1}^{n} \frac{(pq)^m}{2m-1}\binom{2m}{m} \tag{29}$$

In each case the second equality follows from the fact that the $\mathbf{P}_0$ distribution of (X_n) is the distribution of $(|Y_1 + \cdots + Y_n|)$ for independent Y_i with $P(Y_i = 1) = p$, $P(Y_i = -1) = q$. This fact follows from the criterion of Rogers-Pitman [18] for a function of a Markov chain to be Markov; or see Ross [19], Proposition 4.1.1. There does not seem to be any similar representation for the $\tilde{\mathbf{P}}_0$ distribution of (X_n). So without exploiting the duality relation it is not evident why there should be such simple formulae for the $\tilde{\mathbf{P}}_0$ probabilities. Due to the representation in terms of (Y_n), there is a fairly simple formula for n-step transition matrix $\mathbf{P}_i(X_n = j)$.

2. While it does not seem obvious how to extend Wall's duality to birth-death processes in continuous time, the expression of the duality in (5) allows passage to the limit to obtain a duality relation for pairs of reflecting diffusions on $[0, \infty)$ obtained as weak limits of random walks. Compare Siegmund [20] remark (e) on p. 923. It appears that apart from boundary conditions, one obtains exactly the same pairs of dual processes in the limit from either Siegmund's or Wall's duality. The simplest special case of the diffusion result is the following fact about reflecting BM with drift. Let $\mathbf{P}_0^\delta$ govern a BM $(X_t, t \geq 0)$ on $[0, \infty)$ with drift δ and instantaneous reflection at 0. Let $L_t = \sup\{s \leq t : X_s = 0\}$. Then for all $t > 0$ the $\mathbf{P}_0^\delta$

distribution of L_t is identical to the $\mathbf{P}_0^{-\delta}$ distribution of $t - L_t$. This can also be seen easily from the well known representation of $(X_t, t \geq 0)$ with distribution $\mathbf{P}_0^{\delta}$ as $X_t = B_t - M_t$ where $(B_t, t \geq 0)$ is a BM with drift δ, $M_t = \inf_{0 \leq s \leq t} B_s$, by a simple time reversal.
3. Continuing the previous remark, it seems clear that for diffusions the analog of (2) will be a corresponding identity between the transition density $p(0,0,t)$ for one diffusion and the right tail $\tilde{\Lambda}(t,\infty)$ of the Lévy measure $\tilde{\Lambda}$ of the inverse local time process at zero for the dual diffusion. This duality is related to the question, studied by various authors, of what are the possible Lévy measures Λ for the inverse local time of a diffusion process. See e.g. Knight [12], Küchler [13].

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Figure 1: Diagrams for $n = 3$