Large deviations from the McKean-Vlasov limit for super-Brownian motion with mean-field immigration

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Abstract

A large deviation principle is proved for the mean-field measure of a sequence of n-type super-Brownian motions with a mean-field dependent immigration. The rate-function is identified as a pertubation of the rate function for a non-interacting case.

Running title: Large deviations for super-Brownian motion with mean-field immigration.

1 Introduction

In the fundamental paper [DG], Dawson and Gärtner studied large deviations from the McKean-Vlasov limit for weakly interacting $\mathbb{R}^d$-valued diffusions $(\xi^{i,n}, \ldots, \xi^{n,n})$. They considered their empirical distribution $\sum_i \delta_{\xi^{i,n}}$

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as a random function in $C([0,1],M(I\!R^d))$, where $M(I\!R^d)$ denotes a space of measures over $I\!R^d$.

We consider a sequence of N-type interacting superprocesses $(X^{i,N},\dots,X^{N,N}) \in C([0,1],M(I\!R^d)^N)$, where the interaction at time t depends only on the mean-field $\frac{1}{N}\sum_{i=1}^N X^{i,N}$ at time t. The latter random variable takes values in $C([0,1],M(I\!R^d))$ and we give a large deviation result for their distributions as N tends to infinity. As in [DG] we have to restrict to an interaction which arises by a Girsanov-transformation applied to N independent super-processes. In this case the accumulation points of $\frac{1}{N}\sum_{i=1}^N X^{i,N}$ are deterministic weak solutions $\bar{\mu}=(\mu_t)$ of the McKean-Vlasov type non-linear equation

$$\frac{\partial}{\partial t}\mu_t \quad = \quad \frac{1}{2}\Delta\mu_t + b(\mu_t)\mu_t, \tag{1.1}$$

for some interactive *immigration function* b, cf. [O]. It was studied in [COR] and [CR] from the point of view of propagation of chaos for mean-field interacting branching diffusions and associated fluctuations. In more physical terms this equation may describe a system of "particles" with creation and annihilation of masspoint, which may depend on the current state of the particle cloud, and can serve as a description of many biological systems, cf. the references in [GS] and [SP], e.g. of a model for conduction of nerve impulse.

If $b=0$ the present large deviation result is also proved in [FGK] and [Sch]. Our result includes also a new proof of their result, however in different topologies on $C([0,1],M(I\!R^d))$. Our approach emphasizes that already the process without an interactive immigration b can be seen as an empirical distribution of a cloud of interacting particles in $I\!R^d$. Therefore we adopte the techniques of [DG] and the main tools we use are the martingale problem of the interacting superprocesses and a Girsanov transformation associated with (multitype) superprocesses, cf. [O],[D]. The latter reference is also a basic reference for the theory of superprocesses.

In the following section we briefly recall the definition and construction of weakly interacting superprocesses and the Girsanov transformation for multitype superprocesses. Section 2 contains the main technical result, namely the exponential tightness of the sequence of the distributions of $\frac{1}{N}\sum_{i=1}^N X^{i,N}$. The large deviations result is stated in Theorem 4.2 in Section 4. It turns out that if b is uniformly bounded and satisfies a continuity condition the rate function $S(\bar{\mu})$ on $C([0,1], M(\mathbb{R}^d))$ equals

$$\sup_{\phi \in C_c([0,1]\times \mathbb{R}^d)} \Big(\int_0^1 \mu_t(\phi(t))dt - \log E_{\mu_0}^{b(\bar{\mu})}[\exp(\int_0^1 X_t(\phi_t)dt)] \Big), \quad (1.2)$$

where $E_{\mu_0}^{b(\bar{\mu})}$ is the expectation operator associated with a superprocess with fixed immigration function $b(\mu_t, x)$. As in [DG] this follows by a pertubation argument from the case of N independent superprocesses. In the last section we remark on a coupling argument which provides for negative immigration function b a large deviation in the same topology as in [Sch], which is stronger than the one considerd in Section 3 and 4.

2 Superprocesses with mean-field interaction

Let $N \in \mathbb{N}$ be fixed. First we want to describe an N-type superprocess with an interaction depending only on the mean-field (or intensity measure) $\frac{1}{N}\sum X^{i,N}$ of the empirical distribution $\frac{1}{N}\sum \delta_{X^{i,N}}$. We start with real-valued (resp. positive) functions $a_{ij}, d_k, 1 \le i,j,k \le d$, and b (resp. c) defined on $M(\mathbb{R}^d) \times \mathbb{R}^d$. Define for every $\mu \in M(\mathbb{R}^d)$ the operator $L(\mu)$ on $C_0^2(\mathbb{R}^d)$ by

$$L(\mu)f(x) := \sum_{i,j=1}^d a_{ij}(\mu,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d d_i(\mu,x)\frac{\partial f}{\partial x_i}(x). \qquad (2.1)$$

The operator serves as a description of the one-particle motion. The function $b(\mu,x)$ describes the immigration rate and $c(\mu,x)$ the variance in the reproduction rate at place x while $\frac{1}{N}\sum_{i=1}^N X_s^{i,N} = \mu$. Denote $(L(\mu))_{\mu\in M(\mathbb{R}^d)}$ by $\mathcal{L}$.

In order to state now the basic definition we need some notation. If $\vec{\mu} = (\mu_1,\ldots,\mu_N) \in M(\mathbb{R}^d)^N, \vec{f} = (f_1,\ldots,f_N) \in \mathcal{B}_b^{(+)}(\mathbb{R}^d)^N$, where $\mathcal{B}_b^{(+)}$ are bounded measurable (non-negative) functions, then we define $R(\vec{\mu}) := \frac{1}{N}\sum_{i=1}^N \mu_i \in M(\mathbb{R}^d)$ and $\vec{\mu}(\vec{f}) := \sum_{i=1}^N \mu_i(f_i)$, where $\mu(f) = \int f d\mu$. The exponential function $e_{\vec{f}}$ is defined by $e_{\vec{f}}(\vec{\mu}) = \exp(-\vec{\mu}(\vec{f}))$.

Definition *2.1 We call a measure $P^N \in M_1(C_{M(\mathbb{R}^d)^N})$ (the space of probability measures on the continuous paths with values in $M(\mathbb{R}^d)$) an N-type* *superprocess with mean-field interaction* $(\mathcal{L}, b, c)$ *starting at* $\vec{\mu}_0 \in M(\mathbb{R}^d)^N$ *if*

$$\begin{aligned} e_{\vec{f}}(\vec{X}_t) - e_{\vec{f}}(\vec{\mu}_0) \quad &+ \quad \int_0^t e_{\vec{f}}(\vec{X}_s) \sum_{j=1}^N X_s^j (L(R(\vec{X}_s)) f_j + \\ & b(R(\vec{X}_s)) f_j - c(R(\vec{X}_s)) f_j^2) ds \end{aligned} \tag{2.2}$$

is a P^N*-martingale for all* $\vec{f} = (f_1, \ldots, f_N)$ *with non-negative* $f_i \in C_0^2$. *The random variable* $\vec{X}$ *denotes the coordinate process on* $C_{M(\mathbb{R}^d)^N}$.

In [O] such a process is constructed as an accumulation point of a sequence of rescaled interacting branching diffusions, which themselves are limits of interacting branching random walks. If only b depends on the empirical distribution P is given by a Girsanov transformation. Because we are only interested in this case we present it in the following subsection.

2.1 Girsanov transformation for multitype superprocesses

Let us start with the distribution P of N independent super-Brownian motions $\vec{X}$ on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. There exists an $(L^2(\Omega, \mathcal{F}, P))^N$-valued orthogonal martingale measure $\vec{M} = (M_1, \ldots, M_N)$ where M_i has intensity measure $\delta_x(dy) X_t^i(dx)$, i.e.

$$< M_i(f_i), M_j(f_j) >_t = \delta_{ij} \int_0^t \int_{\mathbb{R}^d} f_i^2(x) X_s^i(dx).$$

For the notion of martingale measures we refer to [D]. If

$$\vec{b} = (b_1, \ldots, b_N), b_i : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}$$

such that b_i is M_i*-integrable* then the process

$$\begin{aligned} Z_t^{\vec{b}} \quad := \quad & \exp\{\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} b_i(s,x) M_i(ds, dx) - \\ & \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} b_i(s,x)^2 X_s^i(dx) ds\} \end{aligned} \tag{2.3}$$

is a local martingale. If $Z^{\vec{b}}$ is a martingale, e.g., if b_i is bounded from below, $1 \le i \le N$, then under the probability measure '$P^{\vec{b}} = Z \cdot P$' the process

$$e_{\vec{f}}(\vec{X}_t) - e_{\vec{f}}(\vec{\mu}_0) + \int_0^t e_{\vec{f}}(\vec{X}_s) \sum_{j=1}^N X_s^j (\Delta f_j + b_j(s) f_j - c_j f_j^2) ds \tag{2.4}$$

is a martingale. Hence we can construct a solution of (2.2) if only b depends on the mean-field $\frac{1}{N}\sum_{i=1}^N X^i$ by setting $b_i(s,x,\omega) = b(\frac{1}{N}\sum_{j=1}^N X_s^j(\omega), x), i = 1,\dots,N$.

3 Exponential Tightness

From now on $(X^{1,N},\dots,X^{N,N})$ denotes an N-type super-Brownian motion with mean-field interaction only in the immigration function b started from $(\mu_0,\dots,\mu_0), \mu_0 \in M(\mathbb{R}^d)$, defined on some probability space $(\Omega,\mathcal{F},P_{\mu_0})$. The basic issue of this section is to show that the sequence of probability measures $P_{\mu_0}\circ(t\to\frac{1}{N}\sum_{i=1}^N X_t^{i,N})^{-1}$ on $C([0,1],M(\mathbb{R}^d))$ is exponential tight. Here we equip $M(\mathbb{R}^d)$ with the vague topolopy. Extensions to other topologies on $M(\mathbb{R}^d)$ are discussed in Section 5. The first lemma reminds you how a compact set in $C([0,1],M(\mathbb{R}^d))$ with the topology defined by the supremumsnorm looks like .

Lemma 3.1 *The set*

$$\{\omega\in C([0,1],M(\mathbb{R}^d))|\sup_{t\in[0,1]} X_t(\omega)\in A, X_\cdot(\omega)(f_n)\in K_{f_n}\,\forall\, n\in\mathbb{N}\}$$

is relatively compact, if A is a relatively compact in $M(\mathbb{R}^d)$, $\{f_n\}_{n\in N}$ is a separating sequence for $M(\mathbb{R}^d)$, and K_{f_n} is relatively compact in $C([0,1],\mathbb{R})$. The set A can be of the form

$$A_L := \{\mu|\mu(1)\le L\}.$$

Proof. The proof follows as in [J]. $\diamond$

The next theorem is the main result of this section.

Theorem 3.2 *Suppose that*

$$\sup_{\mu\in M(\mathbb{R}^d)}\sup_{x\in supp(\mu)}|b(\mu,x)| =: K<\infty. \tag{3.1}$$

If $\mu_0\in M(\mathbb{R}^d)$ then for each $\rho>0$ there exists a compact set K_ρ such that

$$\limsup_{N\to\infty}\frac{1}{N}\log P_{\mu_0}[\frac{1}{N}\sum_{i=1}^N X^{i,N}\notin K_\rho]\le-\rho. \tag{3.2}$$

In other words the sequence $(P_{\mu_0}\circ(\frac{1}{N}\sum_{i=1}^N X^{i,N})^{-1})_{N\in\mathbb{N}}$ is exponential tight.

First, we show how the proof follows from the next two lemmas.

Lemma 3.3 *For each $\rho > 0$ there exists $L(\rho) > 0$ such that*

$$\limsup_{N\to\infty} N^{-1} \log P_{\mu_0}[\exists t \in [0,1] : N^{-1} \sum_{i=1}^{N} X_t^{i,N} \notin A_{L(\rho)}] \le -\rho.$$

Lemma 3.4 *For each $\Lambda, r > 0$ and $f \in C_b^2$ there exists a compact set $K_{(f,\Lambda,r)} \subset C([0,1], I\!\!R)$ such that*

$$N^{-1} \log P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N}(f) \notin K_{(f,\Lambda,r)}, \frac{1}{N}\sum_{i=1}^{N} X_t^{i,N} \in A_\Lambda \, \forall t \in [0,1]]$$
$$\le -r.$$

Proof of the Theorem. Take a dense sequence $\{f_k\}_{k\in I\!\!N} \subset C_b^2(I\!\!R^d)$ and define the compact set K_ρ by

$$\begin{aligned} K_\rho \;\; := \;\; & \{\omega \in C([0,1], M(I\!\!R^d)) : \omega(t) \in A_{L(\rho)} \forall t \in [0,1] \\ & \omega(\cdot)(f_k) \in K_{(f_k,\lambda(\rho),r(\rho))} \forall k \in I\!\!N\}, \end{aligned} \tag{3.3}$$

where $K_{(f_k,\lambda(\rho),r(\rho))}$ is an Lemma 3.4 with $\lambda(\rho), r(\rho)$ defined in the course of the proof. Then K_ρ is a compact subset of $C([0,1], M(I\!\!R^d))$ and

$$\begin{aligned} P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N} \notin K_\rho] &\le P_{\mu_0}[\exists t \in [0,1] \text{ such that } \frac{1}{N}\sum_{i=1}^{N} X_t^{i,N} \notin A_{L(\rho)}] \\ &+ \sum_{k=1}^{\infty} P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X_t^{i,N} \in A_{L(\rho)} \forall t \in [0,1], \frac{1}{N}\sum_{i=1}^{N} X_\cdot^{i,N}(f) \notin K_{(f_k,\Lambda(\rho),r(\rho))}]. \end{aligned} \tag{3.4}$$

Fix $\epsilon > 0$. For $N \ge N_\epsilon$ we have that

$$P_{\mu_0}[\exists t \in [0,1] \text{ such that } \frac{1}{N}\sum_{i=1}^{N} X_t^{i,N} \notin A_{L(\rho)}] \le e^{-N(\rho+\epsilon)} \tag{3.5}$$

by Lemma 3.3. Choosing $\Lambda(\rho) = L(\rho)$ and $r(\rho) = \rho k$ we are led to

$$P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X_t^{i,N} \in A_{L(\rho)} \, \forall t \in [0,1], [\frac{1}{N}\sum_{i=1}^{N} X_\cdot^{i,N}(f) \notin K_{(f_k,\Lambda(\rho),r(\rho))}]$$
$$\le e^{-N\rho k}.$$

Hence for $N \geq N_\epsilon$ we have that

$$P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N} \notin K_\rho] \leq e^{-N(\rho+\epsilon)} + \frac{e^{-N\rho}}{1-e^{-N\rho}}$$

and therefore

$$\begin{aligned} & \limsup_{N\to\infty} \frac{1}{N}\log(e^{-N(\rho+\epsilon)} + \frac{e^{-N\rho}}{1-e^{-N\rho}}) \\ = \quad & \max\{-(\rho+\epsilon), \rho - \limsup_{N\to\infty}\frac{1}{N}\log(1-e^{-N\rho})\} \leq -\rho+\epsilon. \end{aligned}$$

Because ϵ was arbitrary small the Theorem is proved. $\diamond$

Now we have to prove the two lemmas.

Basic for all calculations are the martingales

$$\begin{aligned} M_t^N[K,f] \quad := \quad & \frac{1}{N}\sum_{i=1}^{N} X_t^{i,N}(fe^{-Kt}) - \mu_0(f) - \\ & \int_0^t \frac{1}{N}\sum_{i=1}^{N} X_s^{i,N}(e^{-Ks}(\Delta f - Kf + b(\frac{1}{N}\sum_{i=1}^{N} X_s^{i,N})f))ds, \end{aligned} \tag{3.6}$$

which are defined for each $K \in [0,\infty)$ and $f \in C_b^2(I\!R^d)$. They have quadratic variation

$$\int_0^t \frac{1}{N^2}\sum_{i=1}^{N} X_s^{i,N}(f^2 e^{-2Ks})ds. \tag{3.7}$$

Proof of Lemma 3.3.

$$\begin{aligned} & P_{\mu_0}[\sup_{t\leq 1}\frac{1}{N}\sum_{i=1}^{N} X_t^{i,N}(1) \geq L] \\ \leq \quad & P_{\mu_0}[\sup_{t\leq 1}\frac{1}{N}\sum_{i=1}^{N} X_t^{i,N}(e^{-K_1 t}1) \geq Le^{-K_1}]. \end{aligned}$$

If we choose now $K_1 = \sup_\mu \sup_{x\in supp(\mu)} b(\mu,x) + 1$ then $\mu(-K_1 + b(\mu) + \frac{1}{2}e^{-K_1 s}) \leq 0$ and hence the right-hand side of the last inequality is bounded by

$$
\begin{aligned}
& P_{\mu_0}[\sup_{t\le 1} \frac{1}{N}\sum_{i=1}^N X_t^{i,N}(e^{-K_1 t}1) - \mu_0(1) - \qquad\qquad (3.8)\\
& \quad \int_0^t (\frac{1}{N}\sum_{i=1}^N X_s^{i,N}(-K_1 + b(\frac{1}{N}\sum_{j=1}^N X_s^{j,N}))e^{-K_1 s})ds \ge Le^{-K_1} - \mu_0(1)]\\
= \; & P_{\mu_0}[\sup_{t\le 1}(M_t^N[K_1,1] - \frac{N}{2} < M^N[K_1,1]>_t) \ge Le^{-K_1} - \mu_0(1)]\\
= \; & P_{\mu_0}[\sup_{t\le 1}\exp(NM_t^N[K_1,1] - \frac{N^2}{2} < M^N[K_1,1]>_t) \ge e^{N(Le^{-K_1}-\mu_0(1))}]\\
\le \; & e^{-N(Le^{-K_1}-\mu_0(1))},
\end{aligned}
$$

where the last inequality follows by Doob's inequality applied to the exponential supermartingale $\exp(NM^N[K_1,1] - \frac{1}{2} < NM^N[K_1,1]>)$. This completes the proof of the Lemma 3.3. $\diamond$

Proof of Lemma 3.4. We have to compute

$$P_{\mu_0}[\frac{1}{N}\sum_{i=1}^N X^{i,N}(f) \notin K_f, \sup_{t\le 1}\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(1) \le \Lambda] \qquad (3.9)$$

with the relatively compact set $K_f :=$

$$\{x \in C([0,1], \mathbb{R}) : |x(0)| \le ||f||, \sup_{0\le u<v\le 1, v-u\le\delta_n} |x(u) - x(v)| \le \eta_n\}$$

where the sequences of positive numbers (η_n) and (δ_n) converge to zero and will be specified in the proof. For now consider fixed η and δ. We proceed as in [DG, Section 5.3]. By the triangle-inequality (3.9) is bounded by

$$\sum_{k=0}^{[\frac{1}{\delta}]-1} P_{\mu_0}[\sup_{k\delta\le t\le(k+2)\delta\wedge 1} |\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(f) - X_{k\delta}^{i,N}(f)| > \frac{\eta}{2}, \sup_{t\le 1}\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(1) \le \Lambda].$$

By the Markov property this is bounded by

$$\frac{1}{\delta}\sup_{\mu(1)\le\Lambda} P_\mu[\sup_{k\delta\le t\le(k+2)\delta\wedge 1} |\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(f) - \mu(f)| > \frac{\eta}{2}, \sup_{t\le 2\delta}\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(1) \le \Lambda].$$

The assumption (3.1) yields

$$\mu(|\Delta f + b(\mu)f| + \frac{1}{2}f^2) \leq K_{f,\Lambda} \tag{3.10}$$

with some constant $K_{f,\Lambda}$. This implies that for every $\gamma > 0$ and every μ with $\mu(1) \leq \Lambda$

$$\begin{aligned} |\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(f) - \mu(f)| &\leq (1+\gamma)K_{f,\Lambda}t + \int_0^t \frac{1}{N}\sum_{i=1}^N X_s^{i,N}(\Delta f + \\ & b(\frac{1}{N}\sum_{j=1}^N X_s^{j,N})f ds - \frac{\gamma N}{2}\frac{1}{N^2}\int_0^t \sum_{i=1}^N X_s^{i,N}(f^2)ds \\ &= (1+\gamma)K_{f,\Lambda}t + M_t^N[f,0] - \frac{\gamma N}{2} < M^N[f,0] >_t . \end{aligned}$$

Hence, Doob's inequality applied to the supermartingale $\exp(\gamma N M^N[f,0] - \frac{1}{2} < \gamma N M^N[f,0] >)$ gives

$$\begin{aligned} & \sup_{\mu(1)\leq\Lambda} P_\mu[\sup_{k\delta\leq t\leq(k+2)\delta\wedge 1} \frac{1}{N}\sum_{i=1}^N X_t^{i,N}(f) - \mu(f) > \frac{\eta}{2}, \sup_{t\leq 2\delta}\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(1) \leq \Lambda] \\ & \leq \exp(-N\gamma(\frac{\eta}{2} - (1+\gamma)K_{f,\lambda}2\delta)). \end{aligned} \tag{3.11}$$

Minimizing now the right-hand side of inequality (3.11) with respect to γ we obtain that (3.11) is bounded by

$$\exp(-N\frac{(\eta - 4K_{f,\Lambda}\delta)^2}{32K_{f,\Lambda}\delta}). \tag{3.12}$$

With the same arguments we can show that

$$\begin{aligned} & \sup_{\mu(1)\leq\Lambda} P_\mu[\sup_{k\delta\leq t\leq(k+2)\delta\wedge 1} \frac{1}{N}\sum_{i=1}^N X_t^{i,N}(f) - \mu(f) > \frac{\eta}{2}, \sup_{t\leq 2\delta}\frac{1}{N}\sum_{i=1}^N X_t^{i,N}(1) \leq \Lambda] \\ \leq \; & \exp(-N\frac{(\eta - 4K_{f,\Lambda}\delta)^2}{32K_{f,\Lambda}\delta}). \end{aligned}$$

Hence we haved proved that (3.9) is bounded by

$$\frac{2}{\delta}\exp(-N\frac{(\eta - 4K_{f,\Lambda}\delta)^2}{32K_{f,\Lambda}\delta}) \tag{3.13}$$

forall $\delta \in (0, \frac{1}{2})$ and $\eta \geq 4K_{f,\Lambda}\delta$. Now consider a sequence $(\delta_k)_{k\in I\!N} \in (0, \frac{1}{2})$ and $\eta_k > 4K_{f,\Lambda}\delta_k$, $k \in I\!N$. The bound (3.13) yields that (3.9) is bounded by

$$\sum_{k=1}^{\infty} \frac{2}{\delta_k} \exp(-N \frac{(\eta_k - 4K_{f,\Lambda}\delta_k)^2}{32K_{f,\Lambda}\delta_k}). \tag{3.14}$$

If we choose $\delta_k = \frac{1}{2}k^{-2}$, $\eta_k = 10K_{f,\Lambda}(\frac{r}{k})^{\frac{1}{2}}$ the expression (3.14) is bounded by e^{-Nr} which finishes the proof of Lemma 3.4. $\diamond$

Remark: By the assumption of boundedness on b the proofs of Lemma 3.3 and Lemma 3.4 and, as a consequence, the proof of exponential tightness work also if we replace the interactive immigration $b(\frac{1}{N}\sum_{i=1}^N X_s^{i,N}, x)$ by a non-interactive immigration $b(\mu_s, x)$ associated with a *frozen* path $\bar{\mu} = (\mu_s)_{s\in[0,1]} \in C([0,1], M(I\!R^d))$.

4 The large deviations result

Fix a bounded real-valued function $b : M(I\!R^d) \times I\!R^d$. For two elements $\bar{\mu}^{(i)} \in C([0,1], M(I\!R^d)), i = 1, 2$, we define first the rate function along a *frozen* path $\bar{\mu}^{(1)}$.

Definition *4.1*

$$\begin{aligned} S^{\bar{\mu}^{(1)}}(\bar{\mu}^{(2)}) \quad := \quad & \sup_{\phi \in C_c([0,1]\times I\!R^d)} \Big(\int_0^1 \mu_t^{(2)}(\phi(t))dt - \\ & \log E_{\mu_0}^{b^{\bar{\mu}^{(1)}}} [\exp(\int_0^1 X_t(\phi_t)dt)] \Big), \end{aligned} \tag{4.1}$$

where $E_{\mu_0}^{b^{\bar{\mu}^{(1)}}}$ is the expectation associated with $P_{\mu_0}^{b^{\bar{\mu}^{(1)}}}$, the superprocess starting from μ_0 with non-interactiv immigration function $b^{\bar{\mu}^{(1)}}(s,x) = b(\mu_s^{(1)}, x)$. For $\bar{\mu} \in C([0,1], M(I\!R^d))$ we abbreviate $S^{\bar{\mu}}(\bar{\mu})$ by $S(\bar{\mu})$.

Here is the main result of the paper.

Theorem 4.2 *Suppose that (3.1) holds and that the map*

$$\bar{\mu} \to \int_0^1 \mu_s((b(\mu_s^{(1)}) - b(\mu_s))^2)ds \tag{4.2}$$

is continuous on $C([0,1], M(\mathbb{R}^d))$ for every flow $\bar{\mu}^{(1)} \in C([0,1], M(\mathbb{R}^d))$. Then for all open sets $G \subset C([0,1], M(\mathbb{R}^d))$ we have the lower bound

$$\begin{aligned} & \liminf_{N\to\infty} \frac{1}{N} \log P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N} \in G] \\ \geq & \; -\inf\{S(\bar{\mu}) : \bar{\mu} \in G, \bar{\mu}_0 = \mu_0\} \end{aligned} \tag{4.3}$$

and for all closed sets $F \subset C([0,1], M(\mathbb{R}^d))$ we have the upper bound

$$\begin{aligned} & \limsup_{N\to\infty} \frac{1}{N} \log P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N} \in G] \\ \leq & \; -\inf\{S(\bar{\mu}) : \bar{\mu} \in F, \bar{\mu}_0 = \mu_0\}. \end{aligned} \tag{4.4}$$

By exponential tightness proved in Section 3 the Theorem follows from the local large deviation principle stated in the next theorem, cf. [DG, p.295].

Theorem 4.3 *Assume (3.1) and (4.2).*

a. For each open neighborhood V of $\bar{\mu}$ we have

$$\liminf_{N\to\infty} \frac{1}{N} \log P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N} \in V] \geq -S(\bar{\mu}). \tag{4.5}$$

b. For each $\gamma > 0$ there exists an open neighborhood V of $\bar{\mu}$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log P_{\mu_0}[\frac{1}{N}\sum_{i=1}^{N} X^{i,N} \in V] \leq -S(\bar{\mu}) + \gamma. \tag{4.6}$$

The proof of this local result is, similarly as in [DG], carried out by a Girsanov transformation. Fix $\bar{\mu} \in C([0,1], M(\mathbb{R}^d))$ and let $P^{\bar{\mu},N}_{\mu_0}$ be the distribution of N independent super-Brownian motions with immigration function $b^{\bar{\mu}}(s,x) = b(\mu_s, x)$, which can be defined as the unique measure on $C([0,1], M(\mathbb{R}^d)^N)$ such that for all $f \in C_b^2(\mathbb{R}^d), i = 1, \ldots, N$, the processes

$$M_t^{i,\bar{\mu}}(f) \;:=\; X_t^i(f) - \mu_0(f) - \int_0^t X_s^i(\Delta f + b^{\bar{\mu}}(s) f) ds \tag{4.7}$$

are martingales with (co)variation

$$< M^{i,\bar\mu}(f), M^{j,\bar\mu}(g) >_t = \delta_{ij} \int_0^t X^i(fg)ds. \tag{4.8}$$

The density of the N-type process with mean-field interacting immigration $P_{\mu_0}^N$ with respect to the N independent super-Brownian motion with an immigration caused by the frozen path $\bar\mu$ is then given by

$$\frac{dP_{\mu_0}^N}{dP_{\mu_0}^{\bar\mu,N}} = \exp(M_1^{\bar\mu,N} - \frac{1}{2} < M^{\bar\mu,N} >_1), \tag{4.9}$$

where $M^{\bar\mu,N}$ is a martingale with quadratic variation

$$\int_0^t \sum_{i=1}^N X_s^i \Big((b(\frac{1}{N}\sum_{i=1}^N X_s^i) - b(\mu_s))^2 \Big) ds, \tag{4.10}$$

cf. [D,O] and Section 2. By assumption (4.2) we can proceed as in [DG, Section 5.4], if we can show that there is a large deviations result for $Q_N := P_{\mu_0}^{\bar\mu,N} \circ \left(\frac{1}{N}\sum_{i=1}^N X^i\right)^{-1}$ with rate function $S^{\bar\mu}$. We can argue as follows (, cf. also [Sch]). The exponetial tightness of (Q_N) follows by the Remark in Section 3. By a general Cramer-type large deviations result [DZ, Theorem 6.1.3], the sequence $(Q_N)_{N\in \mathbb{N}}$ then satisfies a large deviation principle with rate function

$$J^{\bar\mu}(\bar\nu) = \sup\{< \lambda, \bar\nu > - \log \int_{C([0,1],M(\mathbb{R}^d))} e^{<\lambda,\cdot>} dP_{\mu_0}^{\bar\mu,1}\},$$

where the sup is taken over all λ in the topological dual of $C([0,1], M(\mathbb{R}^d)) - C([0,1], M(\mathbb{R}^d))$. In order to identify $J^{\bar\mu}$ with $S^{\bar\mu}$ we embed $C([0,1], M(\mathbb{R}^d))$ continuously into the set of positive Radon measures $M([0,1]\times\mathbb{R}^d)$ (equipped with the vague topology) via the map $\mathcal{I}$:

$$< \psi, \mathcal{I}(\bar\nu) > := \int_0^1 \nu_s(\psi_s)ds, \tag{4.11}$$

$\psi \in C_c([0,1]\times\mathbb{R}^d)$. By [DZ, Theorem 6.1.3] $(Q_N \circ \mathcal{I}^{-1})_{N\in\mathbb{N}}$ satisfies a weak large deviation principle with rate function $S^{\bar\mu}$ (this is the Fenchel-Legendre transform associated with $(Q_N \circ \mathcal{I}^{-1})_{N\in\mathbb{N}}$). Because $\mathcal{I}$ is continuous we can apply the contraction principle [DZ, Theorem 4.2.1] in order to show that $(Q_N \circ \mathcal{I}^{-1})_{N\in\mathbb{N}}$ satisfies indead a (full) large deviations principle with rate function $\inf_{\nu:\tilde{\nu}=\mathcal{I}(\nu)} J(\nu)$. By the injectivity of $\mathcal{I}$ and the uniqueness of a rate function [DZ, Lemma 4.1.4], we have $J^{\bar\mu} = S^{\bar\mu}$. This proves the large deviation principle with the right rate function.

Now we can prove assertion a. of the Theorem with the same techniques as in [DG, Section 5.4].

Without loss of generality we may assume $S(\bar\mu) < \infty$. Let $\gamma > 0$ arbitrary. It is enough to show that the left-hand side of (4.5) is bounded from below by $-S(\bar\mu) - \gamma$. Let us choose p, q, and $\delta > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{2}(1+\frac{p}{q})\delta + pS(\bar\mu) \leq S(\bar\mu) + \gamma$. By the Remark at the end of Section 3 we can apply Lemma 3.3 to $P^{\bar\mu,N}_{\mu_0}$. Hence there exists $L > 0$ such that

$$\limsup_{N\to\infty} N^{-1} \log P^{\bar\mu,N}_{\mu_0}[\exists t \in [0,1] : N^{-1}\sum_{i=1}^{N} X^{i,N}_t \notin A_L] \leq -S(\bar\mu) - \gamma \quad (4.12)$$

By (4.2) and (4.10) there exists a neighborhood W of $\bar\mu$ in $C([0,1], M(\mathbb{R}^d))$ such that $W \cap \{\mu_s \in A_L, \forall s \in [0,1]\} \subset V$ and $< M^{\bar\mu,N} >_1 < N\delta$ on $\{\frac{1}{N}\sum X^i \in W \cap \{\mu_s \in A_L, \forall s \in [0,1]\}\}$. Hence, by Girsanov's transformation and Hölder's inequality we obtain that $P_{\mu_0}[\frac{1}{N}\sum_{i=1}^N X^{i,N} \in V]$ is bounded from below by

$$e^{-\frac{1}{2}(1+\frac{p}{q})\delta N} E^{\bar\mu,N}_{\mu_0}[\exp(-M^{\bar\mu,N}_1 - \frac{1}{2} < \frac{p}{q} M^{\bar\mu,N} >_1)]^{-\frac{p}{q}} \times$$
$$P^{\bar\mu,N}_{\mu_0}[\frac{1}{N}\sum X^i \in W \cap \{\mu_s \in A_L, \forall s \in [0,1]\}]^p.$$

Because $\exp(-M^{\bar\mu,N} - \frac{1}{2} < \frac{p}{q} M^{\bar\mu,N} >)$ is a supermartingale under $P^{\bar\mu,N}_{\mu_0}$ we have that

$$\begin{aligned} & P^{N}_{\mu_0}[\frac{1}{N}\sum X^i \in V] \\ \geq \; & e^{-\frac{1}{2}(1+\frac{p}{q})\delta N} P^{\bar\mu,N}_{\mu_0}[\frac{1}{N}\sum X^i \in W \cap \{\mu_s \in A_L, \forall s \in [0,1]\}]^p. \end{aligned} \quad (4.13)$$

By the large deviations result for $P^{\bar\mu,N}_{\mu_0}$, inequality (4.12) and the choice of p, q, and δ we obtain (4.5).

Part b. of the Theorem follows in the same way along the lines in [DG, Section 5.4].

5 A coupling argument

Now we want to prove a large deviation result for a different topology on $M(\mathbb{R}^d)$ or rather on a different space. For $p \in \mathbb{N}$ let $M_p(\mathbb{R}^d)$ be the space of all postive Radon measures μ on $\mathbb{R}^d$ such that

$$\int \phi_p d\mu < \infty \qquad \text{with } \phi_p(x) := (1 + |x|^2)^{-p/2}, \tag{5.1}$$

equiped with the topology generated by the mappings

$$\mu \to \mu(f), \qquad f \in C_c(\mathbb{R}^d) \cup \{\phi_p\}.$$

On this space a large deviation result is proved in [Sch] for the empirical mean of N-independent super-Brownian motions.

In order to accomplish a large deviations for super-Brownian motions with a mean-field immigration in this topology we are faced with the difficulty to prove an analogue result to Lemma 3.3.

Lemma 5.1 *Let $\mu_0 \in M_p(\mathbb{R}^d), p > d$.For each $\rho > 0$ there exists $L(\rho) > 0$ such that*

$$\limsup_{N\to\infty} N^{-1} \log P_{\mu_0}[\exists t \in [0,1] : N^{-1} \sum_{i=1}^{N} X_t^{i,N} \notin A^p_{L(\rho)}] \leq -\rho. \tag{5.2}$$

Now A^p_L is a typical compact set in $M_p(\mathbb{R}^d)$ like

$$A^p_L := \{\nu | \nu(\phi_p) \leq L, \nu(\mathbf{1}_{U^c_{R_k}} \phi_p) \leq \frac{L}{a_k} \forall k\}, \tag{5.3}$$

where $\{R_k\}_{k\in\mathbb{N}}, \{a_k\}_{k\in\mathbb{N}} \subset [0,\infty)$ tend to infinity as $k \to \infty$ and U_R is the ball with radius R in $\mathbb{R}^d$.

The method of the proof of Lemma 3.3 can not be carried over to the present case. But it seems to be clear that Lemma 5.1 holds if the interacting process is *smaller* than the non-interacting process. The next theorem which is essentially from Steve Evans and Ed Perkins [EP] shows, that it is possible to find an interacting process which is almost surely smaller then a non-interacting super-Brownian motion. For this theorem as well as for the large deviations result we have to assume that there exists $B > 0$ such that

$$-B \leq b(\mu, x) \leq 0 \quad \forall x \in \operatorname{supp}(\mu) \quad \forall \mu. \tag{5.4}$$

The immigration caused by b is then called an emigration.

Proposition 5.2 *Let $a : M_p(I\!R^d)^N \times I\!R^d \times [0,\infty) \to [-B, 0]$ be a measurable function. Then there are two $M_p(I\!R^d)^N$-valued processes X and X^a defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, I\!P)$, such that $X_s^i(f) \geq X_s^{i,a}(f), \forall s \in [0,1], f \geq 0$ measurable $I\!P$-a.s. Moreover all $X^i, i = 1, \ldots, N$ are super-Brownian motions independent from each other and X^a is an N-type super-Brownian motion which the same interacting emigration function a for each component $X^{a,i}$ $i = 1, \ldots, N$.*

Proof. If $N = 1$ and $B = 1$ we can take $I\!P$ to be the historical process H on its canonical space over the one-particle motion on $I\!R^d \times [0,1]$ defined by the independent product of a Brownian motion and an uniform Poisson process on $[0,1] \times [0,1]$. Then in [EP], there are sets $A_t = A_t(a)$ constructed such that the process H^a defined by $H_t^a(F) := H_t(\mathbf{1}_{A_t} F)$ is a historical process with emigration function defined by a. The projection ΠH^a of H^a to a $M(I\!R^d)$-valued process, i.e. $\Pi H_t^a(f) := H_t^a(F_t^f)$ with $F^f(\omega) = f(\omega(t)), \omega \in C([0,1], I\!R^d)$, is a super-Brownian motion with emigration function a. Obviously, $\Pi H_t^a(f) \leq \Pi H_t(f)$, for $f \geq 0$. Because ΠH is a super-Brownian motion the assertion is proved for $B = N = 1$. The case for general B is obtained by scaling properties of (super)-Brownian motion from the result for $B = 1$. Finally, an extension to multitype processes is straightforward, cf. [O]. $\diamond$.

Proof of Lemma 5.1.

$$\begin{aligned}
& P_{\mu_0}[\exists t \in [0,1] : N^{-1} \sum_{i=1}^N X_t^{i,N} \notin A^p_{L(\rho)}] \\
\leq \;& P_{\mu_0}[\sup_{t\leq 1} \frac{1}{N} \sum_{i=1}^N X_t^{i,N}(\phi_p) \geq L] + \sum_{k=1}^\infty P_{\mu_0}[\sup_{t\leq 1} \frac{1}{N} \sum_{i=1}^N X_t^{i,N}(k^2 \mathbf{1}_{U^c_{R_k}}) \geq kL] \\
& \text{(By Proposition 5.2)} \\
\leq \;& P^0_{N\mu_0}[\sup_{t\leq 1} X_t(\phi_p) \geq NL] + \sum_{k=1}^\infty P^0_{N\mu_0}[\sup_{t\leq 1} X_t(k^2 \mathbf{1}_{U^c_{R_k}}) \geq NkL],
\end{aligned} \tag{5.5}$$

where $P^0_\mu \in M_1(C([0,1], M_p(I\!R^d)))$ is the distribution of the super-Brownian motion starting from $\mu \in M_p(I\!R^d)$ and X is the coordinate process on $C([0,1], M_p(I\!R^d))$. It is shown in [Sch] that the "$\limsup\limits_{N\to\infty} \frac{1}{N} \log$ of (5.5)" is bounded by $-\rho$ for a suitable $L = L(\rho)$. $\diamond$

For the proof of a large deviations result as in Theorem 4.2 it remains to show that for $f \in C_b(I\!R^d)$, for which $\lim_{|x|\to\infty} f(x)/\phi_p(x)$ exists, the distributions of $\frac{1}{N}\sum X^{i,N}_{\cdot}(f)$ under P_{μ_0} restricted to A^p_L are exponential tight. This follows as in Lemma 3.4.

Remark. It is also possible to consider topologies as in [DG] defined by an unbounded Lyapunov function $\phi \geq 0$ (, e.g. $\phi(x) = 1 + |x|^2$). Instead of $M_{(p)}(I\!R^d)$ we can take the set $M_\phi(I\!R^d) := \{\mu \in M(I\!R^d)\} | \mu(\phi) < \infty\}$ and prove an analogue results as in Theorem 3.2 , 4.2 under conditions on b which are related to ϕ instead to the function 1.

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