Non-linear superprocesses

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January 27, 1995

Abstract

Non-linear martingale problems in the McKean-Vlasov sense for superprocesses are studied. The stochastic calculus on historical trees is used in order to show that there is a unique solution of the non-linear martingale problems under Lipschitz conditions on the coefficients.

Mathematics Subject Classification (1991): 60G57, 60K35, 60J80.

1 Introduction

Non-linear diffusions, also called McKean-Vlasov processes, are diffusion processes which are associated with non-linear second order partial differential equation. $\mathbb{R}^d$-valued McKean-Vlasov diffusions are studied in detail in many papers, e.g. [F,Oel,S1,S2]. The main issues are approximation by a sequence of weakly interacting diffusions, associated large deviations and fluctuations

*Supported by an EC-Fellowship under Contract No. ERBCHBICT930682 and partially by the Sonderforschungsbereich 256.

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and finally uniqueness and existence of the non-linear martingale problem associated with McKean-Vlasov process.

In this paper we focus on the latter question in the set-up of branching measure-valued diffusions processes, also called superprocesses. For an excellent introduction to the theory of superprocesses we refer to [D]. In order to formulate the basic definition we need to introduce some notation. The space of finite (resp. probability) measures over a Polish space E is denoted by $M(E)$ (resp. $M_1(E)$) and is equipped with the weak topology. The space of continuous (resp. càdlàg) $E-$valued paths is denoted by C_E (resp. D_E) and $C_b(E)$ is the set of bounded continuous functions on E. The expression $\mu(f)$ with $\mu \in M_{(1)}(E)$ means $\int f d\mu$.

Definition *1.1*

- *Let $\mathcal{L} = (L(m), \mathcal{D})_{m \in M_1(M(E))}$ be a family of linear operators with common domain $\mathcal{D} \subset C_b(E)$, b, c measurable functions on $M_1(M(E)) \times E$ with $c \geq 0$. The function b is called immigration function and the function c measures the variance in the branching behavior.*

- *Fix $\nu \in M(E)$. A measure P_ν on $(C_{M(E)}, \mathcal{F}, \mathcal{F}_t)$ with canonical filtration $\mathcal{F}_t$ and σ-algebra $\mathcal{F}$ generated by the coordinate process X is called a non-linear superprocess with parameter $(\mathcal{L}, b, c)$ started from ν, if for each $f \in \mathcal{D}$ the process $M(f)$ defined by*

$$\begin{aligned} M_t(f) \quad &:= \quad X_t(f) - \nu(f) \\ &\quad - \int_0^t X_s(L(P \circ X_s^{-1})f + b(P \circ X_s^{-1})f) ds \end{aligned} \tag{1.1}$$

is a local martingale with increasing process

$$\int_0^t \int_E f^2(x) c(P \circ X_s^{-1}, x) X_s(dx) ds, \tag{1.2}$$

where $P_\nu \circ X_s^{-1} \in M_1(M(E))$ denotes the distribution of X_s under P_ν.

In terms of partial differential equation the flow of the one-dimensional marginals $u_s := P_\nu \circ X_s^{-1}$ of a solution of the non-linear martingale problem (1.1,1.2) solves the (weak) non-linear equation

$$\dot{u}_s \quad = \quad \mathcal{A}^*(u_s) u_s, \tag{1.3}$$

where for nice functions F on $M(E)$

$$\mathcal{A}(m)F(\mu) \;=\; \mu\Big(L(m)\nabla_{.}F(\mu)+b(m)\nabla_{.}F(\mu)+c(m)\nabla^{(2)}_{.}F(\mu)\Big) \quad (1.4)$$

with $\nabla_x F(\mu) := \lim_{\epsilon\downarrow 0}\frac{F(\mu+\epsilon\delta_x)-F(\mu)}{\epsilon}$. This is one motivation for the study of non-linear superprocesses from the point of view of partial differential equations. Another motivation is a kind of Law of Large Numbers for weakly interacting N-type superprocesses, which provides also an proof of the existence of a non-linear superprocess. A weakly interacting N-type superprocesses $\vec{X}^N = (X^1,\dots,X^N) \in C_{M(E)^N}$ is characterized by the martingale property of the processes

$$\Big(e_{\vec{f}}(\vec{X}^N_t) - e_{\vec{f}}(\vec{X}^N_0) \;+\; \int_0^t e_{\vec{f}}(\vec{X}^N_s)\sum_{j=1}^N X^j_s\big(L(\frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}_s})f_j + b(\frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}_s})f_j - c(\frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}_s})f_j^2\big)ds\Big)_{t\geq 0}, \quad (1.5)$$

where $e_{\vec{f}}(\vec{\mu}^N) := \exp(-\sum_{i=1}^N \mu_i(f_i))$ for $\vec{\mu}^N = (\mu_1,\dots,\mu_N) \in M(E)^N$ and $\vec{f} = (f_1,\dots,f_N) \in C_b(E)^N$. The actual proof the approximation result is based on the *Propagation of Chaos* techniques, cf. [S1,S2]. It needs some machinery on tightness of measure-valued processes. I state the result and an outline of the proof in the appendix. For details I refer to [O1]. In the accompanying papers [O1,O2] I study the large deviations and the fluctuations associated with the approximation if the weakly interacting superprocesses are superprocesses with *mean-field interaction.*

The main result of the present paper is the proof that there is a unique solution to (1.1),(1.2) under Lipschitz conditions on the parameter $(\mathcal{L}, b, c)$. The proof relies on the fact that for two superprocesses $P^i, i = 1,2$, with different parameters there exits a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathcal{F}}, I\!P)$ on which we can define processes X^i with distribution $P^i, i = 1,2$. This follows from the stochastic calculus along historical trees, recently developed by Steven N. Evans and Ed A. Perkins [P1,P2,EP]. Once this is established the proof of existence and uniqueness is carried out by a Picard-Lindelöf approximation.

Basically, there are two different cases. First, if only b depends on $m \in M_1(M(E))$ then $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathcal{F}}, I\!P)$ is the canonical space of a marked historical process as in [EP], cf. Theorem A in Section 2. If all parameters depend on $m \in M_1(M(E))$, we assume that $L(m)$ is a nice differential operator on $\mathbb{R}^d$ and that the coefficients of L, b and c are strongly related, cf. Theorem 3.1 and Theorem B in Section 3. Then we can choose $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathcal{F}}, \mathbb{P})$ as the canonical space of the historical Brownian motion.

In both cases it turns out that the historical process plays the same role for non-linear superprocesses as the Brownian motion plays for non-linear diffusions on $\mathbb{R}^d$, namely as a *driving term for strong stochastic equations.* The fundamental role of the historical process also becomes apparent in several other papers, e.g. in [P1,P2], where interacting measure-valued processes are considered, in [EP], where a Clark-type formula for measure-valued processes is proved, in [LG], where the connections to Brownian excursions are investigated, and in [Dy2] , where the relations to quasi-linear partial differential equation are explored.

2 Non-linearity in the immigration function

In this section we consider the case in which $L(m) = L$ is a generator of a time-homogeneous Hunt process independent of m and $c = 1$. Hence the non-linearity appears only in the immigration function b. Because we need the historical process from now on I will shortly describe it.

2.1 Historical process

The historical process over a one-particle motion ξ, e.g. over a Hunt process with state space E, can be seen as the superprocess constructed over the *path-process* of the one-particle motion. A path process is a path-valued process and evolves from a path of length s to a path of length $t > s$ by pasting on the given path ξ^s a new path of length $t-s$, which is distributed as the underlying one-particle motion started from $\xi^s(s)$. By construction this is a time-inhomogeneous Markov process with state-space D_E and it has a generator $(L^h, D(L^h))$ in the sense of martingale problems, cf. e.g.[P1,P2]. If we superpose a critical branching mechanism to this path-process and take the usual "superprocess limit" we arrive at the historical process, which can then be viewed as the solution of the martingale problem described in (1.1),(1.2) with $c(m) = 1, b(m) = 0, L(m) = L^h$. It is called "historical" because every particle carries all the information about the places it and its anchestor visited. Additionally one can reconstruct from this information the genealogy of a present particle by investigation of the overlap of the paths of two different particles. Because we only use the historical process as a tool and we will not prove theorems about it we will omit an exact definition and refer to [D, Sect. 12] or [DP, P1, P2, Dy1].

2.2 Superprocesses with emigration as functionals of the marked historical process

Let $X = (Y, N) \in D(D(E \times [0,1]))$ denote the path process of the Hunt process ξ generated by L and an independent Poisson process with uniform jumps on $[0,1]$, (i.e., N is the path process of a Poisson point measure on $[0,\infty) \times [0,1]$ with intensity $ds \times dx$). Denote by $\mathbb{P}$ the distribution of the superprocess G over the one-particle motion (Y, N) starting from G_0, i.e., the historical process over the Huntprocess ξ and an independent Poisson process. $\mathbb{P}$ is a measure on $\Omega := C([0,\infty), M(D_{E\times[0,1]}))$ equipped with the canonical filtration $\mathcal{F}_t$ and canonical σ-algebra. The process G is now the canonical process on Ω. Let us denote by $x = (y, n)$ a generic element in $D_{E\times[0,1]}$. Let n also denote the point measure $\sum_{s\le t, n_s \ne n_{s-}} \delta_{s, n_s - n_{s-}}$ on $[0,\infty) \times [0,1]$.

Let b be a predictable function from $[0,\infty) \times E \times \Omega$ to $[0,1]$, the candidate for the emigration term. (Because in Proposition 2.2 and finally in Theorem 2.4 we consider the martingale problem (1.1),(1.2) with $-b$ instead of b, we view b now as an emigration rather then an immigration function.)

In order to meet the formulation of [EP] we define the $[0,1]$-valued function β on $[0,\infty) \times D_E \times \Omega$ by

$$\beta(s, y, \omega) \;=\; b(s, y(s), \omega). \tag{2.1}$$

Further we define the following functions

$$A(t, x, \omega) \;=\; n(\{(s,z) \in]0,t[\times [0,1] \,|\, \beta(s,y,\omega) > z\}) \tag{2.2}$$
$$B(t, x, \omega) \;=\; \mathbf{1}_{\{A=0\}}(t, x, \omega). \tag{2.3}$$

Let K be the martingale measure of the historical process G (for the definition of martingale measures for measure-valued processes cf. [D, Sect. 7] and for historical processes a definition can be found in [P1]). Then we can define a new measure on Ω having the local densities $\frac{d\mathbb{P}^\beta}{d\mathbb{P}}|_{\mathcal{F}_t} =$

$$\begin{aligned} R_t^\beta &= \exp\{\int_0^t \int_{D(E\times[0,1])} \beta(s,y)K(ds,dx) - \qquad (2.4) \\ &\quad \frac{1}{2}\int_0^t \int_{D(E\times[0,1])} \beta^2(s,y)G_s(dx)ds\}, \end{aligned}$$

cf. [D, Sect.7],[EP]. Finally let us define the measure-valued processes

$$H_t^\beta(\Xi) = \int_{D(E\times[0,1])} \mathbf{1}_\Xi(y)B(t,x)G_t(dx) \qquad (2.5)$$

$$H_t(\Xi) = \int_{D(E\times[0,1])} \mathbf{1}_\Xi(y)G_t(dx). \qquad (2.6)$$

From the definition of $\mathbb{P}$ it is obvious that H is the historical process over ξ under $\mathbb{P}$. The following proposition is basic for us:

Proposition 2.1 *[EP, Theorem 5.1] Under $\mathbb{P}^\beta$ the process H^β is the historical process over ξ.*

We need a slightly different version of this result which will be obtained by a Girsanov argument. Let $(L^h, D(L^h))$ be the martingale operator of the path process of ξ.

Proposition 2.2 *For every $\phi \in D(L^h)$ the process $H_t^\beta(\phi)$ is under $\mathbb{P}$ a semimartingale with increasing process $V(\phi) - \int_0^\cdot H_s^\beta(\beta(s)\phi)ds$, where $V(\phi) = \int_0^\cdot H_s^\beta(L^h\phi)ds$ is the increasing process of $H^\beta(\phi)$ under $\mathbb{P}^\beta$. The quadratic variation of the martingale part equals $\int_0^\cdot H_s^\beta(\phi^2)ds$. (Hence under $\mathbb{P}$ the process H^β is a historical process with (negative) immigration $-\beta$, or in other words with an emigration function β.)*

Proof. Applying the Girsanov transformation for martingales we can calculate the semimartingale decomposition of H^β under $\mathbb{P}^\beta$ from the semimartingale decomposition of H^β under $\mathbb{P}$. In order to do that we have to consider the martingale Z of the densities $Z_t = \frac{d\mathbb{P}}{d\mathbb{P}^\beta}|_{\mathcal{F}_t}$. Let M denote the martingale measure associated with the historical Brownian motion H under $\mathbb{P}$. Then we have

$$Z_t = \exp\{-\int_0^t \int_{D(E)} \beta(s,y)M(ds,dy) + \qquad (2.7)$$

$$
\begin{aligned}
& \frac{1}{2}\int_0^t \int_{D(E)} \beta^2(s,y)H_s(dy)ds\} \\
= \; & \exp\{-[\int_0^t \int_{D(E)} \beta(s,y)M(ds,dy) - \int_0^t \int_{D(E)} \beta^2(s,y)H_s(dy)ds] \\
& -\frac{1}{2}\int_0^t \int_{D(E)} \beta^2(s,y)H_s(dy)ds\} \\
= \; & \exp\{-\int_0^t \int_{D(E)} \beta(s,y)N^\beta(ds,dy) - \frac{1}{2}\int_0^t \int_{D(E)} \beta^2(s,y)H_s(dy)ds\},
\end{aligned}
$$

where N^β is the martingale measure associated with H under $I\!P^\beta$, i.e. $(\beta.N^\beta)_t := \int_0^t \int_{D(E)} \beta(s,y)N^\beta(ds,dy)$ is the martingale in the semimartingale decomposition of $H_t(\beta(t))$ under $I\!P^\beta$. This yields in particular, that Z solves under $I\!P^\beta$ the equation

$$
Z_t \;=\; 1 - \int_0^t Z_s d(\beta.N^\beta)_s. \tag{2.8}
$$

According to Proposition 2.1 and Girsanov's theorem (e.g. [RY, p.303]),

$$
\begin{aligned}
& H^\beta(\phi) - V(\phi) - \int_0^\cdot \frac{1}{Z_s} < Z_s, H^\beta(\phi) - V(\phi) >_s \qquad (2.9) \\
= \; & H^\beta(\phi) - V(\phi) + < \beta.N^\beta, H^\beta(\phi) - V(\phi) >
\end{aligned}
$$

is a martingale under $I\!P$. The bracket in the last line equals, again according to [RY, p.303],

$$
\begin{aligned}
& < \text{martingale in the decomposition of } H(\beta) \text{ under } I\!P^\beta, \\
& \quad \text{martingale in the decomposition of } H^\beta(\phi) \text{ under } I\!P^\beta > \\
= \; & < \text{martingale in the decomposition of } H(\beta) \text{ under } I\!P, \\
& \quad \text{martingale in the decomposition of } H^\beta(\phi) \text{ under } I\!P > \\
= \; & < \beta.K, B\cdot\phi.K > \\
= \; & \int_0^\cdot G_s(B\beta\phi)ds \\
= \; & \int_0^\cdot H_s^\beta(\beta\phi)ds.
\end{aligned}
$$

Because the quadratic variation of this martingale remains unchanged under a change of measure the proposition is proved. $\diamond$

2.3 Comparison of two historical processes with different non-interactive emigration

We consider two function $b^i, i = 1,2$, from $[0,\infty) \times E$ to $[0,1]$ and define β^i and A^i by b^i as in (2.1) above. Then $\beta^i, i = 1,2$, do not depend on ω. For a measure H on $\{x^t | x \in D\}$ (, where $x^t(s) := x(s), s < t$, $x^t(s) := x(t), s \geq t$,) and a function $f \in C_b(E)$ we define $H(f) := \int_D f(x_t^t) H(dx)$.

Lemma 2.3 *For every $T > 0$ there exists a constant $C_T < \infty$ such that*

$$\mathbb{E}[(\sup_{||f||_{BL} \leq 1} |H_t^{\beta^1}(f) - H_t^{\beta^2}(f)|)^2] \ \leq \ C_T \int_0^t E[|b^1(s,\xi_s) - b^2(s,\xi_s)|] ds$$

for all $t \leq T$, where ξ is the Hunt process generated by L.

Proof. Let us write $n = \sum_{i=1}^N \delta_{t_i, Z_i}$.

$$\begin{aligned}
& \mathbb{E}[(\sup_{||f||_{BL} \leq 1} |H_t^{\beta^1}(f) - H_t^{\beta^2}(f)|)^2] \\
= \ & \mathbb{E}[(\sup_{||f||_{BL} \leq 1} \int f(x)(\mathbf{1}_{A^1(t,x)=0} - \mathbf{1}_{A^2(t,x)=0}) G_t(dx))^2] \\
\leq \ & \mathbb{E}[(\int |\mathbf{1}_{A^1(t,x)=0} - \mathbf{1}_{A^2(t,x)=0}| G_t(dx)](\mathbb{E}[G_0(1)] + t) \\
\leq \ & \mathbb{E}[\int \mathbf{1}_{\overset{N}{\underset{j=1}{\cup}} \{b^1(t_i, y(t_i)) \leq Z_i, i=1,\ldots,N, b^2(t_j, y(t_j)) > Z_j\}} G_t(dx)](\mathbb{E}[G_0(1)] + t) + \\
& \mathbb{E}[\int \mathbf{1}_{\overset{N}{\underset{j=1}{\cup}} \{b^1(t_i, y(t_i)) \leq Z_i, i=1,\ldots,N, b^2(t_j, y(t_j)) > Z_j\}} G_t(dx)](\mathbb{E}[G_0(1)] + t).
\end{aligned}$$

The term $\mathbb{E}[\int \mathbf{1}_{\overset{N}{\underset{j=1}{\cup}} \{b^1(t_i, y(t_i)) \leq Z_i, i=1,\ldots,N, b^2(t_j, y(t_j)) > Z_j\}} G_t(dx)]$ equals

$$P[\overset{N}{\underset{j=1}{\cup}} \{b^1(t_i, \xi_{t_i}) \leq Z_i, i = 1, \ldots, N, b^2(t_j, \xi_{t_j}) > Z_j\}], \tag{2.10}$$

where Z_j, t_j are uniform distributed on $[0,1] \times [0,t]$, N has a Poisson distribution and all random variables are independent from each other. Because

$$\begin{aligned}
& P[b^1(t_j, \xi_{t_j}) \leq Z_j < b^2(t_j, \xi_{t_j}) \,|\, (t_j, \xi_{t_j})] \\
= \ & (b^2(t_j, \xi_{t_j}) - b^1(t_j, \xi_{t_j})) \mathbf{1}_{b^2(t_j, \xi_{t_j}) \geq b^1(t_j, \xi_{t_j})}
\end{aligned}$$

we obtain by conditioning that the probability (2.10) is bounded by

$$E[N] \cdot \int_0^t E[(b^2(t_j, \xi_{t_j}) - b^1(t_j, \xi_{t_j}))\mathbf{1}_{b^2(t_j,\xi_{t_j}) \geq b^1(t_j,\xi_{t_j})}] ds/t.$$

By the same argument for $P[b^2(t_j, \xi_{t_j}) \leq Z_j < b^1(t_j, \xi_{t_j}) \,|\, (t_j, \xi_{t_j})]$ we can finally prove the assertion. $\diamond$

2.4 Non-linear martingale problem

We define for $p \geq 1$ an appropriate Wasserstein metric:

$$\rho_p(m_1, m_2) := \left(\inf_Q \int_{M(E)\times M(E)} d_{(E,d)}(\mu, \nu) Q(d\mu, d\nu) \right)^{\frac{1}{p}}, \tag{2.11}$$

where the infimum is taken over all $Q \in M_1(M(E) \times M(E))$ whose marginal distributions are m_1 and m_2 and where for a Polish space E with metric d, the metric $d_{(E,d)}$ on $M(E)$ is defined as follows.

$$d_{(E,d)}(\mu, \nu) \quad := \quad \sup\{|\mu(f) - \nu(f)|; ||f||_{BL} \leq 1\} \tag{2.12}$$

where

$$||f||_{BL} = ||f||_\infty \wedge \inf\{K; |f(x) - f(y)| \leq K d(x, y) \quad \forall x, y \in E\}.$$

Notice that if we replace $d_{(E,d)}$ by $d_{(E,d)} \wedge 2$ in the definition of ρ_1 then ρ_1 is smaller than the original ρ_1 and equivalent with the Prohorov metric and also with $d_{(M(E),d_{(E,d)})}$. Recall that by Hölder's inequality $\rho_q \leq K_{p,q}\rho_p$ if $q \leq p$ with some constant $K_{p,q}$.

Fix $R > 0$. Let P^1 and P^2 be two solutions of the non-linear martingale problem (1.1),(1.2) with $0 \leq -b(m, x) \leq R$, $L(m) = L$ and with $c = 1$.

Theorem A *Let $-b : M_1(M(E)) \times E \to [0, R]$ satisfy*

$$|b(m_1, x) - b(m_2, x)| \leq K_b \rho_2^2(m_1, m_2) \tag{2.13}$$

with some constant K_b. Then there is a unique solution to the non-linear martingale problem (1.1), (1.2).

Proof. Let us define the map α on $C([0,T], M_1(M(E)))$ by

$$\alpha(u) \quad := \quad (P^u \circ X_s^{-1})_{0 \le s \le T}, \tag{2.14}$$

where P^u is the superprocess with immigration function $b^u(s,x) := b(u_s, x)$ on the canonical space $C([0,T], M(E))$ with coordinate process X. Define now for $u^i \in C([0,T], M_1(M(E))), i = 1,2$, the processes H^{β^i} and H as in Proposition 2.2 and Lemma 2.3 with $\beta^i(t,y,\omega) = \frac{1}{R} b(u^i_{\frac{t}{R}}, y(\frac{t}{R}))$ and over the one-particle motion generated by $\frac{L}{R}$. By an obvious scaling property the superprocesses projected down form the processes $(H^{\beta^i}_{tR})_{t \le 0}$ have distributions $P^{u^i}, i = 1,2$. Because $H^{\beta_i}, i = 1,2$, satisfy the assumptions of Proposition 2.2 and Lemma 2.3 we can conclude that

$$\begin{aligned} \rho_2^2(\alpha(u^1)_t, \alpha(u^2)_t) &\le E[(\sup_{||f||_{BL} \le 1} |H^{\beta_1}_{tR}(f) - H^{\beta_2}_{tR}(f)|)^2] \\ &\le K_{t,b} \int_0^{tR} \rho_2^2(u^1_{\frac{s}{R}}, u^2_{\frac{s}{R}}) ds \\ &\le K'_{T,b} \int_0^t \rho_2^2(u^1_s, u^2_s) ds. \end{aligned} \tag{2.15}$$

Hence

$$\sup_{r \le t} \rho_2(\alpha(u^1)_r, \alpha(u^2)_r) \le K'' \int_0^t \sup_{r \le s} \rho_2(u^1_r, u^2_r) ds. \tag{2.16}$$

A Picard-Lindelöf approximation yields that there is a solution u^F of the fix-point equation

$$\alpha(u) \quad = \quad u. \tag{2.17}$$

The approximation starts with $u^1 := (P^0 \circ X_s^{-1})_{0 \le s \le T}$ where P^0 is the superprocess with $b^0(s,x) = b(m_0, x)$ with some $m_0 \in M_1(M(E))$ and for $n \in \mathbb{N}$ we define $u^{n+1} = \alpha(u^n)$. Applying successively the inequality (2.16) with u^{n+1} and u^n we obtain that there exists $u^F := \lim_{n \to \infty} u^{n+1}$, which solves (2.17). By the property (2.17) the superprocess P^{u^F} is a solution of the non-linear martingale problem (1.1),(1.2). The measure P^{u^F} is the unique solution because if we denote by $u^i, i = 1,2$, the flow $(P^i \circ X_s^{-1})$ of two solutions P^i of the martingale problem (1.1),(1.2) then both u^1 and u^2 are fix-points of the equation (2.17). The properties (2.17) and (2.16) implies by Gronwall's inequality that $u^1 = u^2$ and therefore $P^i = P^{u^F}, i = 1,2$. $\diamond$

3 Non-linear one-particle motion

Now we consider the non-linear martingale problem (1.1),(1.2) where $E = \mathbb{R}^d$ and $L(m)$ equals some nice partial differential operator $A(m) - b(m)$, see Theorem B, below. As in the last section we have to be able to couple two different solutions of the non-linear martingale problem in order to prove uniqueness. We will use the stochastic calculus "along historical trees" developed by Perkins in [P1,P2]. In order to describe interacting superprocesses he constructs a unique solution of a strong integral equation, in which the stochastic integral is a "H-historical integral".

3.1 Stochastic calculus along historical trees

Let me recall some of the results in [P1,P2] specialized to the case of non-interactive parameters.

We fix a $T \geq 0$.

Let $C = C([0,T], \mathbb{R}^d)$, and let $(\mathcal{C}_t)$ be the canonical filtration on C , $\Omega = C([0,T], M(C))$, $\hat{\Omega} = \Omega \times C$ with product σ-algebra and let the Campbell-type measure $\hat{\mathbb{P}}$ be defined by $\hat{\mathbb{P}}[A \times B] := \mathbb{P}[\mathbf{1}_A H_T(B)]\mathbb{P}[H_T(1)]^{-1}$, where the coordinate process H on the filtered probability space $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbb{P})$ is the historical Brownian motion with branching rate 1 and with starting point H_0. For the definition of H we refer again to [P2, p.3]. Let $\hat{\mathcal{F}}_t := \mathcal{H}_t \times \mathcal{C}_t$.

Let the functions $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $d^0 : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, and $c : [0,T] \times \mathbb{R}^d \to (0,\infty)$ be bounded and Lipschitz continuous in $x \in \mathbb{R}^d$. We assume that $\frac{\partial c}{\partial s}(s,\cdot)$ and $\frac{\partial^2 c}{\partial x_i x_j}(s,\cdot)$ exist and are Lipschitz continuous in x with a Lipschitz constant uniform in s. Define the functions $a := \sigma\sigma^*$, $h(s,x) := \nabla_x c(s,x)$ and $g(s,x) := \frac{\partial c}{\partial s}(s,x) + \frac{1}{2}\sum_{i,j} \frac{\partial^2 c}{\partial x_i x_j}(s,x) a_{ij}(s,x)$,

$$d := d^0 + a h^* c^{-1} \text{ and } b := (g + h \cdot d^0) \cdot c^{-1}. \tag{3.1}$$

Theorem 3.1 *[P2, Theorems 4.10 and 5.1,Example 4.4]*

a) Let $K_0(\cdot) := \int \mathbf{1}_{\{Y_0(y)\in\cdot\}} c(0, Y_0(y)) H_0(dy)$ *where* $Y_0 : \hat{\Omega} \to \mathbb{R}^d$ *is* $\hat{\mathcal{F}}_0$-*measurable. Then there is a* $\hat{\mathcal{F}}_t$-*predictable* $\mathbb{R}^d$-*valued continuous pro-*

cess Y and a $\mathcal{F}_t$-predictable $M(C)$-valued process K such that

$$
\begin{aligned}
Y_t(y) &= Y_0(y) + \int_0^t \sigma(s, Y_s(y))dy(s) + \int_0^t d^\circ(s, Y_s(y))ds \quad (3.2)\\
K_t(\omega)(\phi) &= \int \phi(Y(\omega, y)^t)c(t, Y_t(\omega, y))H_t(\omega)(dy), \quad (3.3)
\end{aligned}
$$

where the first equation holds a.s. with respect to the first component of $\hat{I\!P}$, i.e. w.r.t. Wiener measure with initial distribution $I\!P[H_0(\cdot)]$. The second equality holds for all $\phi \in C_b(C)$ and $0 \leq t \leq T$, $I\!P$-a.s.

b) *We define the $M(I\!R^d)$-valued projection $\Pi(K)$ of the $M(C)$-valued process K by*

$$\Pi_s(K)(f) := \int_C f(y(s))K_s(dy), \; f \in C_b(I\!R^d). \quad (3.4)$$

Under $I\!P$ we have that for every $f \in C_b^2(I\!R^d)$ the process

$$M_t(f) := \Pi_t(K)(f) - \Pi_0(K)(f) - \int_0^t \Pi_s(K)(A(s)f)ds \quad (3.5)$$

is a martingale with quadratic variation

$$\int_0^t \int_{I\!R^d} c(s,x)f^2(x)\Pi_s(K)(dx)ds, \quad (3.6)$$

where $A(s)f(x) = f(x)b(s,x) + \nabla f(x) \cdot d(s,x) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(s,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ for $f \in C_b^2(I\!R^d)$.

Proof. The Theorem is a special case of [P2, Theorems 4.10 and 5.1]. In Example 4.4 in [P2] the case of non-interactive c is considered. The fact that the expression "$H - a.s.$" used in Theorem 4.10 in [P2] is equivalent with "a.s. with respect to Wiener measure" if all coefficients of the stochastic equation (3.2), (3.3) do not depend on the the process K follows by Remarks 3.3a) and 3.13d) in [P1].

3.2 Non-linear martingale problems

In the case of a non-linear martingale problem we want to consider functions a, b, c, d in (3.5) and (3.6) which depend on the external force caused by the distribution of $\Pi_s(K)$ at time s. Hence we consider bounded functions a, b, c, d, d^0 on $M_1(M(I\!R^d)) \times I\!R^d$ instead of functions on $[0,T] \times I\!R^d$. We assume that the functions are Lipschitz continuous with respect to both variables where in the first variable we use the Wasserstein metric ρ_2 on $M_1(M(I\!R^d))$ cf.(2.11), i.e., we assume

$$|r(m_1, x_1) - r(m_2, x_2)| \leq K_r(\rho_2(m_1, m_2) + |x_1 - x_2|) \tag{3.7}$$

for $r = d^0, \sigma, c, g$ and h. (Note that (3.7) is a stronger condition than the condition (2.13) on b.) Caused by the differentiability assumption for the function c there is an additional condition on c which will be formulated in Theorem B, below. Fix now the starting point $\nu \in M(I\!R^d)$. The martingale problem in question is to find a probability measure on $C([0,T], M(I\!R^d))$ such that for every $f \in C_b^2(I\!R^d)$ the process

$$M_t(f) \quad := \quad X_t(f) - \nu(f) - \int_0^t X_s(A(P \circ X_s^{-1})f)ds \tag{3.8}$$

is a local martingale with quadratic variation

$$\int_0^t \int_{I\!R^d} f^2(x) c(P \circ X_s^{-1}) X_s(dx)ds, \tag{3.9}$$

where $A(m)f(x) = f(x)b(m,x) + \nabla f(x) \cdot d(m,x) \frac{1}{2} \sum_{i,j=1}^d a_{ij}(m,x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. First we prove uniqueness.

Theorem B *Let us suppose that there exist bounded and Lipschitz continuous functions $\tilde{c}_0, \tilde{c}_1, \tilde{c}_{jk}, 1 \leq i,j,k \leq d$, on $M_1(M(I\!R^d)) \times I\!R^d$ such that for the flow $u_s := P \circ X_s^{-1}$ of every solution P of (3.8, 3.9) we have*

$$\frac{\partial}{\partial s} c(u_s, x) = \tilde{c}_0(u_s, x), \qquad \frac{\partial}{\partial x_i} c(u_s, x) = \tilde{c}_i(u_s, x) \tag{3.10}$$

$$\text{and } \frac{\partial^2}{\partial x_j \partial x_k} c(u_s, x) = \tilde{c}_{jk}(u_s, x)$$

for $1 \leq i,j,k \leq d$. Assume (3.7). Then there exists at most one probability measure P on $C([0,T], M(I\!R^d))$ which solves (3.8,3.9) with $d = d^0 + ah^* c^{-1}, b = (g + hd^0)c^{-1}$, where

$$h(m,x) \quad = \quad (\tilde{c}_1(m,x), \ldots, \tilde{c}_d(m,x)) \text{ and} \tag{3.11}$$

$$g(m,x) \quad = \quad \tilde{c}_0(m,x) + \frac{1}{2} \sum_{j,k=1}^d \tilde{c}_{jk}(m,x) a_{jk}(m,x). \tag{3.12}$$

Proof. Let $P^i, i = 1,2$, be two solutions of (3.8, 3.9). Define the corresponding flows by $u_s^i = P^i \circ X_s^{-1}$. We can apply Theorem 3.1 with the functions $\sigma^i(s,x) = \sigma(u_s^i, x)$, $d^{0,i}(s,x) = d^0(u_s^i,x)$, $c^i(s,x) = c(u_s^i,x)$ $g^i(s,x) = g(u_s^i,x)$ and $h^i(s,x) = h(u_s^i,x)$. Then the distributions of the processes $\Pi(K^i), i = 1,2$, as defined in (3.3, 3.4) with these functions equals P^i. By Theorem 3.1b) we have that

$$\begin{aligned}
{\rho_2}^2(u_s^1,u_s^2) &\leq I\!\!P[(\sup_{||f||_{BL}\leq 1} |\Pi(K_t^1)(f) - \Pi(K_t^2)(f)|)^2] \\
&\leq I\!\!P[(\sup_{||f||_{BL}\leq 1} \int |f(Y^1(t,y))c(u_t^1,Y^1(t,y)) - f(Y^2(t,y))c(u_t^2,Y^2(t,y))|H_t(dy))^2].
\end{aligned}$$

This can be bounded by

$$\begin{aligned}
I\!\!P[(\int \{ ||c||_\infty |Y^1(t,y) - Y^2(t,y)| + & \\
|c(u_t^1,Y^1(t,y)) - c(u_t^2,Y^2(t,y))| \} H_t(dy))^2]. &
\end{aligned} \tag{3.13}$$

Because $Y_j^i(t) - Y_j^i(0) - \int_0^t d^0(u_s^i, Y^i(s,y))ds$ are continuous martingales for $1 \leq i \leq d$ with covariation $\sum_k a_{jk}(u_s^i, Y^i(s,y))ds$ and because $c \in C_b^{1,2}$ we have by the Itô-formula that

$$c(t,Y^i(t,y)) = c(0,Y^i(0,y)) + \int_0^t h(s,Y^i(s,y))dY^i(s,y) + \int_0^t g(s,Y^i(s,y))ds.$$

Hence (3.13) equals

$$\begin{aligned}
I\!\!P\Big[\Big(\int \Big\{ \Big| \int_0^t & \sigma(u_s^1,Y^1(s,y)) - \sigma(u_s^2,Y^2(s,y))dy(s) + \\
& \int_0^t d^0(u_s^1,Y^1(s,y)) - d^0(u_s^2,Y^2(s,y))ds \Big| \cdot ||c||_\infty + \\
& \int_0^t h\sigma(u_s^1,Y^1(s,y)) - h\sigma(u_s^2,Y^2(s,y))dy(s) + \\
& (hd^0+g)(u_s^1,Y^1(s,y)) - (hd^0+g)(u_s^2,Y^2(s,y))ds \Big| \Big\} H_t(dy)\Big)^2\Big].
\end{aligned}$$

By Cauchy-Schwarz and the formula for the second moment of a superprocess this is bounded by

$$I\!\!P\Big[\int \Big\{ \Big| \int_0^t \sigma(u_s^1,Y^1(s,y)) - \sigma(u_s^2,Y^2(s,y))dy(s) +$$

$$
\begin{aligned}
& \int_0^t d^0(u_s^1, Y^1(s,y)) - d^0(u_s^2, Y^2(s,y))ds\Big| \cdot ||c||_\infty + \\
& \int_0^t h\sigma(u_s^1, Y^1(s,y)) - h\sigma(u_s^2, Y^2(s,y))dy(s) + \\
& (hd^0 + g)(u_s^1, Y^1(s,y)) - (hd^0+g)(u_s^2, Y^2(s,y))ds\Big|\Big\}^2 H_t(dy)\Big] \\
& \cdot (I\!\!P[H_0(1)] + t) \\
= \; & E\Big[\int\Big\{\Big|\int_0^t \sigma(u_s^1, Y^1(s,W)) - \sigma(u_s^2, Y^2(s,W))dW(s) + \\
& \int_0^t d^0(u_s^1, Y^1(s,W)) - d^0(u_s^2, Y^2(s,W))ds\Big| \cdot ||c||_\infty + \\
& \int_0^t h\sigma(u_s^1, Y^1(s,W)) - h\sigma(u_s^2, Y^2(s,W))dW(s) + \\
& (hd^0 + g)(u_s^1, Y^1(s,W)) - (hd^0+g)(u_s^2, Y^2(s,W))ds\Big|\Big\}^2\Big] \\
& \cdot (I\!\!P[H_0(1)] + t),
\end{aligned}
$$

where W is a Brownian motion with initial distribution $I\!\!P[H_0(\cdot)]$ and $Y^i(s,W)$ is a solution of (3.2) with W instead of y and $\sigma^i(s, W_s) = \sigma(u_s^i, W_s)$. Because $hd^0 + g$ and $h\sigma$ also satisfy (3.7) we can bound the last expression by

$$
\begin{aligned}
& 4(I\!\!P[H_0(1)] + t)\max\{K_\sigma^2, tK_{d^0}^2, tK_{(hd^0+g)}^2, K_{\sigma h}^2\}(||c||_\infty^2 \vee 1) \\
& \int_0^t (E[|Y^1(s,W) - Y^2(s,W)|^2]ds + \rho_2(u_s^1, u_s^2)\,)ds \\
\leq \; & K_T' \int_0^t (E[|Y^1(s,W) - Y^2(s,W)|^2]ds + \rho_2(u_s^1, u_s^2)\,)ds.
\end{aligned}
$$

It remains to prove that

$$
E[\sup_{s\leq t} |Y^1(s,W) - Y^2(s,W)|^2] \leq K_T \int_0^t \rho_2^2(u_s^1, u_s^2)ds \tag{3.14}
$$

with a finite constant K_T. We define similarly as in [F],

$$
\begin{aligned}
A_t &:= \int_0^t d^0(u_s^1, Y^1(s,W)) - d^0(u_s^2, Y^2(s,W))ds \\
M_t &:= \int_0^t \sigma(u_s^1, Y^1(s,W)) - \sigma(u_s^2, Y^2(s,W))dW(s).
\end{aligned}
$$

By Burkholder-Davis-Gundy's inequality we obtain

$$\begin{aligned}
&E[\sup_{s\le t}|M_s|^2]\\
&\le\ K(2)\sum_{i=1}^d E[\int_0^t\sum_{j=1}^d|\sigma_{ij}(u_s^1,Y^1(s,W))-\sigma_{ij}(u_s^2,Y^2(s,W))|^2ds\\
&\le\ K(2)K_\sigma^2 tE[\sup_{s\le t}|Y^1(s,W)-Y^2(s,W)|^2]\\
&\quad +K(2)K_\sigma^2\int_0^t\rho_2^2(u_s^1,u_s^2)ds
\end{aligned}$$

with some constant $K(2)$. For A we obtain

$$\begin{aligned}
E[\sup_{s\le t}|A_s|^2] &\le K_{d^0}^2 E\Big[\Big(\int_0^t\rho_2(u_s^1,u_s^2)ds+\int_0^t|Y^1(s,W)-Y^2(s,W)|ds\Big)^2\Big]\\
&\le\ 2K_{d^0}^2 t\int_0^t\rho_2^2(u_s^1,u_s^2)ds+2K_{d^0}^2t^2E[\sup_{s\le t}|Y^1(s,W)-Y^2(s,W)|^2].
\end{aligned}$$

Therefore

$$\begin{aligned}
&E[\sup_{s\le t}|Y^1(s,W)-Y^2(s,W)|^2]\le 2E[\sup_{s\le t}|M_s|^2]+2E[\sup_{s\le t}|A_s|^2]\\
&\le\ (2K(2)K_\sigma^2+4K_{d^0}^2t)tE[\sup_{s\le t}|Y^1(s,W)-Y^2(s,W)|^2]\\
&\le\ (2K(2)K_\sigma^2+4K_{d^0}^2t)\int_0^t\rho_2^2(u_s^1,u_s^2)ds.
\end{aligned}$$

Hence for $t<\frac{1}{2K(2)K_\sigma^2+4K_{d^0}^2}\wedge 1$ we have (3.14). This implies by the previous calculations that

$$\rho_2^2(u_t^1,u_t^2)\le K_T'\int_0^t\rho_2^2(u_s^1,u_s^2)ds$$

for small t. Gronwall's lemma yields that $u_t^1=u_t^2$ for small t. Exploring now the Markov property of the two solutions we obtain uniqueness for all $t\le T$, cf. [F], and the assertion is proved. ◇

Generally, existence of a solution to (3.8), (3.9) is proved by approximation with weakly interacting N-type superprocesses, cf. the appendix. In order to prove an existence result with the present techniques we have to be more specific about the function c, e.g. it suffices that c is a *finitely based* function with *finitely based base functions*.

Corollary 3.2 *Let* $c(m,x) = \Phi(m(F_1),\ldots,m(F_k),x)$ *with* $\Phi \in C_b^3(I\!R^{k+d})$ *and* $F_i(\mu) = \phi_i(\mu(f_{1i}),\ldots,\mu(f_{k_i i}))$ *such that* $f_{ji}, j = 1,\ldots,k_i, \phi_i, i = 1,\ldots,k$. *We keep on assuming (3.7) and the boundedness of the functions* $a_{ij}, d_k, d_k^0, 1 \leq i,j,k \leq d$ *and* b. *Then there exists a unique solution of (3.8), (3.9).*

Proof. Uniqueness follows by Theorem B, if we take for $\tilde{c}$ appropriate differentiations of c. Let P^u denote the superprocess with parameters depending on the flow $u \in C([0,T].M_1(M(I\!R^d)))$, e.g. $a_{ij}^0(s,x) = a_{ij}(u_s,x)$ and $c^0(s,x) = \Phi(u_s(F_1),\ldots,u_s(F_k),x)$. The starting point of the Picard-Lindelöf approximation as in Theorem A is now u^0 the flow of the superprocess $P^{u^{m_0}}$ with parameter depending on some constant flow $u_s^{m_0} = m_0$ for all s. Define $u^{n+1} = \alpha(u^n) = (P^{u^n} \circ X_s^{-1})_{0\leq s\leq T}$. We have by the boundedness assumptions that

$$\begin{aligned} |\frac{\partial c}{s} c^{u^n}(s,x) - \frac{\partial c}{s} c^{u^n}(s,y)| &\leq \sup_{t\leq T} E^{u^n}[\mathcal{A}(u_t^{n-1})F_i(X_t)]K_\Phi |x-y| \\ &\leq \sup_{t\leq T} KE[H_t(1)]|x-y| \end{aligned}$$

with a finite constant $K = K_{\Phi,\phi_i,f_i,a_{ij},d_k,b,c}$ (, where $\mathcal{A}(m)$ is defined in (1.3)). Hence the Lipschitz condition for $\frac{\partial c}{\partial s}$ in the Remark following Theorem 3.1 is proved. It is straight forward to see that the other conditions for Theorem 3.1 are all satisfied. Hence we can construct P^{u^n} and $P^{u^{n+1}}$ as a strong solution of a stochastic equation driven by a historical process. Proceeding now as in the proof of Theorem B with $P^1 = P^{u^n}$ and $P^2 = P^{u^{n+1}}$ we are led to

$$\rho_2^2(u_t^{n+1},u_t^n) \leq K_T' \int_0^t \rho_2^2(u_s^n,u_s^{n-1})ds,$$

which finally yields a solution u^F of $\alpha(u^F) = u^F$. The superprocess P^{u^F} solves (3.8),(3.9). $\diamond$

Of course, all assumptions on c are satisfied for constant c.

Corollary 3.3 *Assume* c *is constant and* σ *and* d *are bounded and Lipschitz continuous in* (m,x) *with respect to* ρ_2 *in the first component. Then there exists a unique probability measure* P *on* $C([0,T],M(I\!R^d)$ *which solves (3.8),(3.9) with* $b = 0$.

Examples. Assume that c satisfies the assumption of Theorem B. Let us now give examples for which we can satisfy condition (3.1) in Theorem 3.1.

First of all a necessary condition is that $hd + g = hah^* c^{-1} + cb$ with h and g as in (3.12) and (3.11), i.e.

$$c(m,x)\Big(\frac{1}{2}\sum_{j,k=1}^{d} \tilde{c}_{jk}(m,x)a_{jk}(m,x) + \sum_{i=1}^{d} \tilde{c}_i(m,x)d_i(m,x)\Big) \tag{3.15}$$
$$= c^2(m,x)b(m,x) + \sum_{j,k=1}^{d} \tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k(m,x).$$

- Hence if c, a, d are given with c strickly positive, a possible choice is

  $$\begin{aligned} b(m,x) &= c(m,x)^{-1}\Big(\frac{1}{2}\sum_{j,k=1}^{d} \tilde{c}_{jk}(m,x)a_{jk}(m,x) + \sum_{i=1}^{d} \tilde{c}_i(m,x)d_i(m,x)\Big) \\ &\quad -c(m,x)^{-2}\sum_{j,k=1}^{d} \tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k. \end{aligned}$$

  It is clear that b is Lipschitz continuous and bounded if $\tilde{c}, a_{jk}, d_i$ are as well. Under the same conditions $d_i^0 := d - c^{-1}\sum_{j=1}^{d} a_{ij}\tilde{c}_j$ is also Lipschitz continuous and bounded. Then the functions a, b, c, d, d^0 satisfy all assumptions of Theorem B.

- If a and b are given, a possible choice for the functions d and d^0 is

  $$\begin{aligned} d_j(m,x) = \Big(\sum_{j=1}^{d} \tilde{c}_j(m,x)\Big)^{-1}\Big(&\sum_{j,k=1}^{d} \tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k(m,x) \\ &-c(m,x)b(m,x) - \frac{1}{2}\sum_{j,k=1}^{d} \tilde{c}_{jk}(m,x)a_{jk}(m,x)\Big) \end{aligned}$$

  and $d_j^0 = d_j - c^{-1}\sum_{i=1}^{d} a_{jk}\tilde{c}_i$ for $1 \le j \le d$.

- If $b = 0$ and a, d are given then c has to satisfy

  $$c(m,x)\Big(\frac{1}{2}\sum_{j,k=1}^{d} \tilde{c}_{jk}(m,x)a_{jk}(m,x) + \sum_{i=1}^{d} \tilde{c}_i(m,x)d_i(m,x)\Big)$$
  $$= \sum_{j,k=1}^{d} \tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k(m,x), \tag{3.16}$$

  which seems to be very restrictive.

- If $c(m,x) = c_0(x)$ with $c_0 \in C_b^3(I\!\!R^d)$ then the functions in (3.10) are computed as follows: $c_0(m,x) = 0, c_i(m,x) = \frac{\partial c_0}{\partial x_i}(x), \tilde{c}_{jk}(m,x) = \frac{\partial^2 c_0}{\partial x_j \partial x_k}(x), 1 \leq i,j,k \leq d$. But nevertheless the conditions an a, b and d are not much simplified. Therefore it seems that only a constant branching variance c leads to reasonable concrete examples, cf. Corollary 3.3.

A Propagation of chaos for weakly interacting superprocesses

Theorem A.1 *Let $L(m)f(x) := \sum a_{ij}(m,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum d_i(m,x)\frac{\partial f}{\partial x_i}(x)$ be a second order partial differential equation operator on $I\!\!R^d$.*

1. *Let the functions $a_{ij}, d_k, 1 \leq i,j,k \leq d, c$ and b satisfy the following assumptions for functions r on $M_1(M(I\!\!R^d)) \times I\!\!R^d$*

$$|r(m_1,x_1) - r(m_2,x_2)| \leq K_r(\rho_1(m_1,m_2) + |x_1 - x_2|) \qquad \text{(A.1)}$$

$$\sup_{x \in I\!\!R^d} r(m,x) < \infty \text{ for each } m \in M_1(M(I\!\!R^d)), \qquad \text{(A.2)}$$

where the Wasserstein metric $\rho_1 = d_{(M_1(I\!\!R^d), d_{(I\!\!R^d, |\cdot|)})}$ is defined in (2.12), below. Additionally we assume that the the vectors $\vec{X}_0^N$ are exchangeable and that one of the following growth conditions is satisfied:

$$\sup_{m \in M_1(M(I\!\!R^d))} \int\int |r(m,x)|\mu(dx)m(d\mu) \leq K_r' < \infty \qquad \text{(A.3)}$$

for all functions $b, c, a_{ij}, d_k, 1 \leq i,j,k \leq d$ or

$$\sup_{m \in M_1(M(I\!\!R^d))} \int_{I\!\!R^d} |r(m,x)|\mu(dx) \leq K_r'\mu(1) + K_r'' \qquad \text{(A.4)}$$

for all functions $b, c, a_{ij}, d_k, 1 \leq i,j,k \leq d$. Then there exists an exchangeable solution of the martingale problem associated with (1.5).

2. *Assume additionally that for each $f \in C_b^2(I\!\!R^d)$ we have $\sup_N P[(X_0^{1,N}(f))^2] < \infty$. Then the sequence $\{\Pi^N\}_{N \in I\!\!N} \subset M_1(M_1(C_{M(I\!\!R^d)}))$ of distributions of $\frac{1}{N}\sum_{j=1}^N \delta_{X^{j,N}}$ is tight and every accumulation point Π^∞ is supported by the set of solutions of the martingale problem (1.1), (1.2).*

3. If we assume finally that there exists only one solution P^∞ to the martingale problem (1.1), (1.2) then the "Propagation of Chaos" holds.

Sketch of proof. An exchangeable solution of (1.5) can be constructed by weak approximation with interacting multitype branching diffusions. The condition (A.3) (resp. (A.4)) ensures the tightness of the interacting branching diffusions as well as its existence as an accumulation point of a sequence of branching random walks. The latter can be constructed as marked point processes, which are exchangeable by construction. It is wellknown that the sequence $\{\Pi^N\}_N$ is tight if sequence of the intensity measures $\{I(\Pi^N)\}_{N\in\mathbb{N}} \subset M_1(C_{M(\mathbb{R}^d)})$ defined by $I(\Pi^N)(F) := \frac{1}{N}\sum_{i=1}^N E[F(X^{i,N})] = E[F(X^{1,N})]$ are tight. By wellknown criteria for tightness of measure-valued processes, cf. [D], we only have to show that the distributions of $\{X^{1,N}_\cdot(f)\}_N$ are tight for each $f \in C_b^2(\mathbb{R}^d)$. The latter follows by tightness criteria as in [EK], e.g. the Aldous-Rebolledo criterium from the growth conditions (A.3) or (A.4). The identification of the limit points of $\{\Pi^N\}_{N\in\mathbb{N}}$ follows from the fact that $\int_{M_1(C_{M(\mathbb{R}^d)})} \Psi^2(Q)\Pi^\infty(dQ) = 0$ with

$$\begin{aligned}\Psi(Q) \;=\; \int_{C_{M(\mathbb{R}^d)}} \Big[e_f(\omega(t)) - e_f(\omega(r)) + \int_r^t \Big\{ \omega(s) \Big(L(s,Q_s)f + b(s,Q_s)f \\ - c(s,Q_s)f^2 \Big) e_f(\omega(s)) \Big\} ds g(\omega(r_1),\ldots,\omega(r_k)) \Big] Q(d\omega).\end{aligned}$$

By assumption (A.1) Ψ is continuous and bounded by the uniform integrable function $K'(\frac{1}{N}\sum_{i=1}^N K_0 X_T^{i,N}(1) + K_1)$. The last part of the Theorem follows by standard arguments of the "Propagation of Chaos" techniques, cf. [S1,S2].

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