Superprocesses and McKean-Vlasov equations with creation of mass

L. Overbeck*

Department of Statistics,

University of California, Berkeley,

367, Evans Hall

Berkeley, CA 94720,

U.S.A.†

Abstract

Weak solutions of McKean-Vlasov equations with creation of mass are given in terms of superprocesses. The solutions can be approximated by a sequence of non-interacting superprocesses or by the mean-field of multitype superprocesses with mean-field interaction. The latter approximation is associated with a propagation of chaos statement for weakly interacting multitype superprocesses.

Running title: Superprocesses and McKean-Vlasov equations .

1 Introduction

Superprocesses are useful in solving nonlinear partial differential equation of the type $\Delta f = f^{1+\beta}$, $\beta \in (0,1]$, cf. [Dy]. We now change the point of view and show how they provide stochastic solutions of nonlinear partial differential

*Supported by an Fellowship of the Deutsche Forschungsgemeinschaft.

†On leave from the Universität Bonn, Institut für Angewandte Mathematik, Wegelerstr. 6, 53115 Bonn, Germany.

equation of McKean-Vlasov type, i.e. we want to find (weak) solutions of

$$\frac{\partial}{\partial t}\mu_t = \sum_{i,j=1}^{d} a_{ij}(x,\mu_t)\frac{\partial^2}{\partial x_i \partial x_j}\mu_t + \sum_{i=1}^{d} d_i(x,\mu_t)\frac{\partial}{\partial x_i}\mu_t + b(x,\mu_t)\mu_t. \quad (1.1)$$

A weak solution $\mu = (\mu_s) \in C([0,T], M(I\!R^d)$ satisfies

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(\sum a_{ij}(\mu_s)\frac{\partial^2}{\partial x_i \partial x_j} f + \sum d_i(\mu_s)\frac{\partial}{\partial x_i} f + b(\mu_s)f)ds.$$

Equation (1.1) generalizes McKean-Vlasov equations of two different types. Firstly, if $b = 0$ it is the classical McKean-Vlasov equations associated with nonlinear $I\!R^d$-valued diffusions and studied in many papers, e.g. [L,Oel,S1,S2]. If only b depends on μ then this equation is studied in [CR], [COR]. It is used in applications for some biological problems, like conduction of nerve impulse, cf. [GS,SP] and the references therein. Equations like (1.1) with non-linear drift in the diffusion and nonlinear reaction term are sometimes called reaction-convection-diffusion equation ([Lo, p. 287]) and can be found in many biological applications, cf. [M, Section 11.4].

Our main result concerning equation (1.1) is Theorem 3.1 in Section 3 and claims that (under Lipschitz conditions on the coefficients of (1.1)) a (unique) solution of (1.1) can be found as the limit of (the interacting one-type superprocess) $\frac{1}{N}\sum_{i=1}^N X^{i,N}$ as $N \to \infty$ where $(X^{1,N}, \ldots, X^{N,N})$ is a N-type superprocess with mean-field interaction. Obviously, this result should be embedded in a *Propagation of Chaos* statement for weakly interacting N-type superprocesses (including in particular tightness of $\frac{1}{N}\sum \delta_{X^{i,N}}$). This is the topic of Section 2. However, this result does not necessarily implies that the limit process gives a solution of (1.1), because the application I from $M_1(M(I\!R^d))$ to $M(I\!R^d)$ defined by

$$I(m)(f) := \int \nu(f)m(d\nu) \quad (1.2)$$

fails to be continuous. Finally, in Proposition 3.3 we construct a unique solution of (1.1) by a Picard-Lindelöf approximation under stronger condition as in Theorem 3.1. This approximation gives a solution of (1.1) as a limit of intensity measures of superprocesses with non-interactive immigration.

2 Weakly interacting and non-linear superprocesses

In this section we construct a non-linear superprocess as an accumulation point of a sequence of weakly interacting multitype superprocesses.

2.1 Weakly interacting superprocesses

First we want to describe an N-type superprocess with an interaction depending only on the empirical process. We start with real-valued (resp. positive) functions $a_{ij}, d_k, 1 \leq i,j,k \leq d$, and b (resp. c) defined on $[0,\infty) \times I\!R^d \times M_1(M(I\!R^d))$, where $M_{(1)}(E)$ denotes the space of (probability) measures on the Polish space E equipped with the topology of weak convergence. Define for every $s \geq 0$, $m \in M_1(M(I\!R^d))$ the (time-inhomogeneous) operator $L(s,m)$ on $C_0^2(I\!R^d)$ by

$$L(s,m)f(x) := \sum_{i,j=1}^{d} a_{ij}(s,x,m)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} d_i(s,x,m)\frac{\partial f}{\partial x_i}(x). \quad (2.1)$$

The operator serves as a description of the one-particle motion. Let us denote the family $(L(s,m))_{s\in[0,\infty),m\in M_1(M(I\!R^d))}$ by $\mathcal{L}$. For the weakly interacting superprocess the function $b(s,x,m)$ describes the mean branching rate and $c(s,x,m)$ the variance in the branching rate while the empirical distribution of the process equals m.

In order to state now the basic definition we need some notation. Let $N \in I\!N$ be fixed. If $\vec{\mu} = (\mu_1,\ldots,\mu_N) \in (M(I\!R^d))^N, \vec{f} = (f_1,\ldots,f_N) \in (\mathcal{B}_b^{(+)}(I\!R^d))^N$, where $\mathcal{B}_b^{(+)}$ are bounded measurable (non-negative) functions, then

$$R(\vec{\mu}) := \frac{1}{N}\sum_{i=1}^{N} \delta_{\mu_i} \in M_1(M(I\!R^d))$$

and $\vec{\mu}(\vec{f}) := \sum_{i=1}^{N} \mu_i(f_i)$, where $\mu(f) = \int f d\mu$. The exponential function $e_{\vec{f}}$ is defined by $e_{\vec{f}}(\vec{\mu}) = \exp(-\vec{\mu}(\vec{f}))$. Finally, we denote for a Polish space E the space of E-valued continuous paths by C_E.

Definition *2.1 We call a measure $P^N \in M_1(C_{M(I\!R^d)^N})$ an N-type weakly* *interacting measure-valued diffusion starting at $\vec{\mu}_0$ if*

$$e_{\vec{f}}(\vec{X}_t) - e_{\vec{f}}(\vec{\mu}_0) + \int_0^t e_{\vec{f}}(\vec{X}_s) \sum_{j=1}^N X_s^j (L(s, R(\vec{X}_s)) f_j + b(s, R(\vec{X}_s)) f_j - c(s, R(\vec{X}_s)) f_j^2) ds \tag{2.2}$$

is a P^N martingale for all $\vec{f} = (f_1, \ldots, f_N)$ with non-negative $f_i \in C_0^2$. $\vec{X}$ denotes the coordinate process on $C_{M(\mathbb{R}^d)^N}$.

2.2 Propagation of Chaos

One main issue in this section is the question under which conditions the sequence $\{P^N\}_{N \in \mathbb{N}}$ is P^∞-chaotic with a measure $P^\infty \in M_1(C_{M(\mathbb{R}^d)})$.

Definition *2.2 Let $P^\infty \in M_1(C_{M(\mathbb{R}^d)})$. We say that the sequence $\{P^N\}_{N \in \mathbb{N}}$ is P^∞-chaotic if for every $k \in \mathbb{N}$*

$$P^N \circ (X_1, \ldots, X_k)^{-1} \Longrightarrow \otimes_{i=1}^k P^\infty \text{ as } N \geq k \text{ tends to infinity,} \tag{2.3}$$

cf. [S1,S2].

Intuitively, this means that the interaction disappears if the number of types N tends to infinity. Typically the measure P^∞ is a solution of a non-linear martingale problem. Roughly speaking, because the law of large numbers implies $R(\vec{X}_s^N(\omega)) \Rightarrow P^\infty \circ X_s^{-1}$, the measure P^∞ solves the martingale problem characterized by the martingale property of the process

$$e_f(X_t) - e_f(X_0) + \int_0^t e_f(X_s) X_s (L(s, P^\infty \circ X_s^{-1}) f + b(s, P^\infty \circ X_s^{-1}) f - c(s, P^\infty \circ X_s^{-1}) f^2) ds \tag{2.4}$$

for every $f \in C_0^2(\mathbb{R}^d)$. Such a measure is called non-linear superprocess with parameter $(\mathcal{L}, b, c)$.

According to [S2] an exchangeable sequence $\{P^N\}_{N \in \mathbb{N}}$ is P^∞-chaotic iff the distribution $\Pi^N \in M_1(M_1(C_{M(\mathbb{R}^d)}))$ of $R(\vec{X}_\cdot^N) = \frac{1}{N} \sum_{j=1}^N \delta_{X^{j,N}}$ converges weakly to δ_{P^∞}. (Here $\{\vec{X}^N\}_{N \in \mathbb{N}}$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $P \circ (\vec{X}^N)^{-1} = P^N$.)

The next theorem gives general conditions under which there is an exchangeable solution P^N to (2.2) and the sequence $\{\Pi^N\}_{N\in\mathbb{N}}$ is tight. Additionally we show that all limits points of $\{\Pi^N\}_{N\in\mathbb{N}}$ are supported by the set of solutions to the martingale problem formulated in (2.4). If (E,d) is a Polish space with metric d we consider on $M_{(1)}(E)$ the Wasserstein metric

$$d_{(E,d)}(\mu,\nu) \quad := \quad \sup\{|\mu(f)-\nu(f)|; ||f||_{BL}\le 1\} \tag{2.5}$$

where $||f||_{BL} = ||f||_\infty \wedge \inf\{K; |f(x)-f(y)| \le Kd(x,y) \quad \forall x,y\in E\}$. Set $\rho_1 = d_{(M(\mathbb{R}^d),d_{(\mathbb{R}^d,|.|)})}$ and $\rho_2 = d_{(\mathbb{R}^d,|.|)}$.

Theorem 2.3 *1. Let the functions $a_{ij}, d_k, 1\le i,j,k\le d, c$ and b satisfy the following assumptions for functions r on $[0,\infty)\times\mathbb{R}^d\times M_1(M(\mathbb{R}^d))$*

$$|r(s,x_1,m_1)-r(s,x_2,m_2)| \le K_r(\rho_1(m_1,m_2)+|x_1-x_2|) \tag{2.6}$$
$$\sup_{x\in\mathbb{R}^d} r(s,x,m) < \infty \text{ for each } m\in M_1(M(\mathbb{R}^d)). \tag{2.7}$$

Additionally we assume that one of the following growth conditions is satisfied:

$$\sup_{s\in[0,\infty)}\sup_{m\in M_1(M(\mathbb{R}^d))}\int\int |r(s,x,m)|\mu(dx)m(d\mu) \le K_0 < \infty \tag{2.8}$$

for all functions $b,c,a_{ij},d_k, 1\le i,j,k\le d$ or

$$\sup_{m\in M_1(M(\mathbb{R}^d))}\int_{\mathbb{R}^d}|r(s,x,m)|\mu(dx) \le K_0\mu(1)+K_1 \tag{2.9}$$

for all functions $b,c,a_{ij},d_k, 1\le i,j,k\le d$.

Then there exists an exchangeable solution of (2.2).

2. Assume additionally that for each $f\in C_b^2(\mathbb{R}^d)$

$$\sup_N P^N[X_0^1(f)^2] < \infty. \tag{2.10}$$

Then the sequence $\{\Pi^N\}_{N\in\mathbb{N}}$ is tight and every accumulation point is supported by the set of solutions $Q\in M_1(C_{M(\mathbb{R}^d)})$ of the martingale problem (2.4) of a non-linear superprocess.

3. *If we assume additionally that the sequence of initial distributions* $\{m_0^N\}_N$ *with* $m_0^N \in M_1(M(\mathbb{R}^d)^N)$ *is* m_0^∞*-chaotic and if there exists only one solution* P^∞ *to the martingale problem (2.4) with initial distribution* m_0^∞ *then* $\{P^N\}_{N\in\mathbb{N}}$ *is* P^∞*-chaotic.*

Proof. 1. An exchangeable solution of (2.2) can be constructed by weak approximation with interacting multitype branching diffusions, cf. [MR, GL, O1]. The condition (2.8) (resp. (2.9)) ensures the tightness of the interacting branching diffusions as well as its existence as an accumulation point of a sequence of branching random walks. The latter can be constructed as marked point processes , cf. [O1], (or as in [RR],) which are exchangeable by construction. This construction is sketched in the appendix. The necessary tightness under condition (2.8) (resp. (2.9)) follows as in the proof of tightness of $\{\Pi^N\}_N$, see part 2 of the present proof.

2. It is well known that the sequence $\{\Pi^N\}_N$ is tight if sequence of the intensity measures $\{I(\Pi^N)\}_N$ is tight in $M_1(C_{M(\mathbb{R}^d)})$, cf. [S1,S2]. The sequence of measures $\{I(\Pi^N)\}_{N\in\mathbb{N}}$ is tight if their one-dimensional projections are tight, i.e. if, by exchangeability, the laws of $X^{1,N}(f)$ build a tight sequence of measures on $C_{\mathbb{R}}$ for every $f \in C_0^2(\mathbb{R}^d)\cup\{1\}$, cf. [D, Sect. 3.6]. The tightness of $\{X^{1,N}(f)\}_{N\in\mathbb{N}}$ may be deduced by the Aldous-Rebolledo criterion. The first condition of that criterion follows by the uniform L^2-boundedness of the initial distribution and (2.9). The second part follows if we can bound

$$E\Big[\Big|\int_S^{S+\theta} X_s^{1,n}\Big(\sum_{i,l=1}^d a_{i,l}(s,R(\vec{X}_s^N))\frac{\partial^2 f}{\partial x_i\partial x_l} + \sum_{i=1}^d d_i(s,R(\vec{X}_s^N)\frac{\partial f}{\partial x_i} + b(s,R(\vec{X}_s^N)f)ds\Big|\Big] \tag{2.11}$$

and

$$E[|\int_S^{S+\theta} X_s^{1,N}(c(s,R(\vec{X}_s^N)f^2)ds|] \tag{2.12}$$

for a bounded stopping time S and $0<\theta<1$. Under (2.9) the term (2.11) is bounded by

$$K_f E[\int_S^{S+\theta}(K_0X_s(1)+K_1)ds] \le \theta K_f'(E[\sup_{s\le L+1} X_s^{1,N}(1)]+1).$$

Applying now the semimartingale decomposition of $X^{1,N}(1)$ and once again assumption (2.9) we obtain

$$E[\sup_{s\leq t} X_s^{1,N}(1)] \leq E[X_0^{1,N}(1)] + \int_0^t K_0' E[\sup_{s\leq r} X_s^{1,N}(1)] + K_1' dr$$

which yields by Gronwall's lemma and the uniform L^2-boundedness of the initial distribution an upper bound for (2.11), which is independent of N. The same procedure applies to (2.12). Under assumption (2.8) we rewrite the term (2.11) by exchangeability as

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N} E\Big[\Big|\int_S^{S+\theta} X_s^{j,n}(\sum_{i,l}^{d} a_{i,l}(s, R(\vec{X}_s^N))\frac{\partial^2 f}{\partial x_i \partial x_l} \\ + \sum_{i=1}^{d} d_i(s,x,R(\vec{X}_s^N)\frac{\partial f}{\partial x_i} + b(s, R(\vec{X}_s^N)f)ds\Big|\Big].\end{aligned}$$

It is immediate that the term (2.11) is then bounded by $\theta K_f K_0$. Together with the same calculations for the quadratic variation, this yields that also in this case the second part of the Aldous-Rebolledo criterion is satisfied and the first part can be proved analogously as under (2.9).

3. *Identification of the sets of accumulations points of* $\{\Pi^N\}_{N\in\mathbb{N}}$.

Let Π^∞ be an accumulation point of $\{\Pi^N\}_{N\in\mathbb{N}}$. For $0 \leq r_1 < \cdots < r_k < r < t$, g bounded and continuous on $M(\mathbb{R}^d)^k$ and $f \in C_0^2 \cup \{1\}$ we define the function F on $M_1(C_{M(\mathbb{R}^d)})$ by

$$\begin{aligned}F(Q) \;=\; \int_{C_{M(\mathbb{R}^d)}} \Big[e_f(\omega(t)) - e_f(\omega(r)) + \int_r^t \Big\{ \omega(s) \Big(L(s,Q_s)f + b(s,Q_s)f - \\ c(s,Q_s)f^2 \Big) e_f(\omega(s)) \Big\} ds g(\omega(r_1),\ldots,\omega(r_k))\Big] Q(d\omega),\end{aligned}$$

where Q_s denotes the distribution of $\omega(s)$ under Q. We will show that

$$\int_{M_1(C_{M(\mathbb{R}^d)})} F^2(Q)\Pi^\infty(dQ) = 0. \tag{2.13}$$

The martingale problem (2.2) implies a formula for the quadratic variation of the exponential martingales $M^{i,N}[e_f]$ defined as in (2.2) with $\vec{f} =$

$(f_1,\ldots,f_N), f_j=\delta_{ij}f$. This formula yields

$$\begin{aligned}\int F^2(Q)\Pi^N(dQ) &\le \frac{K_g}{N^2}E[\sum_{i,j=1}^d\langle M^{i,N}[e_f],M^{j,N}[e_f]\rangle_t]\\ &\le \frac{K_g}{N^2}\sum_{j=1}^d E[\int_r^t X^{j,N}(c(s,R^N(\vec{X}_s))f^2)e_{2f}(X^{j,N})ds]\\ &\le \frac{K_f'''}{N}(t-r)K_f\longrightarrow 0 \text{ as } N\to\infty.\end{aligned}$$

The last inequality follows as in part 2 of the proof by assumption (2.8) (resp. (2.9)) and by the submartingal property of the total mass process $X^{1,N}(1)$. The assertion follows now, if we have that $\int F^2(Q)\Pi^N(dQ)$ converges as $N\to\infty$ to $\int F^2(Q)\Pi^\infty(dQ)$. Because F is under the conditions (2.8) (resp. (2.9)) bounded by K_0 (resp. the uniformly integrable random variables $K'(\frac{1}{N}\sum_{i=1}^N K_0X_T^{i,N}(1)+K_1)$), this can be achieved, if F is continuous on $M_1(C_{M(\mathbb{R}^d)})$. To see the continuity of let us consider the function $F_b(Q):=\int_{C_{M(\mathbb{R}^d)}}Q(d\omega)\int_{\mathbb{R}^d}\omega_s(dx)b(s,Q_s,x)f(x)e_f(\omega_s)$. If $Q_n\to Q$ in $M_1(C_{M(\mathbb{R}^d)})$ then $Q_n\circ\omega(s)^{-1}\in M_1(M(\mathbb{R}^d))$ converges to $Q\circ\omega(s)^{-1}$. Hence we have to show that $m_n\to m$ in $M_1(M(\mathbb{R}^d))$ implies

$$\begin{aligned}&|\int_{M(\mathbb{R}^d)}m_n(d\mu)\int_{\mathbb{R}^d}\mu(dx)b(s,m_n,x)f(x)e_f(\mu)-\\ &\int_{M(\mathbb{R}^d)}m(d\mu)\int_{\mathbb{R}^d}\mu(dx)b(s,m,x)f(x)e_f(\mu)|\to 0.\end{aligned}\tag{2.14}$$

The left hand side of (2.14) is bounded by

$$\int m_n(d\mu)\int\mu(dx)|b(s,m_n,x)-b(s,m,x)||f(x)|e_f(\mu)\tag{2.15}$$

$$+\quad|\int(m_n(d\mu)-m(d\mu))\int_{\mathbb{R}^d}\mu(dx)b(s,m,x)f(x)e_f(\mu)|.\tag{2.16}$$

Using the Lipschitz property we can bound (2.15) by

$$K_b\rho_1(m_n,m)\int_{M(\mathbb{R}^d)}m(d\mu)\int_{\mathbb{R}^d}\mu(dx)f(x)e_f(\mu).$$

If we consider for the moment the one-point compactification $\hat{\mathbb{R}}^d$ of $\mathbb{R}^d$ the integrand is a bounded continuous function on $M(\mathbb{R}^d)$ if $f\ge\epsilon>0$. Hence the second factor is bounded. The first factor converges to 0. The term (2.16) converges because $\mu(dx)b(s,m,x)f(x)e_f(\mu)$ is also a bounded continuous function if $f \geq \epsilon > 0$. The continuity of F follows in the same way. Hence every limit point is supported by the set of solutions of the non-linear martingale problem (2.4), so far viewed as probability measures on $C([0,\infty), M(\hat{I\!R}^d))$. Because every such a probability measure is the distribution of a regular superprocess it does not charge the compactification point and therefore it is indeed a measure on $C([0,\infty), M(I\!R^d))$.

4. The last assertion of the theorem follows now by standard arguments, cf. [S1,S2]. $\diamond$

Remarks. 1. The assumption (2.9) is the usual boundedness condition which is supposed if one considers weak convergence of measure-valued processes whereas condition (2.8) is formulated just for the case of weakly interacting measure-valued processes.

2. A martingale generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ for a non-linear superprocess P^∞ is given by $\mathcal{D}(\mathcal{A}) =$"finitely based functions" and

$$\mathcal{A}(m)F(\mu) = \mu\Big(L(m)\nabla_\cdot F(\mu) + b(m)\nabla_\cdot F(\mu) + c(m)\nabla_\cdot^2 F(\mu)\Big) \tag{2.17}$$

$\nabla_x F(\mu) := \lim_{\epsilon\downarrow 0} \frac{F(\mu+\epsilon\delta_x)-F(\mu)}{\epsilon}$, is the Gâteaux derivative in direction δ_x, the Dirac measure on $x \in E$. Hence, the flow of the one-dimensional marginal distributions $u_s = P \circ X_s^{-1}$ of a non-linear superprocess solves the non-linear 'partial differential equation'

$$\dot{u}_s = \mathcal{A}(u_s)u_s. \tag{2.18}$$

In the special case of mean-field interaction considered in the following section the intensity measure of the non-linear superprocess provides a solution to (1.1).

3. The question of uniqueness of the martingale problem described in (2.4) is discussed in detail in [O2] by means of the Stochastic Calculus on historical trees, cf. [P]. We now give a simple example, where we can show uniqueness directly from the martingale problem (2.4).

Example. Let us consider the sequence of "multitype" superprocesses conditioned on non-extinction, i.e., a_{ij}, d_i, c do not depend on $m, c = 1$ and

$b(s,x,m) = \frac{1}{I(m)(1)}$. Let $\vec{\mu}_0 \in M(I\!R^d)^N$ satisfy $\sum_{i=1}^N (\mu_0)_i(1) = N$. Then there exists only one measure such that

$$e_{\vec{f}}(\vec{X}_t) - e_{\vec{f}}(\vec{X}_0) + \int_0^t \sum_{j=1}^d X_s^{j,N}(Lf_j + \frac{N}{\sum_{i=1}^N X_s^{i,N}(1)} f_j - f_j^2) e_{\vec{f}}(\vec{X}_s) ds$$

is a martingale for all $\vec{f} \in (C_0^2(I\!R^d))^N$, namely the additive H-transform of N-independent superprocesses with the space-time harmonic function $H(s,\vec{\mu}) = N^{-1}\sum_{j=1}^N \mu_j(1)$, cf. [O1]. Let the initial condition be δ_{μ_0}-chaotic with $\mu_0(1) = 1$, e.g. $\vec{\mu}_0^N = (\mu_0, \ldots, \mu_0)$. The function b satisfies assumption (2.9) and therefore, by Theorem 2.3, we have that every limit point of $\{\Pi^N\}_{N\in I\!N}$ is supported by measures P such that

$$e_f(X_t) - e_f(\mu_0) + \int_0^t \sum_{j=1}^d X_s(Lf + \frac{1}{I(P\circ X_s^{-1})(1)} f - f^2) e_f(X_s) ds.$$

Such a measure P gives rise to the linear martingale $X_t(f) - \mu_0(f) - \int_0^t X_s(Lf + \frac{1}{I(P\circ X_s^{-1})(1)} f) ds$. If we apply this with $f = 1$ we obtain $E_P[X_t(1)] = E_P[X_0(1)] + t = 1 + t$ for every solution P. Therefore P is unique. It coincides with the supercritical (time-inhomogeneous) superprocess with branching mean $1 + \frac{1}{1+s}$. It is interesting to notice that P is also an H-transform of the superprocess, namely with the multiplicative space-time harmonic function $H(s,\mu) = e^{-\frac{\mu(1)}{1+s}}$. From the biological point of view one can interpret this result in the following way. If one conditions a particle population on survival, which exhibits a critical branching behaviour and consists of N subpopulations of the same kind, then for large N every subpopulation executes a slightly supercritical branching with time-decreasing rate $1 + \frac{1}{1+s}$.

3 Mean-field interaction

Now we attack the problem to find a solution of (1.1). We have to introduce a special kind of weakly interaction, namely the mean-field interaction. We modify the definition in (2.2) in the following way. We replace the functions a, d, b and c by functions depending on the mean-field of the empirical distribution, e.g. $a_{ij}(s, x, \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}})$ in (2.2) is replaced by $a_{ij}(x, \frac{1}{N}\sum_{i=1}^N X^{i,N})$. A solution of (2.2) is then called a N-type superprocess with mean-field interaction.

3.1 Existence

The next theorem gives existence of a sequence of multitype superprocesses with mean-field interaction $\{(X^{1,N} \dots, X^{N,N})\}_{N \in I\!\!N}$, tightness of $\{\frac{1}{N}\sum_{i=1}^{N}\delta_{X^{i,N}}\}_N$ and a law of large numbers for the sequence $\{\frac{1}{N}\sum_{i=1}^{N} X^{i,N}\}_N$. The proof is sketched because the techniques are the same as in the proof of Theorem 2.3.

Theorem 3.1 *Let the functions $a_{ij}, d_k, 1 \le i,j,k \le d, b$ and c satisfy the following assumptions for functions r on $I\!\!R^d \times M(I\!\!R^d)$*

$$\begin{aligned} |r(x_1,\mu_1) - r(x_2,\mu_2)| &\le K_r(\rho_2(\mu_1,\mu_2) + |x_1 - x_2|) && (3.1)\\ \sup_{x\in I\!\!R^d} r(x,\mu) &< \infty \text{ for each } \mu \in M(I\!\!R^d), && (3.2)\end{aligned}$$

where ρ_2 is defined in (2.5). Additionally we assume that the following growth condition is satisfied for all functions $a_{ij}, d_k, 1 \le i,j,k \le d, b$ and c:

$$\int_{I\!\!R^d} |r(x,\mu)|\mu(dx) \le K_0\mu(1) + K_1. \tag{3.3}$$

Suppose also that $\frac{1}{N}\sum_{i=0}^{N} X_0^{i,N}(1)$ is uniformly in L^2. Then there exists an exchangeable solution $(X^{1,N},\dots,X^{N,N})$ of (2.2) and the distributions of $\{\frac{1}{N}\sum_{i=1}^{N}\delta_{X^{i,N}}\}_N$ build a tight sequence in $M_1(M_1(C_{M(I\!\!R^d)}))$.

If additionally the distribution of $\frac{1}{N}\sum_{i=0}^{N} X_0^{i,N}$ converges to δ_{μ_0}, then every accumulation point of

$$\Big\{ P \circ (\frac{1}{N}\sum_{i=0}^{N} X^{i,N})^{-1} \Big\}_N \tag{3.4}$$

in $M_1(C_{M(I\!\!R^d)})$ is a Dirac distribution on a (deterministic) solution μ of

$$\mu_t(f) \;=\; \mu_0(f) + \int_0^t \mu_s(L(\mu_s)f)ds. \tag{3.5}$$

Hence a weak solution of (1.1) is constructed. Finally, if (1.1) has a unique solution $\mu \in M([0,\infty), M(I\!\!R^d))$, then

$$P \circ (\frac{1}{N}\sum_{i=0}^{N} X^{i,N})^{-1} \Rightarrow \delta_\mu. \tag{3.6}$$

Proof. The construction of an exchangeable solution of (2.2) is the same as in Theorem 2.3, because the mapping $\vec{\mu} \to \frac{1}{N}\sum_{i=0}^{N}\mu^i$ is continuous. The tightness of sequence of the distributions of $\frac{1}{N}\sum_{i=0}^{N}X^{i,N}$ follows as in Theorem 2.3 from the growth conditions (3.3). To prove that every limit point is supported by solutions to (3.5) we follow the same approach as in the proof of Theorem 2.3 with the function

$$F(\mu) \;=\; (\mu_t(f) - \mu_r(f) + \int_r^t \mu_s(L(\mu_s)f)ds)g(\mu_{r_1},\ldots,\mu_{r_l}),$$

$\mu \in C_{M(\mathbb{R}^d)}$. From the continuity and the uniform integrability of F we obtain that $X_t(f) - X_0(f) + \int_r^t X_s(L(X_s)f)ds$ is a martingale under an accumulation point $P^\infty \in M_1(C_{M(\mathbb{R}^d)})$ of the distributions of $\frac{1}{N}\sum_{i=0}^{N}X^{i,N}$. The quadratic variation of this martingale is zero, because $\int F^2(\frac{1}{N}\sum_{i=0}^{N}X^{i,N})dP \;\to\; 0$. Hence $P^\infty = \delta_\mu$, where μ solves (3.5). $\diamond$

Remark. By the martingale property (2.2) the process $\frac{1}{N}\sum_{i=0}^{N}X^{i,N}$ is a solution of a martingale problem associated with a (one-type) superprocess with interaction in the sense of [MR,P]. Therefore Theorem 3.1 says that a solution of (1.1) can be found by weak approximation with interactive superprocesses $Y^N = \frac{1}{N}\sum_{i=0}^{N}X^{i,N}$ whose variances vanish as $N \to \infty$.

3.2 Uniqueness

In this subsection we consider uniqueness of (2.4) with coefficients only depending on $\mu = I(m)$. We first consider the case where the branching is critical, i.e. $b = 0$.

Proposition 3.2 *Let $(A(s,\mu))_{s\geq 0,\mu\in M(\mathbb{R}^d)}$ be a family of linear operators such that for every $\mu_0 \in M(\mathbb{R}^d)$ there is a unique $\mathcal{P}_{\mu_0} \in M(C_{\mathbb{R}^d})$ under which for all $f \in C_0^2$ the process $N(f)$ defined by*

$$N_t(f) = f(\xi_t) - f(\xi_0) - \int_0^t A(s, \mathcal{P}_{\mu_0} \circ \xi_s^{-1})f(\xi_s)ds \tag{3.7}$$

is a martingale and $P_{\mu_0} \circ \xi^{-1} = \mu_0$, where ξ is the coordinate process $C_{\mathbb{R}^d}$.

Let $m \in M_1(M(\mathbb{R}^d))$.

Then there is a unique probability measure P_m on $C_{M(\mathbb{R}^d)}$ such that for all $f \in C_0^2(\mathbb{R}^d)$

$$\left(e_f(X_t) - e_f(X_0) + \int_0^t X_s(A(s, I_s^m)f - c(s, I_s^m)f^2)e_f(X_s)ds\right)_{t \geq 0} \quad (3.8)$$

is a martingale under P_m, where $I_s^m = I(P_m \circ X_s^{-1}) = E_m[X_s(\cdot)]$.

Proof. Let P^1 and P^2 be two solutions of (3.8). It is clear that if the flow of the corresponding intensity measures $(I_s^1)_{s\geq 0}$ and $(I_s^2)_{s\geq 0}$ coincide then $P^1 = P^2 =$ "superprocess with time-inhomegenous one-particle motion generated by $(A(s, I_s^1))_{s\geq 0}$ and branching variance $c(s, I_s^1)$". For $i = 1, 2$, the intensity measure I_s^i equals $\int m(d\mu)\mathcal{P}_\mu^i \circ \xi_s^{-1}$, where $\mathcal{P}_\mu^i \in M_1(C_{\mathbb{R}^d})$ is the one-particle motion of P^i with initial distribution μ which is also the process associated with $(A(s, I_s^i))_{s\geq 0}$. Hence for $i = 1, 2$, the measures $\mathcal{P}^i$ solve (3.7) which implies $I_s^1 = I_s^2 =$ distribution of ξ_i under the unique solution of (3.7).

This proves uniqueness. For existence we see that the superprocess started at m and with one-particle motion generated by the non-linear operator $A(s, \mathcal{P}_\nu \circ \xi^{-1})$ with initial condition $\nu(dx) = \int m(d\mu)\mu(dx)$ solves (3.8). $\diamond$

The next proposition generalizes the uniqueness result in [CR] and shows existence.

Proposition 3.3 *Let us suppose that all functions r=$a_{ij}, d_k, 1 \leq i, j, k \leq d$, and b are bounded and satisfy*

$$\sup_x |r(\mu_1, x) - r(\mu_2, x)| \quad \leq \quad ||\mu_1 - \mu_2|| \quad (3.9)$$

where $||\cdot||$ denotes the norm of total variation. Then there is a unique solution μ^F of (1.1) and the superprocess with coefficients $a_{ij}^{\mu^F}(s, x) = a_{ij}(\mu_s^F, x), d_k^{\mu^F}(s, x) = d_k(\mu_s^F, x), 1 \leq i, j, k \leq d, b^{\mu^F}(s, x) = b(\mu_s^F, x)$ and $c^{\mu^F}(s, x) = c(\mu_s^F, x)$ is the unique solution of (2.4).

Proof. The second assertion follows from the first assertion as in Proposition 3.2. Concerning the first assertion we obtain a solution of (1.1) by a Picard-Lindelöf iteration. $\mu_s^n := P^{\mu^{n-1}} \circ X_s^{-1}$ where P^μ is the superprocess with coefficients $a_{ij}^\mu(s, x) = a_{ij}(\mu_s, x), d_k^\mu(s, x) = d_k(\mu_s, x), 1 \leq i, j, k \leq d, b^\mu(s, x) = b(\mu_s, x)$ and $c^\mu(s, x) = c(\mu_s, x)$. The initial point of the iteration is the flow of a superprocess with coefficient defined by a constant flow

$\mu_s = \mu_0 \quad \forall s$ with some fixed $\mu_0 \in M(I\!R^d)$. We have

$$\mu_t^n - \mu_t^{n-1}(f) = K' \int_0^t \mu_s^n(L(\mu_s^{n-1})f) - \mu_s^{n-1}(L(\mu^{n-2})f)ds$$

which yields by (3.9) that

$$||\mu_t^n - \mu_t^{n-1}|| \leq K'' \int_0^t \mu_s^n(1)||\mu_s^{n-1} - \mu_s^{n-2}||ds. \tag{3.10}$$

Because b is bounded $\sup_{s\leq T}\mu_s^n(1)$ is bounded. From inequality (3.10) we obtain a solution of (1.1) as n tends to infinity. Uniqueness follows by Grownwall's lemma from the same inequality. $\diamond$

Remark. Despite the fact that we constructed already a solution to (2.4) in two different ways we would like to give a construction as a limit of a sequence of multitype superprocesses with mean-field interaction as in Theorem 2.3 in order to give another simulation procedure for the solution of (1.1). The difficulties stem from the fact that the mapping I from $M_1(M(I\!R^d))$ to $M(I\!R^d)$ fails to be continuous, because the function $\nu \to \nu(f)$ is continuous but not bounded. Instead of that I can only construct for every $K \in I\!N$ a solution P^K of the stopped martingale problem:

$$\begin{aligned} e_f(X_t) - e_f(X_0) \quad + \quad & \int_0^{t\wedge T_K} e_f(X_s)X_s(L(s, I(P^K \circ X_s^{-1}))f + \\ & b(s, I(P^K \circ X_s^{-1}))f - c(s, I(P^K \circ X_s^{-1}))f^2)ds \end{aligned} \tag{3.11}$$

is a martingale under P^K for every $f \in C_0^2(I\!R^d)$ and $X_s = X_{s\wedge T_K} \quad P^K$ a.s. with $T_K(\omega) := \inf\{t \geq 0 | X_t(\omega)(1) \geq K\}$ by the chaos-technique as in Theorem 2.3. Because I cannot prove that the stopped martingale problem (3.11) has a unique solution, the techniques in [EK, Theorem 6.3] are not applicable in order to prove existence of a (global) solution to (2.4).

A Construction of an exchangeable weakly interacting superprocess

The starting point is a construction of an exchangeable weakly interacting N-type branching random walk.

Let π be a kernel from $[0,\infty)\times I\!R^d \times M(I\!R^d)^N$ to $I\!R^d$, let $\gamma > 0$ and let for every $\vec{\mu} \in M(I\!R^d)^N$

$$A(\vec{\mu})f(x) \quad := \quad \gamma \int_{I\!R^d} (f(y) - f(x))\pi(s,x,\vec{\mu};dy) \tag{A.1}$$

be the generator of a jump process in $I\!R^d$. The intensity for a particle at $x \in I\!R^d$ and time s to die without children (resp. with two children) is described by $\beta p_0(s,x,m)$ (resp. $\beta p_2(s,x,m)$).

Let $\tilde{\Omega} = \{\tilde{\omega} = \{t_m, \vec{\mu}_m\}_{m\in I\!N}$ with $0 \le t_1 < t_2 < \cdots$, and $\vec{\mu}_m \in M(I\!R^d)^N\}$. We define for every $\tilde{\omega} \in \tilde{\Omega}$ a predictable random measure $v^p \in M([0,\infty)\times M(I\!R^d)^N)$ by

$$v^p(\{t_m,\vec{\mu}_m\}_{m\in I\!N}; ds, d\vec{\nu}) := \sum_{i=1}^N X^i_{s-}\Big(\gamma \int_{I\!R^d} \delta_{\delta_y - \delta_\cdot}(d\nu^i) \otimes_{j\neq i} \delta_O(d\nu^j)$$
$$\pi(s,\vec{X}_{s-};dy) + \beta p_0(s,\vec{X}_{s-})\delta_{-\delta_\cdot}(d\nu^i) + \beta p_2(s,\vec{X}_{s-})\delta_{+\delta_\cdot}(d\nu^i)\Big)\, ds,$$

where $X^i_s(\tilde{\omega}) := X^i_s(\{t_m,\vec{\mu}_m\}_{m\in I\!N}) := \sum_{t_n \le s}(\vec{\mu}_n)_i$, O denotes the zero-measure and $\vec{X}_s = (X^1_s,\ldots,X^N_s)$. Let $v(\cdot,\cdot) = \sum_n \delta_{t_n,\vec{\mu}_n}(\cdot,\cdot)$ be the canonical random measure on $\tilde{\Omega}$. There exists a probability measure $\tilde{P}$ on $\tilde{\Omega}$ such that the stopped processes $\Big(W * (v - v^p)\Big)^{t_n}$ is a uniformly integrable martingale for every n and every predictable $W(\tilde{\omega},s,\mu)$ (- "$*$" indicates integration -).

For a finitely based function $F = \phi(\vec{\mu}\cdot\vec{f}_1,\ldots,\vec{\mu}\cdot\vec{f}_k)$, $\phi \in C^2$ and $\vec{\mu}\cdot\vec{f} = (\mu_1(f_1),\ldots,\mu_l(f_l))$ we consider the predictable function $W(\tilde{\omega},s,\mu) = F(\vec{X}_{s-}+\vec{\mu}) - F(\vec{X}_{s-})$. Then the process $M^{IBRW}[F]$ defined by

$$M^{IBRW}[F]_t := F(\vec{X}_t) - F(\vec{X}_0) -$$
$$\int_0^t ds\, \gamma \sum_{i=1}^N X^i_s\Big(\int_{I\!R^d} (F(\vec{X}_s + e_i(\delta_y - \delta_\cdot)) - F(\vec{X}_s)\,\pi(s,\vec{X}_{s-};dy) +$$
$$\beta p_0(s,\vec{X}_{s-})F(\vec{X}_s - e_i\delta_\cdot) + \beta p_2(s,\vec{X}_{s-})F(\vec{X}_s + e_i\delta_\cdot) - F(\vec{X}_s)\Big)$$

is a local martingale until $T := \sup_n t_n$. (Here $\vec{\mu} + e_i\mu$ means that we add in the i-th component of the vector $\vec{\mu}$ the measure μ.)

Because of our assumptions (2.8) or (2.9) we have $\tilde{P}$-almost surely that $T = \infty$, we get an interacting branching random walk, if we define P as the distribution of $\vec{X}$ under $\tilde{P}$ on $D_{M(\mathbb{R}^d)^N}$.

Starting from a sequence of branching random walks $\{\vec{X}^n\}_{n\in\mathbb{N}}$, whose random walks $\pi^n(\vec{\mu})$ approximates the generator $A(\vec{\mu})$ we can define an $N-$type interacting branching diffusion as an accumulation point of the sequence $\{\vec{X}^n\}_{n\in\mathbb{N}}$ as in [RR,O1]. Then we can rescale the branching behaviour of the $N-$type interacting branching diffusion in order to get an $N-$type interacting superprocess as in [MR,O1]. For a general approach to multitype (non-interacting) superprocesses see [GL].

Acknowledgement: I wish to thank Sylvie Méléard for helpful discussions on the chaos-technique.

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